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COMPENDIUM OF MODAL DENSITIES FOR STRUCTURES

by F. D. Hart and K. C. Shah

Prepared by
NORTH CAROLINA STATE UNIVERSITY
Raleigh, N. C.
for Langley Research Center

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16. Abstract

This paper presents a discussion of the concept of modal density or density of eigenvalues of various structural elements of engineering importance. Expressions and graphs are presented that can be used to estimate the average modal densities of these elements and are valid for elements having any prescribed boundary conditions. The expressions for modal density and their graphical representation were prepared from the information available in the literature, but supplementary data were generated where required.

Cases are considered for rods, beams, solid rectangular and circular plates, thin cylindrical, spherical and conical shells, composite structures, shallow sandwich shells, orthorotropic plates, pretwisted plates, plates subject to in-plane forces and shells on an elastic foundation. For each of the elements up to a composite structure, graphs are plotted using dimensionless parameters to generalize the applications of the results; however, for the rest of the elements, graphs are plotted by choosing some arbitrary dimensions to illustrate the effect on modal density.

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SUMMARY

This paper presents a discussion of the concept of modal density or density of eigenvalues of various structural elements of engineering importance. Expressions and graphs are presented that can be used to estimate the average modal densities of these elements and are valid for elements having any prescribed boundary conditions. The expressions for modal density and their graphical representation were prepared from the information available in the literature, but supplementary data were generated where required.

Cases are considered for rods, beams, solid rectangular and circular plates, thin cylindrical, spherical and conical shells, composite structures, shallow sandwich shells, orthrotropic plates, pretwisted plates, plates subject to in-plane forces and shells on an elastic foundation. For each of the elements up to a composite structure, graphs are plotted using dimensionless parameters to generalize the applications of the results; however, for the rest of the elements, graphs are plotted by choosing some arbitrary dimensions to illustrate the effect on modal density.

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1. INTRODUCTION

Any continuous structure possesses an infinite number of natural modes of vibration and to obtain the information concerning the response of such a structure, it is necessary to express the normal modes in a series form. However during the past few years, there has been an effort to develop a new approach to these multimodal vibration problems that avoids the problem of expanding the response in terms of the mode shapes. In this approach, sometimes referred to as "statistical energy analysis", average response levels in various frequency intervals are estimated without the apparent knowledge of the mode shapes and resonance frequencies. Instead, what is required is a knowledge of the type and number of structural vibration modes occurring in a given frequency interval. This quantity, the number of modes per unit frequency, is called the 'modal density' of the structure. Thus the modal density of a structure is essentially the density of the modes of vibration with respect to frequency. It is an indication of the spacing of the natural modes in the frequency domain.

When dealing with the structures excited in a very complex, or random fashion it is often not only useful but necessary to resort to statistical energy analysis to determine the response of the structure to such loading. In order to apply this type of analysis it is found that the modal density of the structure in question must be known. Moreover statistical energy analysis shows promise of becoming a useful tool for estimating average response levels of multimodal structural vibrations as the modal density of a structure is relatively independent of the boundary conditions. Hence in order to apply a statistical type of analysis to a structural response problem, it is necessary to know the modal density of the basic structural elements such as rods, beams, and shells. It is therefore the purpose of this paper to discuss in a systematic manner the problem of the modal density in vibration problems of some basic structural elements like rods, beams, plates and thin cylindrical, spherical and conical shells, composite structures and certain shallow structural elements and present the expressions and graphs that can be used to estimate the average modal densities of these elements.

The determination of the modal density is essentially a mathematical problem. It involves the determination of the frequency equation for the structure under consideration from the appropriate equation of motion and then the summation of the resonant frequencies over all possible modes of vibration. This yields an expression for the number of resonant modes in terms of frequency. Differentiation of this expression with respect to frequency will then yield the expression for the modal density in terms of the frequency. The k-space integration technique introduced by Courant and Hilbert (1953) is utilized to evaluate the number of resonant frequencies.

In Chapter 3, the modal density for longitudinal and torsional vibrations of circular rods having uniform cross section is discussed and results are compared.

Chapter 4 deals with the beams having constant geometry and properties. The expression for modal density for transverse vibrations of beams is presented. The graph of normalized modal density versus dimensionless frequency is plotted.

For Chapter 5, modal density expressions are presented in flat rectangular and circular plates and the results are discussed.

In Chapter 6, thin cylindrical shell is considered. The modal density expressions are developed for cylindrical shells following three different approaches. These integral expressions are then evaluated and plotted in dimensionless form.

Chapter 7 deals with spherical shells. The modal density expression is developed as a function of dimensionless frequency and a graph is plotted to illustrate the variation of modal density above and below the ring frequency.

In Chapter 8, expressions for modal density of thin conical shells are obtained based on two separate frequency equations and are normalized with respect to cone geometry and presented for the frequency range below the lower ring frequency and above the upper ring frequency of the cone.

In Chapter 9, the additive property of modal density for composite structures is verified analytically by considering an L-shaped frame consisting of two beams joined at right angles and the graph is plotted to illustrate the variation of modal density of the composite structure with respect to frequency.[1]

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[1]Hart, F. D. and V. D. Desai. 1967. Additive properties of modal density for composite structures. Presented at the 74th Meeting of the Acoustical Society of America, Miami, Florida, Paper No. DD 11. Department of Mechanical and Aerospace Engineering, North Carolina State University at Raleigh, N. C.

In Chapter 10, modal density expressions are presented for shallow sandwich shells, orthrotropic plates, pretwisted plates, plates subject to in-plane forces, and shells on an elastic foundation. Graphs are plotted to illustrate the effect on modal density. Results obtained are discussed in detail and compared with some of the basic elements.

Chapter 11 presents a summary of results and conclusions.

2. REVIEW OF LITERATURE

The problem of determining the modal density of any given structure is equivalent to ascertaining the distribution of eigenvalues of large order corresponding to high mode numbers. A general discussion of the asymptotic distribution of eigenvalues for various classes of differential equations is given by Courant and Hilbert (1953). Expressions for the number of eigenvalues up to a given bound are given for differential equations with one, two and three independent space variables. Although the treatment of the subject by Courant and Hilbert is approached from a basic mathematical point of view, the results have direct physical interpretation. It is indicated that boundary conditions have no effect on the asymptotic distribution of the eigenvalues.

Bolotin (1962) has also given considerable attention to the asymptotic method in his studies of eigenvalue determination. In 1962 Bolotin presented a discussion of the asymptotic behavior of the eigenvalues for a generalized rectangular region of arbitrary dimensions. He applied this technique to the problem of plates and shells, where the number of eigenvalues correspond to the number of natural frequencies of vibration. Correction factors were also introduced to extend the work of Courant and Hilbert (1953) to low mode numbers where boundary conditions must sometimes be considered. Bolotin (1960) presented a detailed discussion of the effect of edge conditions on the vibrational modes of elastic shells.

In 1963 Bolotin presented a general treatment of the eigenvalue density problem for a general thin elastic shell of revolution with constant thickness in orthogonal curvilinear co-ordinates coinciding with the curvature lines. Bolotin again used the asymptotic method discussed by Courant and Hilbert (1953) in his work and obtained expressions for the number of natural frequencies and the modal density of a general elastic shell of revolution through elliptic integrals. The results of this work were also extended to the specific cases of the spherical shell and the circular cylindrical shell. Bolotin (1965) presented a discussion which was essentially an extension of his previous work in which he discussed the concentration points of natural modes, as well as the effects of shear and rotary inertia.

