N64   10292

Elastic Models of the Mantle Corresponding to Variations in the External Gravity Field

WILLIAM M. KAULA

NASA, Goddard Space Flight Center
Greenbelt, Maryland

Abstract. Static earth-tide theory was modified to include interior loads expressible as spherical harmonics, and elastic moduli were assumed to be functions of radius only. Variations of density from this model and the corresponding stress distributions in the crust and mantle were calculated to correspond to observed variations in the gravitational field plus the surface topography up to 4th-degree spherical harmonics. These solutions were made determinate by imposing the condition of minimization of strain energy. For the elastic parameters derived from seismology, the maximum stress difference obtained from the discrepancy between the observed and equilibrium flattenings of the earth was 163 bars. The maximum stress difference found for the sum of all other terms was 97 bars in the lower mantle and 300 bars in the crust. Displacements were always less than 70 meters. Modifications of the solution which take into account finite strain, creep, and viscous deformation are discussed. A model consisting of a fluid layer 35 to 400 km deep and a rigidity ½ seismic in the rest of the mantle results in a reduction of the maximum stress difference in the mantle to 54 bars and an increase of the maximum displacement to 1500 meters.

Introduction. Knowledge of the earth's gravity field has been significantly improved in recent years, principally because of the perturbations of close satellite orbits, but also because of the extension of gravimetric coverage and the application of large computers to its analysis. In particular, the long-term perturbations of orbits have yielded very accurate determinations of the low-degree zonal harmonics; the most recent summary and analysis were made by *Kozai* [1963]. These zonal harmonics are incorporated in the analysis of satellite orbits for tesseral harmonics by *Kaula* [1963]. The order of magnitude of satellite results has been confirmed by some recent analyses of terrestrial gravimetry: least-squares determination of harmonic coefficients by *Uotila* [1962] and autocovariance analysis determination of degree variances by *Kaula* [1959]. The principal features of variations in the earth's gravitational potential indicated by these studies are:

1. Zonal harmonics (normalized) up to degree 4 of the order 10^{-6} times the central term.

2. Zonal harmonics of degrees 5 through 9 of the order 10^{-7}.

3. Tesseral harmonics up to degree 4 of comparable magnitude to the zonal harmonics.

4. Negligible correlation with the corresponding harmonics in the topography for all degrees.

Stresses in the mantle. Although there remain some statistically questionable aspects of all these results except the low-degree zonal harmonics, it seems appropriate now to explore their implications with regard to stresses in the mantle. The subject has been discussed generally by *O'Keefe* [1959], *Munk and MacDonald* [1960b], and *MacDonald* [1963]. *Licht* [1960] has applied the linear viscous model of Vening-Meinesz [*Heiskanen and Vening-Meinesz*, 1958] to the explanation of the third-degree zonal harmonic. We start at the opposite extreme of an elastic model.

The elastic model is assumed to have elastic moduli which are functions of the radius only. We seek solutions for variations in density which are functions of latitude and longitude as well, $\delta\rho(r,\Theta,\phi)$, to account for the observed variations in the external gravity field and the surface load constituted by the topography, while at the same time entailing a minimum of shear stress in the elastic model.

The applicable equations in the earth are the equations of equilibrium for a continuous medium:

$$0 = \rho \frac{\partial W}{\partial x_i} + \sum_j \frac{\partial p_{ij}}{\partial x_j} \qquad (1)$$

and Poisson's equation:

$$\sum_i \frac{\partial^2 W}{\partial x_i^2} = -4\pi G\rho \qquad (2)$$

where ρ is density, W is the gravitational potential, x_1, x_2, and x_3 are cartesian coordinates, p_{ij} is the stress tensor, and G is the gravitational constant.

The potential can be considered as consisting of a potential existing in a reference state plus a perturbation:

$$W = W_0 + \Psi \qquad (3)$$

The corresponding equation for the density is

$$\rho = \rho_0 - \int_0^{x_i - x_{0i}} \frac{\partial \rho_0}{\partial x_i} du_i - \rho_0 \Delta + \delta\rho \qquad (3a)$$

where Δ is the dilation and $\delta\rho$ is a density perturbation from outside the problem: a chemical, structural, or thermal inhomogeneity. In $(3a)$, the rule of summation over the repeated subscript applies, and it will apply in all subsequent equations.

We make the assumptions that: (1) the reference state is one of fluid equilibrium, (2) the stress-strain relationship is elastic, and (3) the displacements $u_i = x_i - x_{0i}$ are small enough so that only linear terms must be considered.

These assumptions enable us to write the constitutive equation as

$$p_{ij} = \delta_{ij}\lambda\Delta + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (4)$$

$$\lambda = k - 2\mu/3, \qquad (4a)$$

where δ_{ij} is the Kronecker delta, μ is the modulus of rigidity, λ is the Lamé constant, and k is the bulk modulus.

The problem thus far is identical with that of earth tides, which has been treated by *Takeuchi* [1950], *Molodenskii* [1953], and *Alterman et al.* [1959], with one exception: the term $\delta\rho$ in the expression for the density, $(3a)$. Substituting the expressions from (3) and $(3a)$ into (1) and (2) and neglecting the products of small quantities, we obtain terms additional to those of the earth tide problem of

$$\delta\rho \; \partial W_0/\partial x_i \qquad (1a)$$

on the right of (1) and

$$-4\pi G \delta\rho \qquad (2a)$$

on the right of (2).

