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A NOTE ON LAMBERT'S THEOREM

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This note combines all the various cases of Lambert's Theorem into a single form which is particularly convenient for numerical work. This is made possible by appropriate choice of parameter and independent variable.

Suppose a particle in a gravitational central force field has distances r_1 and r_2 at times t_1 and t_2 from the center of attraction. Let c be the distance and θ the central angle between the positions of the particle at the two times. Define

$$s = (r_1 + r_2 + c)/2$$

$$K = 1 - c/s$$

$$q = \pm K^{1/2}$$

The sign of q is taken care of by the angle θ if we make use of

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta$$

to derive

$$q = \left[(r_1 r_2)^{1/2}/s\right] \cos(\theta/2).$$

We further define

G = universal gravitational constant

M = mass of attracting body

μ = GM

a = semimajor axis of transfer orbit

E = $-s/2a$ for elliptic transfer

= $s/2a$ for hyperbolic transfer

T = $(8\mu/s)^{1/2}\,(t_2 - t_1)/s$

m = number of complete circuits during transfer time.

Note that $-1 \leq q \leq 1$, $0 < E < \infty$ for hyperbolic transfer, $-1 \leq E < 0$ for elliptic transfer, and $E = 0$ for parabolic transfer. Also $0 \leq \theta \leq \pi$ if $0 \leq q \leq 1$ and $\pi < \theta \leq 2\pi$ if $-1 \leq q < 0$.

Lambert's Theorem[1] for elliptic transfer gives

$$T = (-E)^{-3/2} \left[2m\pi + \alpha - \beta - (\sin\alpha - \sin\beta)\right] \tag{1}$$

$$E = -\sin^2(\alpha/2), \quad 0 \leq \alpha \leq 2\pi$$

$$\sin(\beta/2) = q \sin(\alpha/2), \quad -\pi \leq \beta \leq \pi$$

For hyperbolic transfer,

$$T = -(E)^{-3/2} \left[\gamma - \delta - (\sinh\gamma - \sinh\delta)\right] \tag{2}$$

$$E = \sinh^2(\gamma/2)$$

$$\sinh(\delta/2) = q\, E^{1/2}$$

If E is chosen[2] as the independent variable, α is ambiguous. We avoid any ambiguity by choosing as the independent variable

$$x = \cos(\alpha/2), \quad -1 \leq x < 1,$$

$$= \cosh(\gamma/2), \quad x > 1.$$

For both elliptic and hyperbolic transfer

$$E = x^2 - 1.$$

For the elliptic case let

$$y = \sin(\alpha/2) = (-E)^{1/2}$$

$$z = \cos(\beta/2) = (1 + KE)^{1/2}$$

$$f = \sin(1/2)(\alpha - \beta) = y(z - qx)$$

$$g = \cos(1/2)(\alpha - \beta) = xz - qE$$

$$0 \leq \alpha - \beta \leq 2\pi \quad \text{since} \quad 0 \leq f \leq 1$$

$$h = (1/2)(\sin\alpha - \sin\beta) = y(x - qz)$$

$$\lambda = \arctan(f/g), \; 0 \leq \lambda \leq \pi$$

It then follows for the elliptic case that

$$T = 2(m\pi + \lambda - h)/y^3 \tag{3}$$

For the hyperbolic case let

$$y = \sinh(\gamma/2) = E^{1/2}$$

$$z = \cosh(\delta/2) = (1 + KE)^{1/2}$$

$$f = \sinh(1/2)(\gamma - \delta) = y(z - qx)$$

$$g = \cosh(1/2)(\gamma - \delta) = xz - qE$$

$$0 \leq \gamma - \delta < \infty \quad \text{since} \quad 0 \leq f < \infty$$

$$h = (1/2)(\sinh\gamma - \sinh\delta) = y(x - qz)$$

$$\begin{aligned}(1/2)(\gamma - \delta) &= \operatorname{arctanh}(f/g) \\ &= (1/2)\ln[(f + g)/(g - f)] \\ &= (1/2)\ln\left[(f + g)^2/(g^2 - f^2)\right] \\ &= \ln(f + g)\end{aligned}$$