Without apparent knowledge of Bolotin's work, Heckl (1962) developed an expression for the natural frequencies of a cylindrical shell using impedance methods. He then represented the number of natural modes by a finite sum over all possible modes of vibration possible up to some upper frequency. He then replaced the summation by an integral and obtained an approximate expression for the modal density of thin cylindrical shells. Heckl also presented some experimental findings in his report.

In 1965 Smith and Lyon introduced the concept of modal density and discussed its application with regard to structural vibration. The cases of simply supported beams, clamped beams, simply supported rectangular plates and clamped circular plates were considered in particular.

Ungar (1966) discussed the concept of modal density and its application to composite systems. He also presented a list of expressions for the modal density of some simple elastic systems of engineering value.

In 1967 Hart and Desai presented a discussion of the additive property of modal density for composite structures and verified analytically the validity of the additive property by considering the composite structure consisting of two beams joined at right angles to form an L-shaped frame.

Miller and Hart (1967) made a combined analytical and experimental study about the modal densities of a thin cylindrical shell. Expressions for modal density were presented in integral form using three different methods and the validity of the results was discussed in detail.

In 1968 Wilkinson presented the expressions for the modal densities for transverse vibrations of two dimensional structural elements which included shallow sandwich shells, orthrotropic plates, shells on elastic foundation, pretwisted plates and plates subject to in-plane forces. The effect on the modal densities of these elements was illustrated graphically.

In 1969 Miller presented the expressions for modal densities of conical shells based on the two separate frequency equations and applicable to a wide range of cone geometries and valid over a frequency range sufficiently wide to be of engineering value. Miller also presented the findings for the modal densities of conical shells obtained by experimental study.

Erickson (1969) presents expressions to estimate the average modal densities of sandwich beams and flat or cylindrically curved sandwich panels. The effect of transverse shear flexibility, orthrotropic shear moduli of the core, face bending stiffness, rotary inertia and panel curvature on modal density is illustrated graphically over the wide-frequency range. Modal densities of flat rectangular sandwich panels having orthrotropic cores are determined experimentally.

3. MODAL DENSITY OF RODS

3.1 Introduction

In this chapter a circular rod having the uniform cross section with both the ends fixed is considered. A rod can execute longitudinal, torsional or transverse vibrations either individually or in combination. The expressions for modal density are derived considering longitudinal and torsional vibrations individually.

3.2 Longitudinal Vibration

The governing equation of motion for longitudinal vibrations of a rod is given by

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{Eg_c}{\rho} \frac{\partial^2 w(x,t)}{\partial x^2} \tag{3.1}$$

where

E = Young's modulus

ρ = density of material

g_c = gravitational constant

w = longitudinal displacement of a section.

This equation is based on the following assumptions:

1. The rod has a uniform cross section.
2. During the vibratory motion, the cross section normal to the axes of the bar remain plane and normal to the axis.
3. The particle in a normal cross section moves in the axial direction of the bar.

Letting $\frac{Eg_c}{\rho} = C_L^2$, equation (3.1) reduces to

$$\frac{\partial^2 w(x,t)}{\partial t^2} = C_L^2 \frac{\partial^2 w(x,t)}{\partial x^2} , \tag{3.2}$$

where C_L is the longitudinal velocity of wave propagation along the length of the rod.

Assuming that the solution of equation (3.2) is

$$w(x,t) = X(x) \sin\omega t , \tag{3.3}$$

and substituting (3.3) into (3.2) and simplifying, gives

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \tag{3.4}$$

where

$\lambda^2 \quad = (\frac{\omega}{C_L})$

$\omega \quad$ = frequency of vibration

$X(x)$ = the shape of normal mode of vibration.

The solution fo (3.4) is given as

$$X(x) = A\cos\lambda x + B\sin\lambda x . \tag{3.5}$$

For a rod fixed at both the ends, the boundary conditions are

$$w(0,t) = w(\ell,t) = 0 \tag{3.6}$$

or $$X(0) = X(\ell) = 0 . \tag{3.7}$$

Imposing the boundary conditions in (3.7), the frequency equation for this case can be written as

$$\omega = \frac{m\pi C_L}{\ell} \qquad m = 1, 2, 3, \ldots \tag{3.8}$$

where ℓ is the length of the rod.

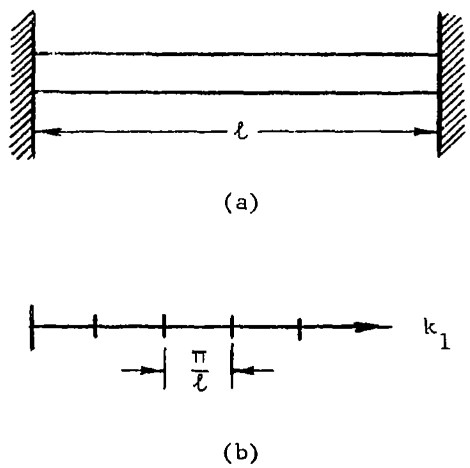


Figure 3.1 Rod with fixed ends and k-space

The wave number k_1 may be defined as

$$k_1 = \frac{m\pi}{\ell} \quad . \tag{3.9}$$

Hence the change in the wave number from one mode to the next is given by

$$\Delta k_1 = \frac{\pi}{\ell} \quad . \tag{3.10}$$

Since the waves in the case of a rod are propagated only along the length of the rod, the k-space is one dimensional and the equation for the number of resonant frequency becomes

$$N(\omega) = \frac{1}{\Delta k_1} \int_0^{k_1} dk_1 \quad . \tag{3.11}$$

This gives

$$N(\omega) = \frac{\ell}{\pi} \; k_1 \; , \tag{3.12}$$

but equations (3.8) and (3.9) give

$$k_1 = \frac{\omega}{C_L} \quad . \tag{3.13}$$

Therefore the expression for the number of resonant frequencies is

$$N(\omega) = \frac{\ell}{\pi} \left(\frac{\omega}{C_L}\right) \quad . \tag{3.14}$$

Defining a dimensionless frequency ν as

$$\nu = \frac{\omega\ell}{C_L} \tag{3.15}$$

equation (3.14) reduces to

$$N(\nu) = \frac{\nu}{\pi} \quad . \tag{3.16}$$

Differentiating (3.16) with respect to ν gives

$$n(\nu) = \frac{1}{\pi} \; . \tag{3.17}$$

This is an expression for modal density for the longitudinal vibration of a rod fixed at both ends and it can be shown to be applicable for arbitrary end conditions.

3.3 Torsional Vibrations

The governing differential equation of motion for torsional vibrations of a rod is given as

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = \frac{G g_c}{\rho} \frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{3.18}$$

where

G = modulus of rigidity

ρ = density of material

g_c = gravitational constant

ϕ = angular displacement of the section.