If the hydrostatic reference state is subtracted out, and the equations converted to spherical coordinates, as in equations 7 through 10 of *Alterman et al.* [1959], the density perturbation will appear only in the radial equation of equilibrium (7) as

$$-\delta\rho \; g_0 \qquad (1b)$$

where g_0 is the negative of the radial gradient of the reference potential. Following *Alterman et al.* [1959] further, we express the displacement as a spheroidal vector spherical harmonic:

$$\mathbf{u} = U(r)\begin{Bmatrix} 0 \\ 0 \\ S_n(\theta, \varphi) \end{Bmatrix}$$
$$+ \; V(r)\begin{Bmatrix} \partial S_n(\theta, \varphi)/\partial\theta \\ [\partial S_n(\theta, \varphi)/\partial\varphi]/\sin\theta \\ 0 \end{Bmatrix} \qquad (5)$$

where $S_n(\theta, \varphi)$ is a surface spherical harmonic, and we express the perturbation of the potential as

$$\psi = P(r)S_n(\theta, \varphi) \qquad (6)$$

Furthermore, we express $\delta\rho$ as

$$\delta\rho = D(r)S_n(\theta, \varphi) \qquad (7)$$

Substituting (5) and (6) into equations 7 through 10 of *Alterman et al.* [1959], and substituting (7) for our extra terms $(1b)$ and $(2a)$ we get the additional term $-Dg_0$ on the left side of their equation 23 and the additional term $-4\pi GD$ on the right of their equation 25.

Finally, following their conversion of variables, we get the six first-order equations:

$$y_1' = -\frac{2\lambda y_1}{(\lambda + 2\mu)r}$$

$$+ \frac{y_2}{(\lambda + 2\mu)} + \frac{\lambda n(n + 1)}{(\lambda + 2\mu)r} \; y_3$$

$$y_2' = \left[-4\rho_0 g_0 r + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)}\right]\frac{y_1}{r^2}$$

$$- \frac{4\mu}{(\lambda + 2\mu)r} \; y_2 + \left[n(n + 1)\rho_0 g_0 r\right.$$

$$\left. - \frac{2\mu(3\lambda + 2\mu)n(n + 1)}{(\lambda + 2\mu)}\right]\frac{y_3}{r^2}$$

$$+ \frac{n(n + 1)}{r} \; y_4 - \rho_0 y_6 + Dg_0$$

$$y_3' = -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu}$$

$$y_4' = \left[g_0 \rho_0 r - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \right] \frac{y_1}{r^2}$$

$$- \frac{\lambda}{(\lambda + 2\mu)} \frac{y_2}{r} + \frac{2\mu}{(\lambda + 2\mu)}$$

$$\cdot [\lambda(2n^2 + 2n - 1)$$

$$+ 2\mu(n^2 + n - 1)] \frac{y_3}{r^2} - \frac{3y_4}{r}$$

$$- \frac{\rho_0 y_5}{r}$$

$$y_5' = 4\pi G \rho_0 y_1 + y_6$$

$$y_6' = -\frac{4\pi G \rho_0 n(n + 1)}{r} y_3$$

$$+ \frac{n(n + 1)}{r^2} y_5 - \frac{2 y_6}{r} - 4\pi G D$$

$$\Biggr\} \quad (8)$$

where the y_i's are the radial factors, respectively, of the radial displacement, the radial stress, the tangential displacement, the tangential stress, the potential perturbation, and the gradient of the potential perturbation less the radial displacement contribution thereto. Primes denote derivatives with respect to r.

We abbreviate (8) as

$$y_i' = Q_{ij} y_j + w_i D \qquad (9)$$

At least one Q_{ij} is of the order of $1/r$ for all j, so all y_j's must be zero at the origin. Furthermore, some Q_{i1}, Q_{i3}, and Q_{i5} are $0(1/r^3)$, so $y_1'(0)$, $y_3'(0)$, and $y_5'(0)$ must all be zero, leaving three constants of integration, $y_2'(0)$, $y_4'(0)$, and $y_6'(0)$, for a solid core. For a fluid core, there are only two independent equations (cf. *Longman* [1963]):

$$y_5' = y_6$$

$$y_6' = \left[\frac{n(n + 1)}{r^2} - \frac{4\pi \rho^2}{\lambda} \right] y_5 - \frac{2}{r} y_6$$

$$\Biggr\} \quad (10)$$

For a fluid core with radius c, we took as constants of integration $y_6'(0)$, $y_1(c)$, and $y_3(c)$. The conditions at the surface $r = a$ with a layer of surface density σ_n are

$$\left. \begin{array}{c} y_2(a) = -g_0(a)\sigma_n \\ y_4(a) = 0 \\ [(n + 1)/a]y_5(a) + y_6(a) = 4\pi G \sigma_n \end{array} \right\} \quad (11)$$

If we consider the external gravitational field as known, with coefficient C_n for the spherical harmonic component of the potential, there is a fourth surface condition,

$$y_5(a) = C_n \qquad (11')$$

If it is also assumed that $D(r)$ is known throughout, the problem is overdetermined. The simplest assumption is that $D(r)$ is known except for a constant multiplier κ, which becomes a fourth unknown in addition to the three constants of integration. If σ_n is zero and C_n is the exterior potential due to $D(r)$ in a perfectly rigid earth, the load Love number k_n'' for the internal load $D(r)$ comparable to the k_n' calculated for surface loads by *Takeuchi et al.* [1962], and *Longman* [1963] is

$$k_n'' = (1 - \kappa)/\kappa \qquad (12)$$

κ, like k_n'', is thus a measure of the response of the earth to the internal load $\delta\rho$; the exterior potential perturbation C_n will be a combination of the potential due to $\delta\rho$ itself and that due to the deformation of the earth by the load $\delta\rho$.

