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THE ATTITUDE DETERMINATION SYSTEM FOR THE ORBITING ASTRONOMICAL OBSERVATORY

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ABSTRACT

Various applications of attitude estimation as applied to the operation of the Orbiting Astronomical Observatory are enumerated and explained in some detail. The techniques used, the results of these techniques, and the problems encountered during the first nine months of the mission are delineated. The operation of the Orbiting Astronomical Observatory including the spacecraft and the supporting ground system is briefly described.

INTRODUCTION

On December 7, 1968 the NASA's Orbiting Astronomical Observatory (OAO) was placed into a near circular orbit inclined 35 degrees from the equator by an Atlas-Centaur booster. The OAO, orbiting 480 miles above the earth's surface, doubled existing stellar data clear of the earth's absorbing atmosphere in its first thirty days of operation. After nine months of successful operations this first astronomical observatory in space has made over 3000 observations of stellar objects and has attained some 5000 different attitudes distributed over the entire celestial sphere.

Two experiments are aboard the OAO spacecraft: One designed by the University of Wisconsin and the other by the Smithsonian Astrophysical Observatory (SAO). The Wisconsin Experimental Package (WEP) includes four stellar photometers, two scanning spectrometers, and one nebula photometer. The primary function of the WEP is to gather spectral energy distributions on selected stars and nebulae in the ultraviolet range of 1000 to 4000 Angstrom (A). The SAO experiment contains four 12-inch telescopes each of which images a star field onto an ultraviolet sensitive photocathode. The scientific objective of SAO is to measure the brightness of many celestial bodies in four separate ultraviolet spectral bands ranging between 1100 A and 3000 A.

THE SPACECRAFT AND ITS CONTROL

The OAO spacecraft is an octagonally shaped aluminum structure ten feet long and six and two-thirds feet across any two parallel sides of the octagon. There is a hollow central tubular area four feet in diameter running the length of the spacecraft which is used to house the experimental equipment. Two hinged sunshades approximately four feet square are attached to each end of the spacecraft to protect the experimental equipment from sunlight. Attached to each of two opposite sides of the octagon at an angle of 34 degrees from the longitudinal (optical) axis are four solar cell paddles whose combined dimensions are nearly eleven feet by nine feet. Solar cells are mounted on both sides of all eight paddles. The major subsystems of the OAO are stabilization and control, data processing, communications, and power. Thermal control is passive. The entire spacecraft weighs over two tons which includes 1000 pounds of experimental equipment.

The primary mode of control is by stellar guidance so as to point either of the experiments to within one arc-minute of a specified target star. Control perpendicular to the optical axis is also required for maneuvering, power, thermal, and shading considerations. This three dimensional control is accomplished by using various combinations of six orthogonally mounted gimballed startrackers each having a field of view of one degree and an excursion of approximately forty degrees in any direction. A seventh tracker, boresighted with the optical axis, is also available which provides single axis (optical) control. Averaged error signals from the trackers drive three orthogonally mounted "fine" momentum wheels which absorb the extraneous torques that would cause attitude errors. The momentum in the fine wheels is continuously "dumped" by a magnetic unloading system consisting of three flux gate magnetometers, three torquer coils (electromagnets), and a processor. The momentum in the fine wheels may also be removed by a gas jet system.

The OAO may also be controlled by three rate gyros which actuate either the fine momentum wheels (Hold On Wheels) or the gas system (Hold On Jets). The maximum drift rate for any one gyro as obtained from in-flight data is about 0.2 degrees per hour.

The spacecraft is maneuvered from one attitude (two axes) to another by means of three "coarse" inertia wheels in an open loop manner. Two or more trackers are required to track continuously during the slew and averaged errors from preset gimbal angles settle the spacecraft when the slew has terminated. Slewing plus settling times vary with the axis and angle of rotation but is usually better than three degrees per minute. Small closed loop slews (several degrees) may also be accomplished with the fine wheels by changing the commanded gimbal angles.

The OAO has two special modes of control: "sunbathing" and "sun-pointing". In the former, sun sensors are used to actuate gas jets which align the normal of the solar cell array to the sun when in sunlight. Rate gyros then hold this position while in darkness. Sun-pointing is similar except in this case the optical axis is pointed toward the anti-sun line. This is the pointing the spacecraft automatically obtains upon separation from the launch vehicle.

In addition to the sensors mentioned above there are four sun sensors mounted so as to give solar aspect data at any attitude. It is the purpose of these sensors to provide independent attitude information while in sunlight.