This equation is also based on the assumption of equation (3.1). Defining

$$\frac{G g_c}{\rho} = C_T^2 \tag{3.19}$$

where C_T is the torsional wave velocity, equation (3.18) becomes

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = C_T^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} . \tag{3.20}$$

Assuming the solution of equation (3.19) is

$$\phi(x,t) = \theta(x) \sin\omega t , \tag{3.21}$$

substitution of (3.21) into (3.20) and further simplification gives

$$\frac{d^2\theta(x)}{dx^2} + \lambda^2\theta(x) = 0 \tag{3.22}$$

where

$$\lambda^2 = \left(\frac{\omega}{C_T}\right)^2$$

ω = frequency of vibration

$\theta(x)$ = the shape of normal modes of vibration.

Again for the fixed rod, the boundary conditions are

$$\theta(0) = \theta(\ell) = 0 \ . \tag{3.23}$$

Applying the boundary conditions to (3.22) gives the following frequency equation:

$$\omega = \frac{m\pi C_T}{\ell} \qquad m = 1,\ 2,\ 3\ .\ .\ .\ . \tag{3.24}$$

Hence as in the previous section, the number of resonant frequencies obtained by k-space integration is given as

$$N(\omega) = \frac{\ell}{\pi}\left(\frac{\omega}{C_T}\right) \tag{3.25}$$

where

$$C_T^2 = \frac{Gg_c}{\rho} = \frac{g_c}{\rho}\left[\frac{E}{2(1+\mu)}\right] \tag{3.26}$$

or

$$C_T = \frac{C_L}{[2(1+\mu)]^{\frac{1}{2}}} \quad . \tag{3.27}$$

Substitution of equation (3.27) into (3.25) gives

$$N(\omega) = \frac{\omega\ell[2(1+\mu)]^{\frac{1}{2}}}{\pi C_L} \quad . \tag{3.28}$$

Now introducing the dimensionless frequency, equation (3.28) reduces to

$$N(\nu) = \frac{\nu[2(1+\mu)]^{\frac{1}{2}}}{\pi} \quad . \tag{3.29}$$

Differentiating (3.29) with respect to ν gives

$$n(\nu) = \frac{[2(1+\mu)]^{\frac{1}{2}}}{\pi} \quad . \tag{3.30}$$

This gives an expression for modal density for the torsional vibrations of a rod fixed at both the ends.

3.4 Discussion

The expressions developed for modal density of rods for both longitudinal and torsional vibrations show that modal density of a rod is constant and is independent of geometry of cross sections in dimensionless form.

Moreover, the modal density of the rod for torsional vibrations is about 1.5 times that for longitudinal vibrations.

Expressions generated by considering different boundary conditions give the same answer and hence it is independent of boundary conditions also.

4. MODAL DENSITY OF BEAMS

4.1 Introduction

In this chapter, the modal density for transverse vibration of beams having constant geometry and constant properties are discussed. The simply supported beam is considered for deriving the expression.

The problem of simply supported beams was discussed in the literature. It is reproduced here to illustrate an exact way for developing the expression. Modal density is expressed as a function of dimensionless frequency and the graph is plotted in terms of dimensionless parameters.

4.2 Simply Supported Beams

The governing equation of motion for transverse vibrations of a beam is given by the differential equation

$$EI \frac{\partial^4 V(x,t)}{\partial x^4} + \frac{\rho A}{g_c} \frac{\partial^2 V(x,t)}{\partial t^2} = 0 \tag{4.1}$$

where

E = Young's modulus of elasticity

I = moment of inertia of cross section

ρ = density of the material

g_c = gravitational constant

A = area of the cross section

$V(x,t)$ = deflection of the beam at any section.

The equation (4.1) is based on the following assumptions:

1. The effect of rotary inertia is neglected.
2. Shear displacement due to a vibratory force is negligible.

3. Cross sections are plane before strain and remain plane after strain.
4. Beam is slender.

Assuming that the solution of equation (4.1) is

$$V(x,t) = X(x) \sin \omega t \ , \tag{4.2}$$

substitution of equation (4.2) into (4.1) and further simplification gives

$$\frac{d^4X(x)}{dx^4} - \lambda^4 X(x) = 0 \tag{4.3}$$

where

$$\lambda^4 = \frac{\rho\omega^2 A}{EIg_c}$$

ω = frequency of vibration

$X(x)$ = the shape of the normal modes of vibration.

The general solution fo the differential equation (4.3) is given as

$$X(x) = A \sin \lambda x + B \cos \lambda x + C \sinh \lambda x + D \cosh \lambda x \tag{4.4}$$

where A, B, C and D are arbitrary constants.

For the beam under consideration, the boundary conditions are

$$V(0,t) = V''(0,t) = 0 \tag{4.5}$$

$$V(\ell,t) = V''(\ell,t) = 0 \ . \tag{4.6}$$

The above conditions can be written as

$$X(0) = X''(0) = 0 \tag{4.7}$$

$$X(\ell) = X''(\ell) = 0 \; . \tag{4.8}$$

Applying the boundary conditions (4.7) and (4.8) to equation (4.4), the frequency equation for the beam can be expressed as (Smith and Lyon, 1965),

$$\omega = \frac{m^2 \pi^2}{\ell^2} K C_L \tag{4.9}$$

where

C_L = the longitudinal velocity of wave propagation in the beam material along the length of beam

ℓ = length of beam

K = radius of gyration of the cross section.

Let the wave number k_1 be defined as

$$k_1 = \frac{m\pi}{\ell} \tag{4.10}$$

Therefore the change in the wave number from one mode to the next is given as

$$\Delta k_1 = \frac{\pi}{\ell} \tag{4.11}$$

Again in case of a beam, waves are propagated only along the length, hence the k-space is one dimensional and the equation for the number of resonant frequencies is

$$N(\omega) = \frac{1}{\Delta k_1} \int_0^{k_1} dk_1 \quad . \tag{4.12}$$

This gives

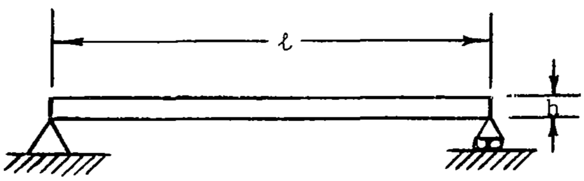
$$N(\omega) = \frac{\ell k_1}{\pi} \quad . \tag{4.13}$$

Combining equations (4.9) and (4.10) gives

$$k_1 = \sqrt{\frac{\omega}{KC_L}} \quad . \tag{4.14}$$

Therefore the number of resonant frequencies is given as

$$N(\omega) = \frac{\ell}{\pi} \sqrt{\frac{\omega}{KC_L}} \quad . \tag{4.15}$$



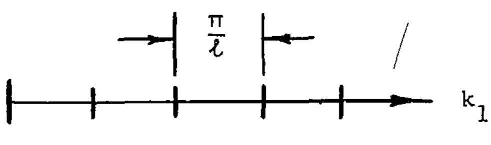


Figure 4.1 Simply supported beam and k-space

Let the dimensionless frequency ν be defined as

$$\nu = \frac{\omega \ell}{C_L} . \tag{4.16}$$

Equation (4.15) reduces to

$$N(\nu) = \frac{1}{\pi} \sqrt{\frac{\nu \ell}{K}} . \tag{4.17}$$

Differentiating the above expression with respect to ν gives

$$n(\nu) = \frac{1}{2\pi} \sqrt{\frac{\ell}{K\nu}} . \tag{4.18}$$

This is the expression for the modal density for the transverse vibration of beams in terms of dimensionless frequency.