If $D(r)$ is assumed to involve two or more unknown parameters, further assumptions are required in order to make the problem determinate. The most logical assumption is that the shear strain energy is minimized; i.e.,

$$\int \mu(r) e_{ij}' e_{ij}' \, dV = \text{minimum} \qquad (13)$$

where μ is the rigidity, the integration is over the volume of the earth, and e_{ij}' is the deviatoric strain tensor [*Jeffreys*, 1959 p. 12]

$$e_{ij}' = e_{ij} - \tfrac{1}{3} \delta_{ij} e_{kk} \qquad (14)$$

To connect e_{ij} with the y_i's, we use the equations for the strain in spherical coordinates [*Love*, 1927, p. 56], (applying a factor $\tfrac{1}{2}$ to the off-diagonal components of strain to be consistent with tensor convention) and eliminate dy_1/dr and dy_3/dr by (8)

$$e_{\theta\theta} = 2 \frac{S_{nm}}{r} y_1 + \frac{2}{r} \frac{\partial^2 S_{nm}}{\partial \theta^2} y_3$$

$$e_{\varphi\varphi} = 2 \frac{S_{nm}}{r} y_1 + \frac{2}{r \sin \theta}$$

$$\cdot \left(\frac{1}{\sin \theta} \frac{\partial^2 S_{nm}}{\partial \varphi^2} + \cos \theta \frac{\partial S_{nm}}{\partial \theta} \right) y_3$$

$$e_{rr} = -\frac{4\lambda}{\lambda + 2\mu}\frac{S_{nm}}{r}y_1 + \frac{2n(n+1)\lambda}{(\lambda + 2\mu)}$$
$$\left. \cdot\frac{S_{nm}}{r}y_3 + \frac{2S_{nm}}{\lambda + 2\mu}y_2 \right\}\quad (15)$$

$$e_{\theta\varphi} = \frac{2}{r\sin\theta}\left(\frac{\partial^2 S_{nm}}{\partial\theta\partial\varphi} - \cot\theta\,\frac{\partial S_{nm}}{\partial\varphi}\right)y_2$$

$$e_{\varphi r} = \frac{1}{\mu\sin\theta}\frac{\partial S_{nm}}{\partial\varphi}y_4$$

$$e_{r\theta} = \frac{1}{\mu}\frac{\partial S_{nm}}{\partial\theta}y_4$$

We express (15) in matrix form

$$\mathbf{e}(\theta, \varphi, r) = \mathbf{M}(\theta, \varphi)\mathbf{y}(r) \qquad (16)$$

The shear strain energy per unit volume can then be expressed as

$$s(\theta, \varphi, r) = \mu(r)\mathbf{y}^{\tau}\mathbf{M}^{\tau}\mathbf{J}\mathbf{M}\,\mathbf{y} \qquad (17)$$

where the τ superscript denotes the transpose and

$$\mathbf{J} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

The Runge-Kutta or other numerical method of solution of (9) (see, e.g., *Kopal* [1955]) can be adapted to express the variables y_i at level k in terms of the y_i at level $k - 1$ and the density function $D(r)$ at levels $k - 1$, $k - \frac{1}{2}$, and k:

$$\mathbf{y}_k = \mathbf{T}_k\mathbf{y}_{k-1} + \mathbf{w}_k D_k + \mathbf{w}_{2k}D_{k-1/2} + \mathbf{w}_{3k}D_{k-1} \qquad (18)$$

If the density function $D(r)$ is expressed as a function of a few parameters $\mathbf{d}$, (18) can be simplified to the form

$$\mathbf{y}_k = \mathbf{T}_k\mathbf{y}_{k-1} + \mathbf{W}_k\mathbf{d} \qquad (19)$$

By repeated substitution of the $k - 1$, $k - 2$, etc., equations into (19) we obtain

$$\mathbf{y}_k = \mathbf{U}_k\mathbf{c} + \mathbf{V}_k\mathbf{d} \qquad (20)$$

where $\mathbf{c}$ is a vector of the constants of integration, $y_6{}'(0)$, $y_1(c)$, and $y_3(c)$,

$$\mathbf{U}_k = \prod_{i=1}^{k}\mathbf{T}_i$$

and

$$\mathbf{V}_k = \sum_{i=2}^{k}\prod_{j=i}^{k}\mathbf{T}_j\mathbf{W}_{j-1} + \mathbf{W}_k$$

and $\mathbf{T}_1$ is appropriately modified to fit the initial conditions.

The surface conditions (equations 11 and 11') for m levels of integration are expressible as

$$\mathbf{a} = \mathbf{C}\mathbf{y}_m = \mathbf{C}\mathbf{U}_m\mathbf{c} + \mathbf{C}\mathbf{V}_m\mathbf{d} \qquad (21)$$

The system of equations (17), (20), and (21), along with the condition expressed by (13), is mathematically identical to that of generalized least squares (see, e.g., *Arley and Buch* [1950, pp. 196–198]) with the vector of corrections to observations

$$\mathbf{x} = (\mathbf{c}\,\vdots\,\mathbf{d})^{\tau} \qquad (22)$$

condition equation coefficient matrix

$$\mathbf{F} = (\mathbf{C}\mathbf{U}_m\,\vdots\,\mathbf{C}\mathbf{V}_m) \qquad (23)$$

and covariance matrix

$$\mathbf{W} = \sum_k \mu(r_k)(\mathbf{U}_k\,\vdots\,\mathbf{V}_k)^{\tau}$$
$$\cdot\left(\int\mathbf{M}^{\tau}\mathbf{J}\mathbf{M}\,d_{\sigma}\right)(\mathbf{U}_k\cdots\mathbf{V}_k) \qquad (24)$$

where the integration is over the surface of the sphere. The solution of the system of equations (21) through (24), subject to (13), by the method of Lagrangian multipliers is

$$\mathbf{x} = \mathbf{W}\mathbf{F}^{\tau}(\mathbf{F}\mathbf{W}\mathbf{F}^{\tau})^{-1}\mathbf{a} \qquad (25)$$