The data processing subsystem can accept 168 different control commands in addition to the gimbal and slewing commands. All commands may be executed in real time or stored (up to 256 commands) and executed at a specific time later as a function of an on-board clock.

THE GROUND SYSTEM

For the most part, the actual functions performed by the OAO are initiated by one of the experimenters although several major elements of the ground system lie between his request and the commands executed by the spacecraft. The experimenter specifies a sequence of target stars (pointings) and the associated experimental equipment commands necessary to obtain the desired scientific objectives. This information (Experimenter's Target List) is passed on to the Mission Computing Group where it and other inputs are entered into a complex computing system known as the Support Computer Program System (SCPS). The SCPS (which resides in a large scale-high speed computer) determines the total attitude as a function of the target and other geometric considerations. From this, the number of turns for each inertia wheel required to slew from the previous attitude to the present one is then determined. The gimbal angles for each tracker and their on-off schedule as a function of occultations by sun, moon, or earth are also generated. One of the final outputs of the SCPS is the ordered list of commands in spacecraft format which are to be executed and their time of execution. Based upon the schedule of ground contacts with the spacecraft, the number of commands to be executed, and the current memory assignments the SCPS also determines when and where the commands are to be loaded into the spacecraft memory.

The image of the OAO command memory along with certain ground procedures generated by the SCPS is then routed to the OAO Operations Control Center (OCC) which in turn transmits this "contact message" to one of five remote sites. These sites are: Rosman, North Carolina; Quito, Ecuador; Santiago, Chile; Tananarive, Madagascar; and Orroral, Australia. At the scheduled contact the remote station establishes communications with the OAO, loads new commands in memory (if necessary), and gathers telemetered data (real time and stored). The OCC monitors these contacts (in the case of a Rosman contact it also replaces the remote station functions), and displays the returning telemetered data. During the contact, real time commands already at the remote site may be executed and real time commands from Goddard Space Flight Center (where the SCPS and OCC are located) can be received by the site and relayed to the spacecraft.

ATTITUDE ESTIMATION APPLICATIONS

The applications for attitude estimation techniques in the OAO program range from coarse estimates (several degrees) from the solar and magnetic sensors to a precise attitude determination (seconds of arc) from as many as six startrackers.

A coarse attitude estimate is used as an intermediate step in obtaining stellar guidance. The transition between the coarse attitude estimate and stellar control is obtained by generating a sequence of startracker gimbal angles consistent with a sequence of attitudes. This sequence of attitudes sweeps out the most probable region associated with the coarse attitude estimate. Stellar guidance is then obtained when two or more trackers acquire their preassigned stars during the search maneuver.

The coarse estimate of the OAO's attitude is obtained by placing the spacecraft in an attitude hold using the rate gyros and fine momentum wheels. Sun sensor and magnetometer data are then collected while the spacecraft is in contact with a ground station (approximately ten minutes). This data is then relayed, by high speed data links, to the control center where it is passed to the SCPS. Since the attitude of the spacecraft is fairly stable during the collection of the data (relative to the accuracy of the data), it is assumed that all the data refers to the same attitude and is thus used collectively to obtain a weighted least squares estimate of th t attitude. The star search commands, based upon the attitude estimate, are then generated by the SCPS and transmitted (via the total ground system) to the spacecraft at some subsequent contact in real time.

Even though the magnetometers were not calibrated and give noisy data (their original purpose was solely to unload momentum from the fine wheels) the above procedure yields an adequate solution and has been used to place the OAO under stellar control approximately thirty times in a nine month period. The success of this technique has saved valuable time and extended the life of the OAO's gas supply since it is applicable at any orientation whereas the original design concept required a reorientation to align a particular axis with the sun. Stellar control would then be accomplished by rolling about this axis until two or more startrackers simultaneously acquired stars.

The OAO's capability to hold an unknown attitude under gyro control has also been used to derive an estimate of attitude using only magnetometer data. To date such rough estimates have had very limited application. However, on one occasion such an estimate was used to verify that the telemetry from a sun sensor had been misinterpreted. With calibrated magnetometers this capability could provide useful attitude information while the spacecraft is in darkness.

When under stellar control, attitude determination techniques using startracker data have been used for an independent verification of the commanded attitudes and to determine the accuracy of the various sensors. Least squares solutions using all tracker data (usually from two to four trackers) as an aggregate and in various combinations provides an excellent means for evaluating the performance of each tracker and whether any misalignments exist between them.