4.3 Graphical Results and Discussion

The result of the analytical development in the preceding section is represented graphically in Figure (4.2). From the graph it is seen that modal density for a beam decreases as the dimensionless frequency ν increases and asymptotically approaches zero value as ν becomes large.

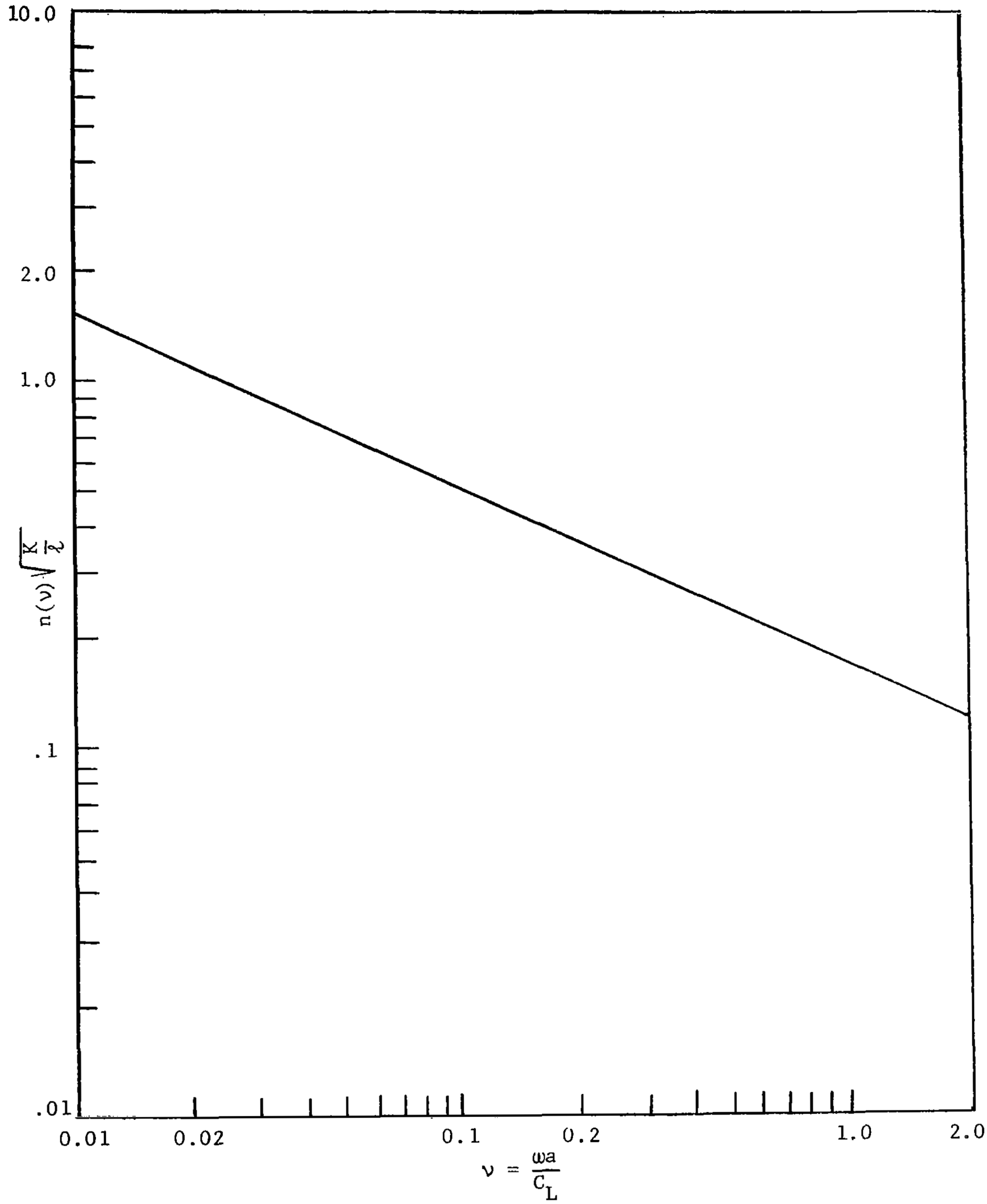


Figure 4.2 Normalized modal density versus dimensionless frequency for beams

5. MODAL DENSITY OF PLATES

5.1 Introduction

In this chapter, modal density expressions are developed for the free vibrations of rectangular and circular plates. To simplify the derivation, a rectangular plate with simply supported edges and a circular plate with clamped edge are considered. The expressions obtained also hold good for other boundary conditions (Bolotin, 1960).

Both the cases discussed in this chapter were readily available in the literature and the information was gathered for systematic representation. The differential equations governing free vibrations of plates are obtained by modifying the equations describing the static equilibrium to account for the inertia forces introduced.

5.2 Rectangular Plates

The governing equation of motion for free vibration of a rectangular plate is given as

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + \frac{\rho h}{g_c}\,\frac{\partial^2 w}{\partial t^2} = 0 \tag{5.1}$$

where

$D = \dfrac{Eh^3}{12(1-\mu^2)}$ = the flexural rigidity

ρ = density of the material

g_c = gravitational constant

$w(x,y,t)$ = displacement normal to the x-y plane

E = Young's modulus

μ = Poisson's ratio

h = the plate thickness.

Equation (5.1) is based on the following assumptions:

1. Rotary inertia is neglected.
2. Cross sections are plane before strain and remain plane after strain.
3. The thickness of the plate is small as compared to the other dimensions.
4. No strain is suffered by the middle surface.
5. Deflections are small in relation to the plate thickness.

In general, when a plate vibrates, there are an infinite number of natural frequencies, and each of them has a specific mode or shape of vibration associated with it. These modes are called normal modes or principal modes.

Let the solution of equation (5.1) be assumed as

$$w(x,y,t) = W(x,y)e^{i\omega t} \quad . \tag{5.2}$$

Substituting equation (5.2) into (5.1) and simplifying gives

$$D\nabla^4 W(x,y) - k^4 W(x,y) = 0 \tag{5.3}$$

where

$$k^4 = \frac{\rho h \omega^2}{D g_c} \quad .$$

Considering the rectangular plate of dimensions ℓ_1 and ℓ_2 as shown in Figure 5.1, the boundary conditions for the plate under consideration are

$$w(x,y,t) = \frac{\partial^2 w(x,y,t)}{\partial x^2} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad x = \ell_1 \tag{5.4}$$

$$w(x,y,t) = \frac{\partial^2 w(x,y,t)}{\partial y^2} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = \ell_2 \ . \tag{5.5}$$