Thus for the hyperbolic case

$$T = 2\,[h - \ln(f + g)]/y^3 \tag{4}$$

When $m = 0$, equations (1), (2), (3) and (4) break down for $x = 1$ and suffer from a critical loss of significant digits in the neighborhood of $x = 1$. To remedy this (1) is written

$$T = \phi(-E) - qK\phi(-KE), \tag{5}$$

$$\phi(u) = 2\left[\arcsin u^{1/2} - u^{1/2}(1 - u)^{1/2}\right]\Big/u^{3/2}.$$

Replacing $\arcsin u^{1/2}$ and $(1 - u)^{1/2}$ by series[3],

$$\phi(u) = 4/3 + \sum_{n=1}^{\infty} a_n u^n, \quad |u| < 1,$$

$$a_n = 1 \cdot 3 \cdot 5 \cdots (2n - 1)\Big/2^{n-2}(2n + 3)\,n!$$

A similar procedure produces the same series for the hyperbolic case. In fact (5) holds for the elliptic $(m = 0)$, parabolic, and hyperbolic cases provided $0 < x < 2$.

It is now apparent that, given q and x, the following steps produce T for all cases:

1. $K = q^2$

2. $E = x^2 - 1$

3. $\rho = |E|$

4. If ρ is near 0, compute T from (5).

5. $y = \rho^{1/2}$

6. $z = (1 + KE)^{1/2}$

7. $f = y(z - qx)$

8. $g = xz - qE$

9. If $E < 0$, $\lambda = \arctan(f/g)$, $d = m\pi + \lambda$, $0 \leq \lambda \leq \pi$

If $E > 0$, $d = \ln(f + g)$

10. $T = 2(x - qz - d/y)/E$

The following formula for the derivative holds for all cases except for $x = 0$ with $K = 1$ and for $x = 1$.

$$dT/dx = (4 - 4qKx/z - 3xT)/E$$

If x is near 1, the series representation should be differentiated. If $q = 1$ we have a left-hand derivative of -8 and a right-hand derivative of 0 at $x = 0$. If $q = -1$ we have a left-hand derivative of 0 and a right-hand derivative of -8 at $x = 0$. (See Figure 1.)

In the derivation of Lambert's Theorem for the elliptic case α and β are defined in such a way that

$$E_2 - E_1 = \alpha - \beta + 2m\pi \tag{5}$$

where E_1 and E_2 are the values of the eccentric anomaly at times t_1 and t_2. Thus from equation (1)

$$E_2 - E_1 = (-E)^{3/2}T + \sin\alpha - \sin\beta$$

$$= y^3 T + 2y(x - qz). \tag{6}$$

We now obtain a formula for the scalar product

$$S_1 = \mathbf{r}_1 \cdot \mathbf{v}_1 = r_1 v_1 \sin\psi_1$$

of the position and velocity vectors at time t_1, v_1 and ψ_1 being the speed and flight path angle.

Kepler's equation can be written[4]

$$(\mu/a^3)^{1/2}\,(t_2 - t_1) = E_2 - E_1 + S_1\left[1 - \cos(E_2 - E_1)\right]\Big/(\mu a)^{1/2}$$
$$-(1 - r_1/a)\,\sin(E_2 - E_1).$$

Substituting $a = -s/2E$, $t_2 - t_1 = s^{3/2}T/(8\mu)^{1/2}$, and making use of (5) and (6) we have, after some algebra,

$$S_1 = (2\mu s)^{1/2}\left[qz(s - r_1) - x(s - r_2)\right]\Big/c$$

A similar procedure produces the same formula for S_1 in the hyperbolic case. It also holds for the parabolic case.

Figures 1 and 2 show T as a function of x for elliptic and hyperbolic transfer, the parabolic case occurring for $x = 1$. We suggest the reader compare these curves with those in Reference 2 showing T as a double-valued function of E with infinite slope at $E = -1$.