More sophisticated techniques have been employed to determine the values of the tracker misalignments from inflight data. This has been necessary since shifts due to launch stresses and other effects have been greater than expected. By use of this procedure the pointing accuracy has been better than the design specification of one arc-minute.

Attitude determination from tracker gimbal angles is also performed in the processing of the experimental data. This allows a correlation of the data to a known star and aids in the evaluation of the scientific results.

Stellar control has provided the opportunity to study the correlation between the errors of the sensors and the resulting error in attitude. In this manner a good deal of experience has been obtained in the selection of weighting factors for the various sensors as well as interpreting the results based on residual errors and the angular separation between the solar and magnetic vectors. A technique has recently been developed which will determine from in-flight data any systematic errors caused by: the misalignment of the magnetometers, the permanent magnetic moment of the spacecraft, and the effects of the torquer coils on the magnetometers. It is felt that this information will help eliminate the present wide dispersion in the magnetic data.

THE ATTITUDE ESTIMATE

All of the attitude estimates referred to in the preceding sections are obtained from the same mathematical algorithm which yields a weighted least squares estimate of attitude. The only difference between the various applications is the selection of the weights and/or the type of data to be used.

The method (delineated in an appendix) is based upon two different vector parametrizations of three-dimensional rotations. One vector has been referred to as the Gibbs vector and the other's components are three of Euler's symmetrical parameters. The direction of both vectors defines the axis of rotation and their lengths are trigonometric functions of the angle of rotation. One vector has a length equal to the tangent of the half-angle of rotation and is denoted as the Y vector. The other vector is denoted as Z and has length equal to the sine of the half-angle of rotation.

The Z vector is used to obtain a vector expression for the "smallest" rotation which will align an estimate of a rotated vector to its true value. A generalized weighted least squares criterion is thus established by requiring the sum of squared lengths of all such Z's (premultiplied by a symmetric weight matrix) to be a minimum. The resulting equation is then simplified by assuming that the weight matrix is the identity times a scalar (this implies that the component errors of each Z vector are equal and independent).

The Y vector is used to express the least squares condition in vector notation where the only variables are the three components of the Y vector. The necessary conditions for an extremum are then applied to this function and the result is a vector equation in terms of the Y vector. The least squares solution can then be obtained by finding the largest zero of a fourth degree polynominal and then solving a linear system of three equations. In practice, this approach is not taken since the vector equation is also amenable to a simple successive substitutions iteration which converges rapidly. Thus an approximate solution is constructed to start the iteration.

When only two data vectors are given the least squares rotation can be expressed as a product of two rotations, each of which are obtained by juxtaposing vectors that are linear combinations of the given vectors. The scalar coefficients of these linear combinations are explicitly given as functions of the lengths and dot products of the given vectors. This method yields the attitude estimate when only two data vectors are available and provides the first approximation when more than two vectors have been measured.

RESULTS AND PROBLEMS

Attitude error signals from the experiments are not available on the present OAO. Therefore, it is difficult to ascertain the absolute error in attitude while under stellar control. However, attitude determinations using telemetered gimbal angles from several trackers over a ten minute interval usually yields residual errors of one arc-minute or less (the smallest command increment is twenty arc-seconds). To maintain this precision, however, it has been necessary to occasionally re-evaluate the misalignments between the trackers. More elaborate misalignment models are being considered to eliminate this minor problem and to improve the overall accuracy. Since the misalignments have been as good as thirty arc-seconds over extended periods it is felt that much of the variation can be accounted for with additional parameters in the model.

In the following table are the results of fifteen typical attitude estimates using only sun sensor and magnetometer data. Column one contains the number of data points used in the estimate-each data point gives two vectors (solar and magnetic). The second column shows the angular separation between the solar and magnetic vectors. The length of time over which the data were taken is given in column three. Columns four and five give the actual angular error of the solar and magnetic data respectively (the attitude obtained from tracker data serves as the reference). The next two columns displays the solar and magnetic errors using the solar-magnetic estimate as a reference. The last column contains the angle of that rotation necessary