These boundary conditions can also be written as

$$W(0, y) = W''(0, y) = W(\ell_1, y) = W''(\ell_1, y) = 0 \tag{5.6}$$

$$W(x, 0) = W''(x, 0) = W(x, \ell_2) = W''(x, \ell_2) = 0 \ . \tag{5.7}$$

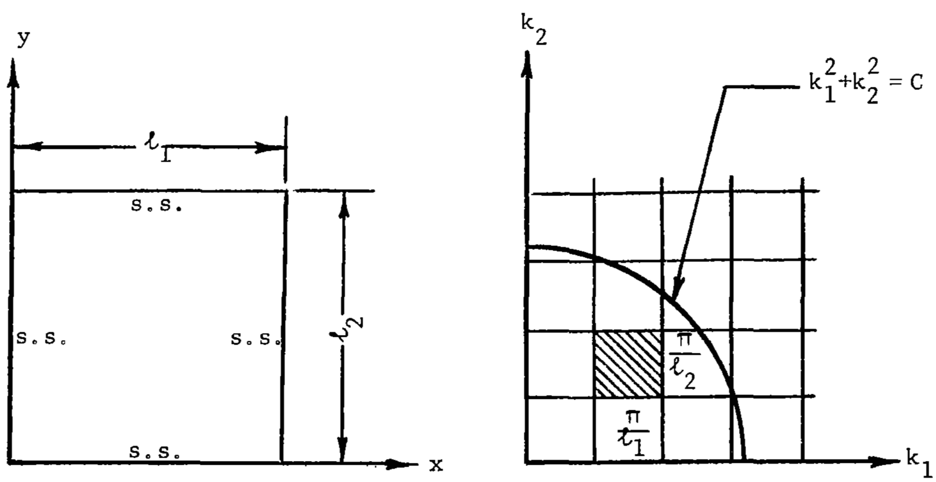


Figure 5.1 Simply supported rectangular plate and k-space

Let

$$W(x,y) = \sin\left(\frac{m\pi x}{\ell_1}\right) \sin\left(\frac{n\pi y}{\ell_2}\right) \tag{5.8}$$

where m and n are integers.

This function satisfies the boundary conditions for a simply supported rectangular plate.

Substitution of equation (5.8) into (5.3) gives

$$k_{mn}^2 = \left(\frac{m\pi}{\ell_1}\right)^4 + 2\left(\frac{m\pi}{\ell_1}\right)^2 \left(\frac{n\pi}{\ell_2}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^4 . \tag{5.9}$$

Equation (5.9) can also be expressed as

$$k_{mn}^2 = \left[\left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2\right]^2 . \tag{5.10}$$

Hence the frequency equation for the simply supported rectangular plate is given as [Smith and Lyon, 1965]

$$\omega_{mn} = \left(\frac{m^2\pi^2}{\ell_1^2} + \frac{n^2\pi^2}{\ell_2^2}\right) KC_L \qquad m = n = 1, 2, 3, \ldots \tag{5.11}$$

where

ℓ_1 and ℓ_2 are the length and width of the plate

$K = \left(\sqrt{\frac{1}{12(1-\mu^2)}}\right) h \approx 0.289h$ = radius of gyration for the plate

$C_L = \sqrt{\frac{Eg_c}{\rho}}$ = longitudinal wave velocity .

Defining the wave numbers k_1 and k_2 as

$$k_1 = \frac{m\pi}{\ell_1} \quad \text{and} \quad k_2 = \frac{n\pi}{\ell_2} \tag{5.12}$$

the changes in the two wave numbers from one mode to the next is given as

$$\Delta k_1 = \frac{\pi}{\ell_1} \quad \text{and} \quad \Delta k_2 = \frac{\pi}{\ell_2} \quad . \tag{5.13}$$

Since the waves in the case of a rectangular plate are propagated along the length and width of the plate, the k-space is two dimensional.

The expression for the number of resonant frequencies can be expressed as

$$N(\omega) = \frac{1}{\Delta k_1 \cdot \Delta k_2} \iint_s dk_1 \cdot dk_2 \ . \tag{5.14}$$

Cylindrical coordinates can be utilized to integrate over the surface of the k-space.

Letting

$$k_1 = r \cos \theta$$

$$k_2 = r \sin \theta$$

equation (5.14) takes the form

$$N(\omega\ ) = \frac{\ell_1 \ell_2}{\pi^2} \int_0^r \int_0^{\pi/2} r d\theta dr \quad . \tag{5.15}$$

Writing the equation (5.15) as

$$N(\omega) = \frac{\ell_1 \ell_2}{\pi} \int_0^{\pi/2} [\int_0^{\ell} r dr] \, d\theta \tag{5.16}$$

and carrying out the integration with respect to r first, and then with respect to θ, gives

$$N(\omega) = \frac{\ell_1 \ell_2}{4\pi} r^2 . \tag{5.17}$$

Combining (5.11) and (5.12) gives

$$r^2 = \frac{\omega}{K C_L} . \tag{5.18}$$

Substitution of equation (5.18) into (5.17) gives

$$N(\omega) = \frac{\ell_1 \ell_2}{4\pi} \frac{\omega}{K C_L} . \tag{5.19}$$

Define $\nu = \frac{\omega \ell_1}{C_L}$, then equation (5.19) reduces to

$$N(\nu) = \frac{\ell_2 \nu}{4\pi K} \tag{5.20}$$

where ν is a dimensionless frequency parameter.

Differentiating (5.20) with respect to ν gives

$$n(\nu) = \frac{dN(\nu)}{d\nu} = \frac{\ell_2}{4\pi K} . \tag{5.21}$$

This is an expression for the modal density of a rectangular plate.

If the plate thickness is h, the radius of gyration is $h/\sqrt{12}$ and the expression is given as

$$n(\nu) = \frac{\ell_2 \sqrt{3}}{2\pi h} . \tag{5.22}$$

5.3 Circular Plates

The differential equation for the free transverse vibrations of a circular plate is given as

$$D\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]^2 w + \frac{\rho h}{g_c}\frac{\partial^2 w}{\partial t^2} = 0 \qquad (5.23)$$

where

D	$= \frac{Eh^3}{12(1-\mu^2)}$ = flexural rigidity of the material
ρ	= mass density of the material
E	= Young's modulus
μ	= Poisson's ratio
g_c	= gravitational constant
h	= plate thickness
$w(r,\theta,t)$	= displacement of a point on the middle surface of the plate.

Equation (5.23) is based generally on the assumption of equation (5.2).

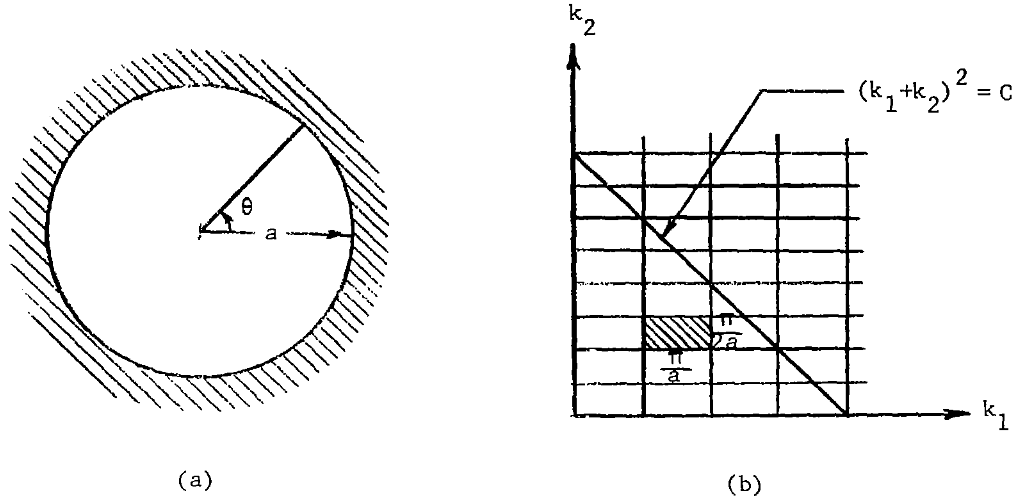


Figure 5.2 Clamped circular plate and k-space

The method of separation of variables can be used to solve equation (5.23).