Solutions were made for density distributions corresponding to spherical harmonics of the external gravitational field up to the harmonic $Y_{4,4}$ given by *Kaula* [1963] and the Gutenberg model density and elasticity parameters as given by *Takeuchi et al.* [1959]. The surface layer coefficients were derived from the harmonic analysis of the topography by G. J. Bruins, as described by *Vening-Meinesz* [1959]. In these solutions the density parameters for the mantle and the crust were kept separate. Two different types of parameters $\mathbf{d}$ were used to represent

TABLE 1. Mantle Model Corresponding to the External Gravitational Field
Minimum shear strain energy, Gutenberg model of elastic parameters.

Coefficient	External Potential, earth units	Topography Surface Density, equivalent earth units	Crustal Density Anomaly, g/cm³	Maximum Mantle Density Anomaly, g/cm³	Maximum Mantle Stress Difference at Various Radii		Total Shear Strain Energy,* ergs
					Δp, bars	R, earth radii	
$\Delta \bar{C}_{20}$	5.0×10^{-6}	4.65×10^{-6}	0.012	5.7×10^{-4}	163	0.56	22.4×10^{28}
$\bar{C}_{22}$	1.84	-3.9	0.021	-1.2	70	0.56	4.6
$\bar{S}_{22}$	-1.71	-0.34	-0.002	-1.3	65	0.56	2.4
$\bar{C}_{30}$	0.98	-2.50	0.014	-1.4	32	0.56	2.4
$\bar{C}_{31}$	1.77	-1.52	0.012	-2.2	60	0.56	5.4
$\bar{S}_{31}$	-0.11	1.16	-0.006	1.2	5	0.78	0.1
$\bar{C}_{32}$	0.34	-4.45	0.022	-2.9	15	0.56	2.3
$\bar{S}_{32}$	0.08	3.94	-0.018	2.7	3	0.56	1.6
$\bar{C}_{33}$	-0.31	0.72	-0.004	0.3	11	0.56	0.2
$\bar{S}_{33}$	0.74	4.47	-0.019	3.3	55	0.56	2.8
$\bar{C}_{40}$	-0.41	2.68	-0.014	2.1	18	0.78	0.6
$\bar{C}_{41}$	-0.21	-1.68	0.007	-1.2	8	0.87	0.4
$\bar{S}_{41}$	0.46	-2.46	0.013	-1.8	19	0.77	1.2
$\bar{C}_{42}$	-0.03	-4.0	0.018	-2.9	9	0.98	1.6
$\bar{S}_{42}$	0.32	0.6	-0.002	0.5	12	0.56	0.1
$\bar{C}_{43}$	0.50	3.0	-0.012	2.2	15	0.86	1.4
$\bar{S}_{43}$	0.16	-1.8	0.009	-1.3	7	0.98	0.4
$\bar{C}_{44}$	-0.24	-0.19	0.000	-0.2	9	0.56	0.2
$\bar{S}_{44}$	0.55	4.25	-0.018	3.1	20	0.56	2.5

* Neglecting interactions between different harmonics.

the density anomalies in the mantle: layers and polynomial coefficients. About the same answers were obtained by the two methods, but those from the polynomial coefficients varied more smoothly. In general, no appreciable reduction in the summation of strain energy was obtained by using more than four parameters to represent the density variations for each harmonic in the mantle. One parameter was used to represent the density anomaly in the crust.

As a test of the program, tidal and surface-load Love numbers were also calculated by fixing $D(r) = 0$ and appropriately modifying the surface conditions. The answers agreed more closely with those of Longman [1963] than with those of Takeuchi et al. [1962].

The results for the polynomial density-variation solution are summarized in Table 1. The potential and surface density coefficients are in units such that the radius of the earth, the mass of the earth, and the gravitational constant are all unity, and all coefficients apply to normalized spherical harmonics $\bar{S}_{nm}$ so that the integral of $\bar{S}_{nm}{}^2$ over the unit sphere is 4π. The $\Delta \bar{C}_{20}$ is the discrepancy between the observed value and that corresponding to fluid equilibrium [O'Keefe, 1959]. The maximum shear stress obtained for $\Delta \bar{C}_{20}$ is consistent with the 100 bars strength estimated by Munk and MacDonald [1960a, p. 280].