No. of Data Points	Angular Separation	Duration min: sec.	Actual Sun Error	Mag. Error Mean, Max.	Residual Sun Error	Mag. Residual Mean, Max.	Attitude Error
6	36 to 38	0:47	0.02	3.9, 4.9	0.18	3.9, 4.8	1.1
6	62 64	1:03	0.25	6.4, 8.6	0.32	5.8, 8.0	1.1
10	39 44	1:50	0.14	3.6, 4.5	0.20	3.4, 4.4	1.2
11	37 29	9:42	0.06	2.6, 4.0	0.08	2.2, 3.3	1.8
11	56 49	2:21	0.06	4.7, 6.0	0.37	4.4, 5.9	1.8
14	81 82	4:11	0.08	7.1, 9.2	0.31	6.8, 9.9	2.8
16	40 50	3:25	0.26	2.2, 3.5	0.15	2.3, 3.8	0.9
17	37 28	3:56	0.06	3.4, 5.4	0.16	2.6, 5.3	3.5
18	47 25	8:23	0.02	1.4, 3.3	0.05	1.4, 3.3	0.5
24	69 45	7:52	0.06	4.5, 6.3	0.30	4.2, 6.0	0.3
25	80 107	7:07	0.35	5.4, 6.4	0.31	5.4, 6.4	0.8
25	88 78	7:52	0.09	6.1, 8.1	0.32	5.8, 8.1	1.0
25	29 41	6:49	0.22	1.7, 3.4	0.08	2.2, 4.0	1.5
25	61 46	6:34	0.21	6.1, 8.5	0.33	5.4, 8.4	1.9
25	74 61	6:49	0.26	5.5, 8.1	0.30	4.7, 7.6	2.1

All errors and angular separation are in degrees.

In all attitude estimates the sun sensor data was given a weight of 4 and the magnetometer data a weight of 1.

to bring the solar-magnetic estimate in agreement with the stellar determination. These figures clearly show the dispersion in the magnetic data that was referred to previously. The errors range from as good as our capability to predict the magnetic vector (the time of the data is only known to within 16 seconds) to a factor of five times worse. This makes it very difficult to determine the accuracy of the estimate in practice, especially when the angular separation is poor (less than twenty degrees). In this case, small residual errors do not necessarily mean a good estimate. On the other hand, a separation of less than twenty degrees may yield a good estimate if the magnetic data are good.

It is rather apparent from the table that attitude estimates based upon the magnetic data only would be quite poor in general. However, we made such an estimate for what was considered to be the best case (the one with eighteen data points) merely to see what type of accuracy could be obtained by this method. The attitude error in this case was 0.95 degrees. Thus, if the larger errors in the magnetic data can be eliminated, useful attitude estimates could be obtained while the spacecraft is in darkness.

Of the thirty some star searches attempted, only three have failed due to poor attitude estimates. The present search capability, due to a ground constraint, is about 3.5 degrees which is the maximum error of the fifteen cases discussed earlier. However, in the failure cases data was limited to a small interval of time where the angular separation was also small. Two searches have also failed because one or more trackers locked onto wrong stars (a bright neighbor of the selected star). Normally, logic aboard the OAO prevents this by requiring simultaneous acquisition of stars. This logic which is set whenever all trackers lose their stars, has been by-passed several times by ground commands when it was apparent that stellar control had been lost (only one tracker tracking). This action is justifiable in that it also prevents further loss of control.

The occasional large errors in the magnetic data have alerted us to a potential problem in estimating attitude from multiple data representing essentially only two directions. When one direction is weighed heavily over the other the solution is weak in one dimension geometrically and in another statistically. The net result is that there is another solution (a saddle point) to the necessary conditions for extremum near the desired least squares solution. This is especially troublesome for small angular separations.

CONCLUSION

In every respect, the present OAO mission must be considered a resounding success. Although the OAO is the most complex scientific satellite ever built it has operated almost flawlessly for nine continuous months. The ground system has been able to keep the satellite productive around the clock and both experimenters are continuing to receive valuable scientific data. The problems, thus far, have been minor and solved with work-around procedures on the ground.

With future OAO spacecraft carrying such equipment as integrating as well as rate gyros, ultra precision experiments, and an on-board computer it appears that the OAO program will continue to offer challenging and exciting possibilities in the field of attitude determination and control.

ACKNOWLEDGMENT

I would like to take this opportunity to express my sincere thanks to the many hundreds of people who have contributed to the success of the Orbiting Astronomical Observatory which has made this paper possible. A special "thank you" goes to Mr. and Mrs. Joseph Hennessy for the computer programs associated with this effort.