Assuming that the solution of equation (5.23) is

$$w(r,\theta,t) = W(r,\theta)\, e^{i\omega t} \tag{5.24}$$

and substituting equation (5.24) into (5.23), yields

$$\left[\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right]^2 = \pm\, \lambda^2 W \tag{5.25}$$

where

$$\lambda^4 = \frac{\omega^2}{\beta^2}$$

$$\omega^2 = \text{a constant}$$

$$\beta^2 = \frac{Eh^2 g_c}{12(1-\mu^2)\rho} \quad .$$

Again applying the separation of variables method for solving equation (5.25) gives

$$W(r,\theta) = R(r)\,\phi(\theta) \quad . \tag{5.26}$$

Substituting equation (5.26) into (5.25) gives

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - (\pm\lambda^2 + \frac{n^2}{r^2})R = 0 \tag{5.27}$$

$$\frac{d^2\phi}{d\theta^2} + n^2\phi = 0 \tag{5.28}$$

where n^2 is a constant. Solving the equations (5.27) and (5.28) and substituting into (5.26) yields

$$W(r,\theta) = [A_nJ_n(\lambda r) + B_nJ_n(i\lambda r) + C_nY_n(\lambda r) + D_nY_n(i\lambda r)]$$

$$[E_n \cos n\theta + F_n \sin n\theta] \qquad n = 1,\ 2,\ 3,\ \ldots \tag{5.29}$$

where A_n, B_n, C_n, D_n, E_n and F_n are the arbitrary constants and depend on the boundary conditions of the plate.

J_n and Y_n are the Bessel functions of the first kind of order n and the Bessel functions of second kind of order n respectively.

Since the functions $Y_n(r)$ become infinite for $r = 0$, the constants C_n and D_n must be zero in the case of a solid plate. Hence equation (5.29) becomes

$$W(r,\theta) = [A_n J_n(\lambda r) + B_n J_n(i\lambda r)][E_n \cos n\theta + F_n \sin n\theta] \ . \tag{5.30}$$

For the plate under consideration, the boundary conditions are

$$W(r,\theta) = \frac{\partial W(r,\theta)}{\partial r} = 0 \qquad \text{at} \quad r = a \ . \tag{5.31}$$

By applying the boundary conditions to equation (5.30), the frequency equation for a circular plate can be expressed as

$$\lambda_{m,n} = \sqrt{\frac{\omega}{\beta}} \tag{5.32}$$

where ω is a natural frequency corresponding to the mode characterized by the eigenvalue $\lambda_{m,n}$ and β is a constant defined as

$$\beta = \sqrt{\frac{Eh^2 g_c}{12(1-\mu^2)\rho}} \ .$$

Thus, the exact angular frequency is given as

$$\omega = \lambda_{m,n}^2 \sqrt{\frac{Eh^2 g_c}{12(1-\mu^2)\rho}} \quad \text{rad/sec} \ . \tag{5.33}$$

Now, for large values of λ,

$$\lambda_{m,n} = \frac{\pi}{2a} (m + 2n)^2 \tag{5.34}$$

Therefore, for high frequencies, the frequency equation for a plate with free edge or clamped edge conditions is given as

$$\omega = \frac{\pi^2}{4a^2}(m+2n)^2\sqrt{\frac{Eh^2 g_c}{12(1-\mu^2)\rho}} \tag{5.35}$$

$$\approx \frac{\pi^2}{4a^2}KC_L(m+2n)^2 \qquad m,n = 1,\ 2,\ 3,\ \ldots \tag{5.36}$$

where

a = radius of the plate

K = radius of gyration $= \sqrt{\dfrac{h^2}{12(1-\mu^2)}} \approx \dfrac{h}{\sqrt{12}}$

C_L = longitudinal velocity of wave propagation.

The frequency equation can also be written as

$$\omega = \left(\frac{m\pi}{2a} + \frac{n\pi}{a}\right)^2 KC_L\ . \tag{5.37}$$

Defining the wave numbers k_1 and k_2 as

$$k_1 = \frac{m\pi}{2a} \quad \text{and} \quad k_2 = \frac{n\pi}{a}\ , \tag{5.38}$$

the change in the wave numbers from one mode of vibration to the next is given as

$$\Delta k_1 = \frac{\pi}{2a} \quad \text{and} \quad \Delta k_2 = \frac{\pi}{a}\ . \tag{5.39}$$

The wave propagation, in the case of a circular plate, takes place in the two directions so that the k-space is also two dimensional. The equation for the number of resonant frequencies is given as

$$N(\omega) = \frac{1}{\Delta k_1 \Delta k_2} \int_s\int dk_1 dk_2 \ . \tag{5.40}$$

In this case, cylindrical co-ordinates can be utilized to integrate over the surface of k-space. Letting $k_1 = r\cos\theta$ and $k_2 = r\sin\theta$ and substituting the values for the change in wave numbers and converting to cylindrical co-ordinates, equation (5.40) becomes

$$N(\omega) = \frac{a^2}{\pi^2} \int_{\theta_1}^{\theta_2} r^2 d\theta \ . \tag{5.41}$$

The frequency equation (5.37) can be written as

$$(k_1 + k_2)^2 = \frac{\omega}{KC_L} \ . \tag{5.42}$$

Changing equation (5.42) to cylindrical co-ordinates and solving for r^2 gives

$$r^2 = \frac{\omega}{KC_L} \times \frac{1}{(\sin\theta + \cos\theta)^2} \ . \tag{5.43}$$

Substituting the value of r^2 into equation (5.41) and carrying out integration over values of θ in the quadrant $0 \leq \theta \leq \frac{\pi}{2}$ for which the integrand is real, gives

$$N(\omega) = \frac{a^2}{\pi^2} \times \frac{\omega}{KC_L} \ . \tag{5.44}$$

Defining $\nu = \frac{\omega a}{C_L}$, equation (5.44) reduces to

$$N(\nu) = \frac{a\nu}{\pi^2 K} \tag{5.45}$$

where ν is a dimensionless frequency.

Differentiating equation (5.45) with respect to ν gives

$$n(\nu) = \frac{a}{\pi^2 K} \quad . \tag{5.46}$$

In case of a circular plate, there are two modes of vibration for each frequency, hence the modal density of a circular plate is doubled for any frequency and expression (5.46) reduces to

$$n(\nu) = \frac{2a}{\pi^2 K} \quad . \tag{5.47}$$

This is an expression for the modal density of circular plates.

If the plate thickness is h, radius of gyration is $h/\sqrt{12}$, then expression is given as

$$n(\nu) = \frac{4a\sqrt{3}}{\pi^2 h} \quad . \tag{5.48}$$

5.4 Discussion

From the expressions developed for the modal density of flat rectangular and circular plates, it can be concluded that the modal density of a flat plate is a constant for a given plate and thus is independent of frequency.