Figures 1 and 2 sum up the effects of all coefficients except $\Delta \bar{C}_{20}$ in the form of maps of the maximum stress difference and the radial component of displacement at three selected levels within the mantle. The displacements have, in general, a negative correlation with the variations in the external field [Kaula, 1963, Figure 1]. The stress differences, which are quadratic functions of the displacement gradients, do not show such a clear pattern, although maximums generally occur in areas where the correlation between the topography and the gravitational field is most negative. The appreciable variability of the stress differences suggests that a significantly different solution might be obtained by applying a yield stress limit. Also calculated were strain energy densities; the maximum shear strain energy density found in the mantle was 1100 ergs/cm³.
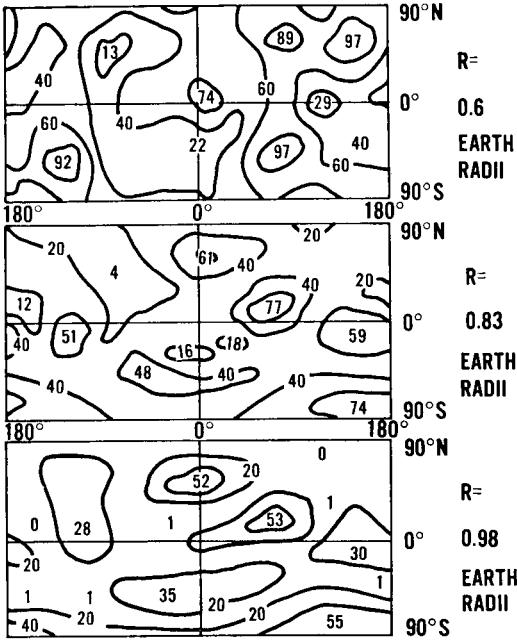
As might be expected under the criterion of

Fig. 1. Principal stress differences in bars, rigidity and bulk modulus assumed from seismology (Gutenberg model).
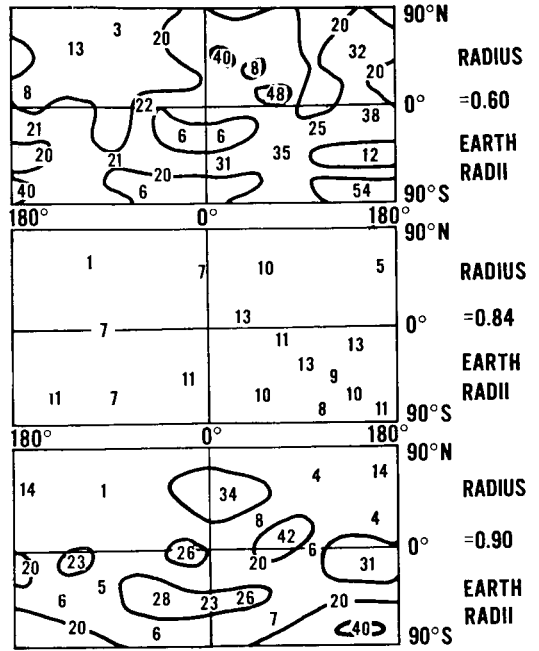
Fig. 3. Principal stress differences in bars, bulk modulus assumed from seismology throughout; fluid layer 35 to 400 km deep; and ½ seismic rigidity 400 to 2900 km deep.
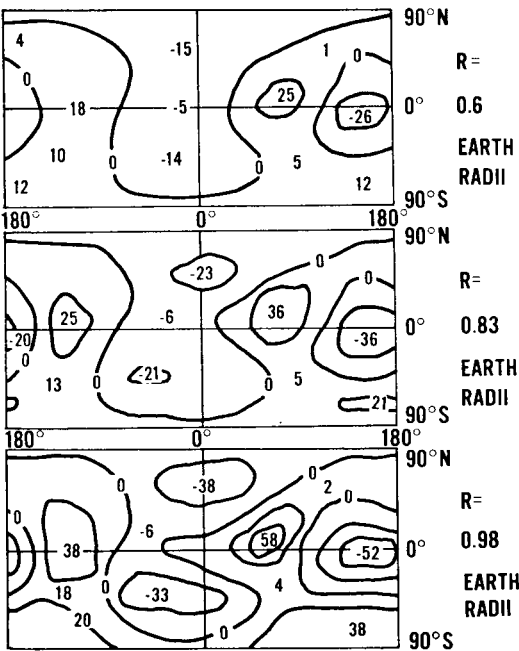
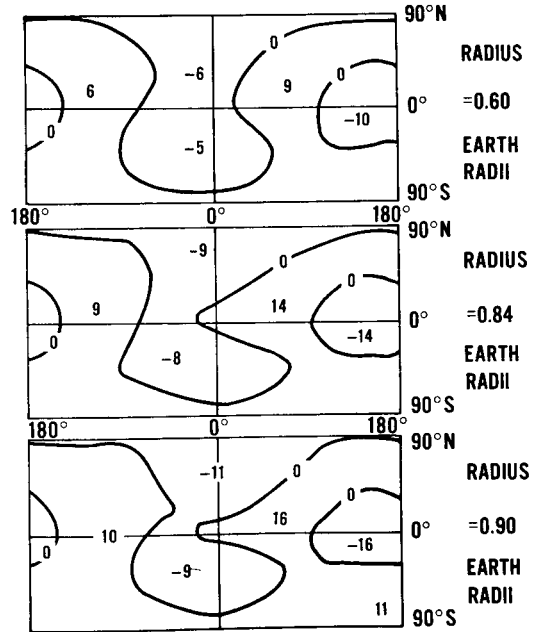Fig. 2. Radial displacements in meters, rigidity and bulk modulus assumed from seismology (Gutenberg model).

Fig. 4. Radial displacements in meters, bulk modulus assumed from seismology throughout; fluid layer 35 to 400 km deep; and ½ seismic rigidity 400 to 2900 km deep.

strain energy minimization, for most terms the density anomaly in the crust accomplished almost complete isostatic compensation of the surface topography. However, in satisfying the condition of the known external gravitational field the compensation is never exact: generally, for terms in which the surface layer has a sign opposite to that of the gravitational field term, there is an overcompensation, and if it has the same sign, there is an undercompensation. The maximum density anomaly in the mantle nearly always fell immediately below the crust.