No solutions of Lambert's equation exist in the shaded regions of figures 1 and 2. $x = 1$ $(m > 0)$ and $x = -1$ are vertical asymptotes. $T \to 0$ as $x \to \infty$.

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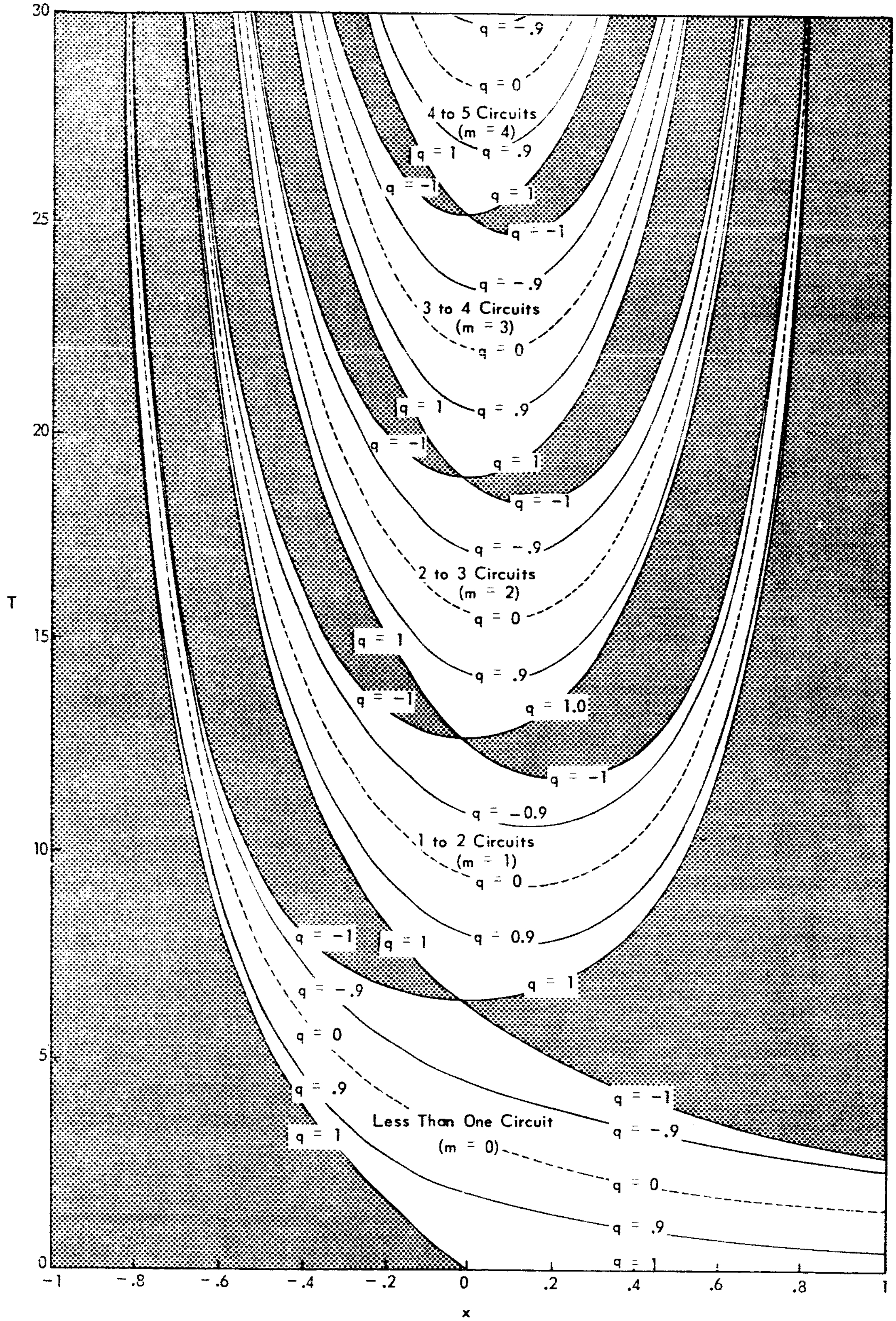