Now for a rectangular plate for each frequency, there is just one mode of vibration, whereas for the circular plate, there are two modes of vibration for each frequency. Therefore for two plates of equal area, thickness and of the same material, one circular and the other rectangular, the ratio of modal densities is found to be

$$\frac{n(\omega)_c}{n(\omega)_r} = 2 \times \frac{\sqrt{12}\;(\pi a^2)}{\pi^3\, h\, C_L} \Big/ \frac{\ell_1 \ell_2 \sqrt{3}}{2h\,\pi\, C_L} = \frac{8}{\pi^2} \approx 1\ .$$

Thus for a given frequency (high frequency due to an assumption in the circular plate derivation), the modal density of the rectangular plate will be approximately equal to that of the circular plate.

6. MODAL DENSITY OF THIN CIRCULAR CYLINDERS

6.1 Introduction

In this chapter, thin walled circular cylinders are considered and the expressions for the modal density developed in three different ways are discussed in detail. The first presentation is that of Heckl (1962) in which the expressions are found for the cylindrical shell alone. The second representation is that of Bolotin (1963) in which the general shell of evolution is discussed and then applied to the case of the thin cylindrical shell. The third representation is essentially a modification of Bolotin's work for the specific case of the cylindrical shell.

Shells simply supported at their edges are considered for developing the expressions. However the effects of boundary conditions on the vibrational modes are limited, and hence the edge conditions are of little significance in the modal density expressions except for the first few modes.

All the three representations are discussed in the literature (Miller and Hart, 1967) and they are reproduced here and expressed as a function of dimensionless frequency. The graphs of normalized model density versus frequency are plotted.

6.2 First Representation

For a thin infinitely long cylindrical shell, the equations are given as

$$\alpha V + n_o V_t + \mu k a V_a = i\nu^2 P_o g_c / \omega \rho h \tag{6.1}$$

$$n_0 C + [n_0^2 + \nu^2 + \tfrac{1}{2}(1-\mu) k^2 a] V_t + \tfrac{1}{2}(1+\mu) n_0 k a V_a = 0 \tag{6.2}$$

$$\mu k a V + \tfrac{1}{2}(1+\mu) n_0 k a V_t + [k^2 a^2 + \tfrac{1}{2}(1-\mu) n_0^2 - \nu^2] V_a = 0 \tag{6.3}$$

where

V, V_t and V_a are all radial, tangential and axial components of velocity amplitude

n_0 = the half number of modes in the circumferential direction

k = wave number in the axial direction

μ = Poisson's ratio

h = shell thickness

g_c = gravitational constant

ω = frequency of vibration

P_o = amplitude of excitation

a = cylinder radius

ν = $\frac{\omega a}{C_L}$ = dimension frequency of vibration

a = velocity of wave propagation in the shell material

α = $1 - \nu^2 + \{(n_0^2 + k^2 a^2)^2 - \tfrac{1}{2}[n_0^2(4-\mu) - 2 - \mu](1-\mu)^{-1}\} h^2/12a^2$.

Equations (6.1), (6.2) and (6.3) can be solved for the impedance of the cylindrical shell and letting the impedance go to zero, the following frequency equation can be obtained:

$$\nu^2 = (1-\mu^2)\left(\frac{m\pi a}{\ell}\right)^4 \times \left[\left(\frac{m\pi a}{\ell}\right)^2 + n_0^2\right]^{-2} + \{\left[\left(\frac{m\pi a}{\ell}\right)^2 + n_0^2\right]^2 - \tfrac{1}{2}[n_0^2(4-\mu) - 2 - \mu](1-\mu)^{-1}\} \frac{h^2}{12a^2} . \tag{6.4}$$

The approximate frequency equation obtained by neglecting some of the terms is given as

$$\nu^2 = [\sigma^2(n_0^2+\sigma^2)^{-1} + \beta(n_0^2+\sigma^2)^2]^2 \tag{6.5}$$

where

$$\sigma = \frac{m\pi a}{\ell}$$

$$\beta = \frac{h}{2\sqrt{3}\,a} \quad .$$

The terms neglected have little effect on the frequency expression for frequencies above the ring frequency ($\nu = 1$). However, below the ring frequency the effect may be as much as forty percent of the actual value.

Now solving for σ and then summing over all possible values of n, the equations as obtained by Heckl for the number of resonant frequencies and the modal density are given as

$$N(\omega) = \sum_{n_0=0,1}^{n_0=(\frac{\nu}{\beta})^{\frac{1}{2}}} \tag{6.6}$$

and

$$n(\omega) = \frac{\ell}{\pi a}\int_{0,1}^{(\frac{\nu}{\beta})^{\frac{1}{2}}} \frac{\partial\sigma}{\partial\nu}\,d\nu \tag{6.7}$$

where the lower limit is 1 for $\nu < 1$, and 0 for $\nu > 1$.

Simplifying further results into the followings:

For $\nu > 1$

$$N(\omega) = \frac{\sqrt{3}\,\ell\, a\omega}{2C_L h} \tag{6.8}$$

$$n(\omega) = \frac{\sqrt{3}\,\ell\, a}{2C_L h} \tag{6.9}$$

For $\nu < 1$

$$N(\omega) = \{\tfrac{1}{2}(2\nu-1)[\tfrac{1}{2}\pi + \text{arc sin }(2\nu-1) + (\nu-\nu^2)^{\frac{1}{2}}\}\ \frac{\ell}{4a\beta} \tag{6.10}$$

$$n(\omega) = [\tfrac{1}{2}\pi + \text{arc sin }(2\nu-1)]\ \frac{\ell}{4\pi a\beta}\quad . \tag{6.11}$$

These are the final expressions obtained by Heckl for the number of resonant modes and the modal density of a thin cylindrical shell.

These expressions as functions of dimensionless frequency are given as:

For $\nu > 1$

$$N(\nu) = \frac{\sqrt{3}\,\ell\,\nu}{2h} \tag{6.12}$$

$$n(\nu) = \frac{\sqrt{3}\,\ell}{2h} \tag{6.13}$$

For $\nu < 1$

$$N(\nu) \approx \frac{3\ell\nu^{3/2}}{8\pi a\beta} = \frac{3\sqrt{3}\,\ell\,\nu^{3/2}}{4\pi ah} \tag{6.14}$$

$$n(\nu) = \frac{9\sqrt{3}\,\ell\,\nu^{1/2}}{8\pi ah}\quad .$$

6.3 Second Representation

The diff. equation of motion for thin shell of revolution is given as

$$D\Delta\Delta w - \left(\frac{1}{R_2}\frac{\partial^2\phi}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2\phi}{\partial x_2^2}\right) - \frac{\rho h\omega^2}{g_c} = 0 \qquad (6.15)$$

$$\frac{1}{Eh}\Delta\Delta\phi + \frac{1}{R_2}\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2 w}{\partial x_2^2} = 0 \qquad (6.16)$$

where

x_1 and x_2 are the general curvilinear co-ordinates

R_1 and R_2 are the principal radii of curvature

$D = \frac{Eh^3}{12(1-\mu^2)} \approx \frac{Eh^3}{12}$ = the plate stiffness

ρ = density of the material

h = thickness of the shell

w = normal deflection

ϕ = stress function for the middle surface

E = Young's modulus of elasticity

ω = frequency of vibration

g_c = gravitational constant.