A manifestly oversimplified physical model has two possible values: to give a notion of the quantities involved and to suggest how a more realistic (and hence probably more complicated) model might be developed. The stresses and energies found for the elastic model are probably a crude approximation to those in the actual earth, but the deformations obviously fall far short of explaining what occurs. We therefore want to examine in turn the possible modifications of the model in the categories of (1) finite strain elasticity: the strain is constant, but the stress is significantly affected by higher-order terms than those in equation 4; (2) creep: the strain rate is appreciable, but the stress remains a function of only the strain; and (3) viscous deformation: the stress is a function of the strain rate.

Finite strain elasticity. Finite strain is significant within the earth in two respects: first, the density and elastic moduli for small superimposed loads, such as seismic waves, must vary with pressure and temperature in a way consistent with actual materials; second, the higher-order terms may contribute appreciably to the stress: i.e., products of displacement gradients may be significant in calculating strain, and products of strain components may be significant in calculating stress.

The first aspect of the finite strain theory has been applied most extensively by *Birch* [1952, 1961] to the problem of deducing composition or phase changes in the mantle from seismic velocities. It is of concern in our present problem only to the extent that the density and elasticity parameters assumed should be materially plausible.

The second aspect of finite strain theory seems of dubious applicability to the earth's mantle, since the strains involved in the minimum energy

solution described above are less than 10^{-4}. If higher-order terms were taken into account, the initial state of stress would no longer be simply additive; hence the solution according to strain energy minimization or other criterion would be a function of the initial state of stress. This problem was investigated by *Jeffreys* [1943], who assumed an implicit nonlinear constitutive equation in the form of a stress matrix with one higher-order term of the form $x_i x_k S_n$. The model was a homogeneous shell with density layers at the upper and lower boundaries. The radial factors of the displacements were expressed in terms of four functions of the form $A_s + B_s r^2$. The eight parameters A_s and B_s were adjusted to minimize the strain energy or stress difference while satisfying the boundary conditions separately for each harmonic. The reduction in stress obtained was moderate, even though interactions between different harmonics were neglected and it was not made clear whether the constitutive equation implied by the solution was realistic.

In our formulation of the problem, it seems very probable that considerations of nonelasticity are of much greater concern that nonlinear elasticity terms.

Creep. There have been several recent discussions of the geophysical application of experimental evidence and theoretical models of creep: *Orowan* [1960], *Griggs and Handin* [1960], *Jeffreys* [1958, 1959], *Jeffreys and Crampin* [1960], *MacDonald* [1961], *Lomnitz* [1962], *Weertman* [1962], *Scheidegger* [1963], *Stacey* [1963], and *Donath and Faill* [1963]. There appear to be appreciable differences of terminology, interpretation, and opinion in these discussions, so we attempt to summarize the principal conclusions in order to decide how our mathematical model should be modified.

1. Transient creep, or elastic afterworking. Consideration of the phase lags and dampings in the response of the earth to periodic disturbances leads to creep models of the form [*Jeffreys*, 1958; *Jeffreys and Crampin*, 1960; *MacDonald*, 1961; *Lomnitz*, 1962]

$$e(t) = p\mu^{-1}[1 + \psi(t)] \qquad (26)$$

for the strain e at a time $t > 0$ due to a constant stress p applied at $t = 0$. 'Creep functions' $\Psi(t)$ which appear to fit phenomena having periods up to the 430 days of the free nutation

are

$$\psi(t) = q \log (1 + at) \qquad (27)$$

and

$$\psi(t) = [(1 + at)^{\alpha} - 1]q/\alpha \qquad (28)$$

The time scale of these phenomena falls far short of that in our problem, but *Jeffreys and Crampin* [1960] find that their numerical values in (28) would permit an appreciable part of a second-degree harmonic to survive for more than 10^9 years. *MacDonald* [1961] objects to the rules (27) and (28) because they imply an infinite population of relaxation times in a linear superposition model and because they do not lead to frequency independence of dissipation, as observed for frequencies above 1 cps, but these considerations have no apparent bearing on the long term problem.

2. Delayed elasticity. Glasses at low temperature can be regarded as elastic, but their full response to loads is appreciably delayed. If the irregularities of grain boundaries, etc., are such that the material of the mantle can be regarded as amorphous, there may be a significant amount of delayed elasticity. *Orowan* [1960] suggests that delayed elasticity may account for the post-glacial rise. In this case, elasticity would have to prevail to a considerable depth in the mantle, rather than only in the crust, as considered by Vening-Meinesz [*Heiskanen and Vening-Meinesz*, 1958].

3. Steady-state creep, or elastoviscosity. Crystalline substances at low shear stresses above a 'creep strength' deform very slowly. This steady-state creep can usually be expressed as [*Weertman*, 1962; *Stacey*, 1963]

$$\dot{e} = C(\mathbf{p}) \exp (-Q/kT) \qquad (29)$$

Where $\dot{e}$ is the strain rate, Q is an activation energy, k is the Boltzmann constant, T is the absolute temperature, and C is an exponential or, near melting, a power of the shear stress $\mathbf{p}$. *Jeffreys* [1958] considers steady-state creep to be of no importance in the earth; *Stacey* [1963] suggests that it is the dominant rheological mode in the mantle.

4. Uniform flow. If the shear stress in a crystalline substance exceeds a certain yield point, rapid plastic deformation or uniform flow takes place. At confining pressures above 1000 bars, the zone of failure widens, but cohe-

sion is retained after failure [*Griggs and Handin*, 1960; *Donath and Faill*, 1963].