Figure 1—E vs. T for elliptic case

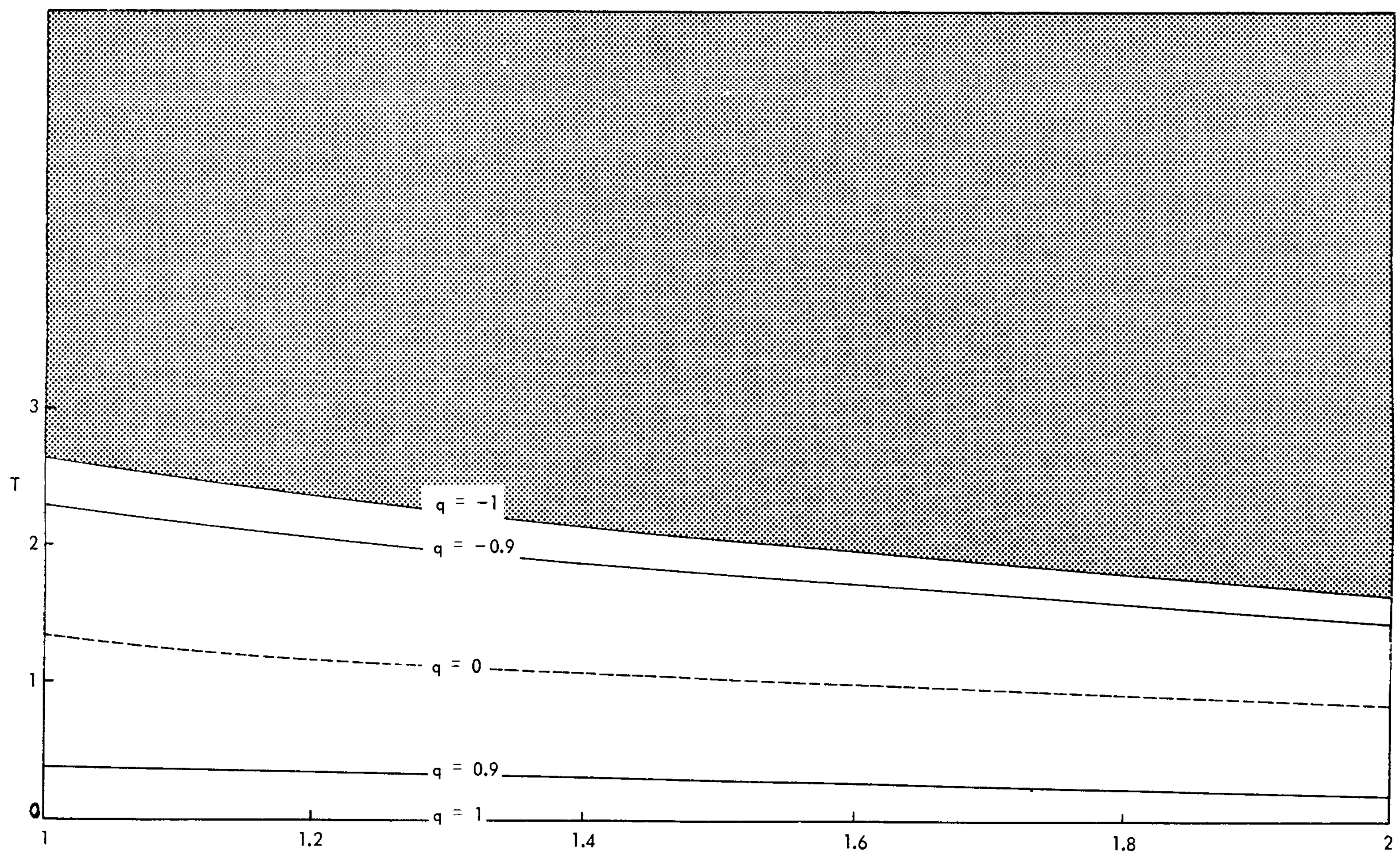

Figure 2–E vs. T for hyperbolic case