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APPENDIX

A VECTOR APPROACH TO WEIGHTED LEAST SQUARES ATTITUDE ESTIMATION

Given the angle of a rotation, θ, and the rotation axis defined by the unit vector X it is well known that the matrix, R, of the rotation can be expressed as

$$R_X(\theta) = \cos\theta\, I + (1-\cos\theta)\, XX^T - \sin\theta\, \tilde{X} . \quad (1)$$

Here $\tilde{X}$ denotes the skew-symmetric matrix formed from the components of X such that for any vector V, $\tilde{X}V = X \times V$ (I is the 3×3 identity matrix and the superscript T denotes transpose). Furthermore, any orthogonal matrix with determinant equal to plus one can be expressed in the form of Eq. (1). Except when otherwise stated, it is assumed that X has been selected so $0 \leq \theta \leq \pi$.

If we define the vectors Y and Z as

$$Y = \tan(\theta/2)\, X , \qquad Z = \sin(\theta/2)\, X ,$$

then we find, by standard trigonometric identities, that R may be written as

$$R = \frac{1}{1 + Y \cdot Y}\left[(1 - Y \cdot Y)\, I + 2YY^T - 2\tilde{Y}\right] ,$$

or

$$R = (1 - 2Z \cdot Z)\, I + 2ZZ^T - 2\sqrt{1 - Z \cdot Z}\, \tilde{Z} .$$

In this manner, the vector components relative to a rotated system can be expressed, by vector relations, as a function of Y or Z and the components relative to a fixed system. For every rotation there exists a Z vector, uniquely except for 180 degree rotations. A unique Y vector exists for any rotational matrix whose trace is different from minus one.

With the above definitions it then follows that if T and U are any two vectors, the rotation, R, which will align U and RT with the minimum angle of rotation can be expressed by the Z rotation vector as

$$Z = \frac{1}{\sqrt{2(1 + U_n \cdot T_n)}}\, U_n \times T_n ,$$

where $U = |U|\, U_n$ and $T = |T|\, T_n$. In other words, the angle of the rotation is the angle between U and T, and the rotation axis is perpendicular to the plane containing these two vectors. If $|U| = |T|$ then $U = RT$. In particular, if U is a measurement (including error) of vector components relative to a local right-handed orthonormal coordinate system, T is the same physical vector but relative to a fixed reference system (also right-handed and orthonormal), and R is an estimate of the rotation (attitude) relating the two systems then the vector error that can be corrected by a <u>rotation</u> is given by

$$\Delta Z = \frac{1}{\sqrt{2(1 + U_n \cdot RT_n)}}\, U_n \times RT_n . \quad (2)$$

The quantity $U - RT$ is the error in a true vector sense—it cannot be removed, in general, by a rotation.

For each such measurement, U_i, there corresponds a ΔZ_i. Thus, the weighted sum of "rotational errors" squared is given by

$$f(R) = \sum_{i=1}^{m} (P_i \Delta Z_i)^T P_i \Delta Z_i , \quad (3)$$

where m is the number of measured vectors and P_i is the weight matrix for the i^{th} measurement. If $P_i = p_i I$ (p_i a scalar) and equation (2) is substituted into equation (3) we obtain

$$f(R) = \sum_{i=1}^{m} (W_i - RV_i)^2 , \quad (4)$$

where

$$W_i = \frac{p_i}{2} \frac{U_i}{|U_i|} ,$$

and

$$V_i = \frac{p_i}{2} \frac{T_i}{|T_i|}$$

The function $f(R)$, the weighted squared errors, is minimized when the function

$$g(R) = \sum_{i=1}^{m} W_i \cdot RV_i$$

is a maximum. This expression may be written as

$$g(R) = \sum_{i=1}^{m} (RV_i) W_i^T = \sum_{i=1}^{m} tr(RV_i W_i^T)$$

$$tr\left[R \sum_{i=1}^{m} V_i W_i^T\right] = tr(RA)$$

where tr denotes "trace of" and

$$A = \sum_{i=1}^{m} V_i W_i^T .$$

Hence, the least squares rotation is obtained by maximizing the function $g(R) = tr(RA)$.

Except for notation, Eq. 4 is the same as the equation in Ref. 4 to be minimized and the solutions there may also be applied to Eq. 4. There is, however, considerable difference in the definitions of the vectors used in the equations. As applied to attitude estimation, the vectors in Ref. 4 are direction cosines (although the solution also applies to unnormalized vectors) whereas here they are generated from the measurements (normalized or not) so as to minimize the weighted rotation error. The following solution to minimizing Eq. 4 is similar to that given in Ref. 4.