Instability of creep in metals has been observed to result in the rapid propagation of intense shear bands, or faulting without fracture [*Orowan*, 1960]. This 'shear melting' is suggestive of a mechanism for earthquakes or magma formation.

Estimates of the creep strength or yield point at the temperatures and pressures prevailing in the mantle seem to be rather speculative. *Orowan* [1960] states that the creep strengths of crystalline materials close to the melting point are of the order of 100 or 10 bars, or even less. *Stacey* [1963], on the other hand, obtains creep strengths (defined as the 'stress at which creep reaches [a] just observable value') in excess of 500 bars, mainly, it seems, from the observation of *Scheidegger* [1963, p. 159] that departures from isostasy which result in stresses smaller than about 4000 bars can exist 'almost indefinitely.' *Weertman* [1962] does not give a creep strength, but he does say that a steady-state creep law of the form of (29) breaks down at stresses above 100 to 1000 bars, which presumably defines the yield point. *Griggs and Handin* [1960] make no sharp distinction between creep and uniform flow but suggest that shear melting could occur in the mantle at shear stresses as low as 100 bars.

We seem to have authority for a wide range of hypotheses as to creep strengths and yield points in the mantle. In particular, in comparing the stress differences in Figure 1 with the values suggested for the strengths, it appears entirely possible that the mantle can be assumed to be in a state in which creep is taking place, but in which stress is a function of the strain, not of the strain rate. Such a state would occur if the density irregularities $\delta\rho$ were not removed by the creep in the 10^8 or 10^9 years time scale of our problem. Taking the situation at any instant $\tau = 0$ as a reference state, we find that the rigidity μ in (4) would be replaced by a pseudo-rigidity

$$\mu_p(t) = \mu/[1 + \psi(t)] \qquad (30)$$

where

$$\psi(t) = \int_0^t A(t - \tau)\mathbf{p}(\tau)\,d\tau > 0$$

which is formally similar to the transient creep

TABLE 2.   Mantle Model Corresponding to the External Gravitational Field
Minimum shear strain energy, fluid layer 35 to 400 km deep, $\frac{1}{2}$ Gutenberg model rigidity in lower mantle.

Coefficient	External Potential, earth units	Topography Surface Density, equivalent earth units	Crustal Density Anomaly, g/cm³	Maximum Mantle Density Anomaly, g/cm³	Maximum Mantle Stress Difference at Various Radii		Total Shear Strain Energy,* ergs
					Δp, bars	R, earth radii	
$\Delta \bar{C}_{20}$	5.0×10^{-6}	4.65×10^{-6}	0.0146	-2.0×10^{-4}	31	0.56	1.84×10^{25}
$\bar{C}_{22}$	1.84	-3.9	0.0019	-1.0	13	0.56	0.19
$\bar{S}_{22}$	-1.71	-0.34	-0.0042	0.7	12	0.56	0.19
$\bar{C}_{30}$	0.98	-2.50	0.0034	-3.5	22	0.937	0.40
$\bar{C}_{31}$	1.77	-1.52	0.0080	-6.1	35	0.937	1.39
$\bar{S}_{31}$	-0.11	1.16	0.0002	0.4	2	0.937	0.05
$\bar{C}_{32}$	0.34	-4.45	0.0010	-1.4	7	0.937	0.06
$\bar{S}_{32}$	0.08	3.94	0.0028	0.03	1	0.56	0.03
$\bar{C}_{33}$	-0.31	0.72	-0.0011	1.1	6	0.937	0.04
$\bar{S}_{33}$	0.74	4.47	0.0065	-2.3	13	0.937	0.32
$\bar{C}_{40}$	-0.41	2.68	-0.0013	2.7	16	0.937	0.14
$\bar{C}_{41}$	-0.21	-1.68	-0.0024	1.2	7	0.937	0.05
$\bar{S}_{41}$	0.46	-2.46	0.0017	-3.0	16	0.937	0.08
$\bar{C}_{42}$	-0.03	-4.0	-0.0025	-0.09	5	0.82	0.03
$\bar{S}_{42}$	0.32	0.6	0.0025	-1.9	9	0.937	0.10
$\bar{C}_{43}$	0.50	3.0	0.0052	-2.9	13	0.937	0.26
$\bar{S}_{43}$	0.16	-1.8	0.0006	-1.1	5	0.937	0.02
$\bar{C}_{44}$	-0.24	-0.19	-0.0018	1.5	7	0.937	0.05
$\bar{S}_{44}$	0.55	4.25	0.0062	-3.1	16	0.937	0.34

* Neglecting interactions between different harmonics.

function of *MacDonald* [1961]. Equation 30 makes $\mu_p(t)$ a function of latitude and longitude, which in turn would not permit development of the solution in the form of (8). To see how the stresses and displacements might be modified in an extreme case, however, a solution of the system of equations 21 to 24 subject to (13) was made for a model in which $\mu_p(t)$ was zero in the low-velocity zone 35 to 400 km deep, and equal to $\frac{1}{2}$ the Gutenberg model rigidity throughout the rest of the mantle. This solution is summarized in Table 2. The principal result was that the incompressibility of the fluid layer, in addition to that of the core, reduced the shear stresses required by supporting much of the load. Specifically, there was (1) an increase in the radial displacements of the crust by a factor of about 30, (2) a decrease in the shear stresses in the crust to about $\frac{1}{4}$, (3) a reduction of the stresses in the mantle to about $\frac{1}{5}$ for the second degree, $\frac{1}{2}$ for the third degree, and $\frac{4}{5}$ for the fourth degree, (4) a concentration of stresses in the mantle near the solid-liquid boundaries, and (5) a density-anomaly maximum at the top of the elastic part of the mantle. The reduction of the $\Delta \bar{C}_{20}$ stress to a value comparable

to that for $\bar{C}_{31}$ suggests that reduction of rigidity similar to that of the hypothesized model could take place in the 10^7 years indicated by the lag in adjustment of the rotational bulge. The relatively small reduction in stresses for the fourth-degree terms suggests that higher-degree variations could not be reasonably supported by such a model, and it is consistent with the indications from satellite zonal harmonics and autocovariance analysis of gravimetry that there is an appreciable drop-off in the magnitude of variations of the gravitational field above the fourth degree.

Viscous deformation. We may ask two questions: Is large-scale convective motion now taking place in the mantle? If convection is taking place, is the stress which supports the density irregularities primarily a function of the strain rate—i.e., is it viscous (Newtonian or otherwise)? Answers to the first question appear to depend on the background of the answerer, the two extremes being the geothermists [*Lubimova*, 1960; *MacDonald*, 1963], who consider that convection must be limited in time and place (of the nature of shear melting) in order that

the observed heat flow not be greatly exceeded, and the paleomagnetists [*Runcorn*, 1962; *Stacey*, 1963], who need to explain evidence of continental drift of the order of 3 cm/year. Answers to the second question are even vaguer because of the lack of experimental data, the uncertainty as to the appropriate rheological theory, and the lack of an adequate mathematical solution for convection with significant variation of parameters and a finite yield stress, as discussed by *MacDonald* [1963]. If it is assumed that stress is a function of strain rate d_{ij} with constitutive equation $p_{ij} = \rho d_{kk}\delta_{ij} + 2\eta\, d_{ij}$, where η is the viscosity, a solution for the displacement rate field analogous to (4) to (25) for the displacement field would be somewhat more complicated, since the external gravitational field would still be a function of the displacement field. We should expect, however, that the order of magnitude of the rates **v** could be estimated from the displacements **u** by

$$\mathbf{v} \approx \mathbf{u}\mu/\eta \qquad (31)$$

The pseudo-viscosity η most commonly quoted is 10^{22} poises, deduced by Vening-Meinesz [*Heiskanen and Vening-Meinesz*, 1958, pp. 365–370] and others from a decay time of about 5300 years and a linear dimension of about 1200 km for postglacial uplift. The Newtonian viscous model requires that the decay time vary inversely as the diameter of the load; however, *Crittenden* [1963] recently obtained a decay time of about 4000 years for the 180-km-diameter Bonneville Lake area uplift, or a pseudo-viscosity η of 10^{21} poises.

Taking $\mu = 10^{12}$, $\mathbf{u} = 3 \times 10^3$, and $\eta = 10^{22}$ cgs in (31), we obtain velocities of the order of 10 cm/year. *Licht* [1960] objected to such a high rate because it required improbably high efficiency of heat transport from core to surface; however, his discussion seems to be based on the implicit assumption that the radiogenic heat in the mantle is less than 10^{-2} of the 1.6 ergs/g/yr estimated for chondritic composition [*MacDonald*, 1959]. *Runcorn* [1962] also appears to neglect radiogenic heating in the mantle in showing that convective velocities of the order of 10 cm/yr are obtained with temperature differences of only 0.2 degree centigrade, thus yielding a surface heat flow much less than observed. A more fundamental weakness of mantle-wide convection models is the assumption that the pseudo-

viscosity of 10^{22} poises can be applied to organized convection with a characteristic length of 2900 km; Crittenden's result emphasizes the danger of such an extrapolation.

To reconcile the gravitational and thermal evidence with a Newtonian viscous model appears to require either a mantle-wide system with viscosity coefficient in excess of 10^{23} poises or a system with a characteristic length of 180 km or less and a viscosity of 10^{21} poises or less. The first model leaves unexplained many surface evidences of large-scale motion; the second model has not been worked out. Therefore, further development of convective models seems worth while, as well as critical re-examination of the evidences of continental drift [*Munk and MacDonald*, 1960a, pp. 251–262, 282–285] and study of the problem of distribution of radiogenic heating.

Conclusions. The indications we have found as to the state of stress in the earth's interior from the low-degree, or long-wave, variations in the external gravitational field are minimal in two respects. First, the condition of strain energy minimization has been imposed; the actual density distribution could conceivably be quite different. Second, superimposed on the low-degree variations may be higher-degree variations of sufficient magnitude to increase appreciably the stress differences above those here calculated. If the scale of significant change in the earth is appreciably smaller than the wavelengths of the low-degree harmonics—as suggested by the narrowness of the belts of surface manifestation of tectonic activity and the relatively shallow origin of seismic activity—then the long-wave variations in the gravity field supply evidence of only the passive background for the processes currently important.

The development of models more realistic than the elastic cannot be purely mechanical because of the rheological uncertainties. To provide some limitation to the possible solutions, we should include energy flows and distribution of heat sources in the problem. The desirability of incorporating the thermal aspects, as well as shorter-wave variations in the gravitational field, suggests a more statistical approach to keep the problem manageable. The inputs would be the spectrums of gravity, topography, heat flow, etc., variations in the form of degree variances

$$\sigma_n{}^2 = \sum_{m=0}^{n} (\bar{C}_{nm}{}^2 + \bar{S}_{nm}{}^2)$$

together with some measure of the correlation between the different observed quantities. The solutions sought would be the spectrums of density, displacement, displacement rate, heat sources, etc., at various levels within the earth under specified conditions such as the strain energy minimization.

Acknowledgment. I am grateful to W. D. Putney for doing much of the programming for the computations described in this paper.

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