The matrix $A^T A$ is symmetric and positive-semi-definite. Let d_i^2 denotes the i^{th} ordered ($d_1^2 \geq d_2^2 \geq d_3^2 \geq 0$) non-negative eigenvalue of $A^T A$ corresponding to the normalized eigenvector N_i. The vectors AN_i ($i = 1, 2, 3$) then constitute an orthogonal system of vectors with $|AN_i| = d_i$. Thus, an orthonormal system of vectors T_i ($i = 1, 2, 3$) can be constructed such that $AN_i = d_i T_i$. If $d_1 = 0$ then T_1 may be any arbitrary unit vector; otherwise

$$T_1 = \frac{AN_1}{d_1} .$$

T_2 and T_3 are then similarly constructed. Let N and T denote the orthogonal matrices obtained by juxtaposing the vectors N_i and T_i respectively. Then from the constructions above it follows that $T^{-1} AN = D$, where D is a diagonal matrix with entries d_i (ordered). Introducing the orthogonal matrix $M = N^{-1} RT$ (R is the desired least squares rotation) we find that the expression $tr(RA)$ as a function of M becomes

$$tr(RA) = tr(NMT^{-1} A) = tr(MD) = \sum_{i=1}^{3} m_{ii} d_i .$$

For M orthogonal and $d_i \geq 0$, the maximum value of $tr(RA)$ is obtained when $M = I$. Since $|M| = |N| |R| |T|$, R will be a rotation matrix when $M = I$ if $|N| |T| = 1$. If $|N| |T| = -1$ then the maximum value of $tr(RA)$ for $|R| = 1$ is obtained when

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

This latter case is equivalent to changing the sign of T_3 in the definition of T. Thus, $R = NT_0^T$, where $T_0 = (T_1, T_2, |N| |T| T_3)$ is the desired solution to the least squares condition. This solution can easily be generalized for vectors of arbitrary dimensions.

When only two measured vectors are given it can be shown that

$$N_1 = \frac{W_1 - yW_2}{|W_1 - yW_2|} , \qquad N_2 = \frac{W_1 + xW_2}{|W_1 + xW_2|} ,$$

$$T_1 = \frac{xV_1 - V_2}{|xV_1 - V_2|} , \qquad T_2 = \frac{yV_1 + V_2}{|yV_1 + V_2|} .$$

$N_3 = N_1 \times N_2$, and $T_3 = T_1 \times T_2$ gives the least squares solution if x and y satisfy the condition $x + y > 0$ as well as the simultaneous equations:

$$(V_1 \cdot V_1)\,xy + (V_1 \cdot V_2)(x - y) - V_2 \cdot V_2 = 0$$

$$(W_1 \cdot W_1) + (W_1 \cdot W_2)(x - y) - xy\,W_2 \cdot W_2 = 0\,.$$

The solution to this pair of equations is given by

$$x = \frac{a \pm \sqrt{a^2 + 4bc}}{2c}\,,$$

$$y = \frac{-a \pm \sqrt{a^2 + 4bc}}{2c}\,,$$

with

$$a = (V_2 \cdot V_2)(W_2 \cdot W_2) - (V_1 \cdot V_1)(W_1 \cdot W_1)\,.$$

$$b = (V_1 \cdot V_2)(W_1 \cdot W_1) + (W_1 \cdot W_2)(V_2 \cdot V_2)\,.$$

$$c = (W_1 \cdot W_2\ (V_1 \cdot V_2) + (V_1 \cdot V_2)(W_2 \cdot W_2)\,.$$

If Eq. 4 is written as a function of the Y rotation vector and the conditions for an extremum applied, the resulting three scalar equations can be expressed by a single vector equation as

$$2\left[\sum (W_i \cdot V_i) \cdot Y + (V_i \cdot Y)(W_i \cdot Y) + V_i \cdot W_i\right] Y$$

$$= (1 + Y \cdot Y) \sum \left[(V_i \cdot Y)\,W_i + (W_i \cdot Y)\,V_i + W_i \times V_i\right]\,.$$

Ref. 2 gives a solution of this equation which requires obtaining the largest zero of a fourth order polynomial and then solving a linear system of three equations. It has also been solved successfully by a successive substitutions iteration (divide both sides of the equation by the coefficient of Y on the left side). In this case, the V_i are first rotated by an approximation of R. The rotation corresponding to Y is then the correction to the approximation necessary to obtain the least squares solution.