The solution of the equations (6.15) and (6.16) gives the following frequency equation:

$$\omega^2 = \frac{Dg_c}{\rho h}\left[(k_1^2 + k_2^2) + \frac{Eh}{DR_1^2}\frac{(k_1^2\chi + k_2^2)}{(k_1^2 + k_2^2)}\right] \qquad (6.17)$$

where

$$\chi = \frac{R_1}{R_2}$$

k_1 and k_2 are the wave numbers and are given as

$$k_1 = \frac{m\pi}{a_1} \quad \text{and} \quad k_2 = \frac{n\pi}{a_2} \qquad m,n = 1,\ 2,\ 3,\ \ldots \tag{6.18}$$

where a_1 and a_2 are the principal dimensions of the shell surface.

Now the number of resonant modes in the shell is given as

$$N(\omega) = \frac{1}{\Delta k_1 \cdot \Delta k_2} \int_s\int dk_1 dk_2 \quad . \tag{6.19}$$

The change in the wave numbers Δk_1 and Δk_2 from one mode of vibrations to the next is given by

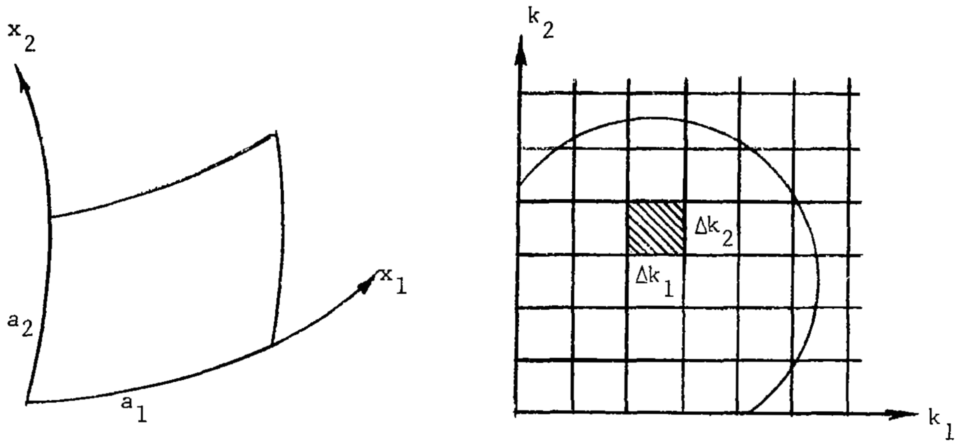
$$\Delta k_1 = \frac{\pi}{a_1} \quad \text{and} \quad \Delta k_2 = \frac{\pi}{a_2} \tag{6.20}$$

where a_1 and a_2 are the principal dimensions of the shell surface.

Substituting the values for the change in wave numbers and converting to cylindrical co-ordinates, equation (6.19) reduces to

$$N(\omega) = \frac{a_1 a_2}{\pi^2} \int_s\int r dr d\theta$$

$$= \frac{a_1 a_2}{2\pi^2} \int_{\theta_1}^{\theta_2} r^2 d\theta \ . \tag{6.21}$$



(a) (b)

Figure 6.1 Generalized rectangular region and k-space

Converting the frequency equation (6.17) to cylindrical coordinates and solving for r^2, gives

$$r^2 = [\omega^2 - \Omega_R^2(X\cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}} \left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} \tag{6.22}$$

where

$$\Omega_R = \frac{1}{R_1}\left(\frac{Eg_c}{\rho}\right)^{\frac{1}{2}} .$$

Thus the expression for the number of resonant modes becomes

$$N(\omega) = \frac{a_1 a_2}{2\pi^2}\left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} \int_{\theta_1(\omega)}^{\theta_2(\omega)} [\omega^2 - \Omega_R^2(X\cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}}\, d\theta . \tag{6.23}$$

Equation (6.23) is an integral expression for the number of resonant frequencies up to a boundary frequency for a thin shell of revolution. The integration is taken over the values of θ in the quadrant $0 \leq \theta \leq \frac{\pi}{2}$ for which the integrand is real and positive.

Differentiating equation (6.23) with respect to frequency ω under the integral sign using Leibnitz's rule gives

$$n(\omega) = \frac{a_1 a_2}{2\pi^2} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \int_{\theta_1(\omega)}^{\theta_2(\omega)} \frac{d\theta}{[\omega^2 - \Omega_R^2 (X \cos^2 \theta + \sin^2 \theta)^2]^{\frac{1}{2}}} \,. \tag{6.24}$$

This is an expression for the modal density of thin shells of revolution.

These equations as written by Bolotin can be represented in the form as shown:

$$N(\omega) = \frac{a_1 a_2}{4\pi} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \omega\, H\left(\frac{1}{\nu}, X\right) \tag{6.25}$$

$$n(\omega) = \frac{a_1 a_2}{4\pi} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} H_1\left(\frac{1}{\nu}, X\right) \tag{6.26}$$

where

$$\frac{1}{\nu} = \frac{\Omega_R}{\omega} = \frac{C_L}{\omega R_1}$$

$$H\left(\frac{1}{\nu}, X\right) = \frac{2}{\pi} \int_{\theta_1(\omega)}^{\theta_2(\omega)} [1 - \frac{1}{\nu^2} (X \cos^2 \theta + \sin^2 \theta)^2]^{\frac{1}{2}}\, d\theta$$

$$H_1\left(\frac{1}{\nu}, X\right) = \frac{2}{\pi} \int_{\theta_1(\omega)}^{\theta_2(\omega)} \frac{d\theta}{[1 - \frac{1}{\nu^2} (X \cos^2 \theta + \sin^2 \theta)^2]^{\frac{1}{2}}}$$

Now for cylindrical shell in particular

$\chi = 0$

a_1 and a_2 are the dimensions of the shell surface

hence $a_1 = \ell$ - length of cylinder

$a_2 = \pi a$ - one-half the circumference of the cylinder.

The reason for taking only half of the cylinder into account is that the cylinder is a closed surface and that the vibrational modes are limited to one-half by that fact.

Hence equations (6.25) and (6.26) take the form

$$N(\omega) = \frac{a\ell}{4} \left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} \omega H\left(\frac{1}{\nu}, 0\right) \tag{6.27}$$

$$n(\omega) = \frac{a\ell}{4} \left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} H_1\left(\frac{1}{\nu}, 0\right) . \tag{6.28}$$

Now rewriting the expression (6.27) as a function of dimensionless frequency ν gives

$$N(\nu) = \frac{\sqrt{3}\,\ell}{2h} \nu\, H\left(\frac{1}{\nu}, 0\right) \tag{6.29}$$

and

$$n(\nu) = \frac{\sqrt{3}\,\ell}{2h} H_1\left(\frac{1}{\nu}, 0\right) . \tag{6.30}$$

The expressions (6.28) and (6.30) are expressions for the modal density of a cylindrical shell of length ℓ and radius a.

The expressions for H_1 may be used in the elliptical integral form as follows:

For $\nu > 1$

$$H_1(\frac{1}{\nu}, 0) = \frac{2}{\pi\sqrt{1 + \frac{1}{\nu}}} \; k_\epsilon \left(\sqrt{\frac{2/\nu}{1 + 1/\nu}}\right) \tag{6.31}$$

For $\nu < 1$

$$H_1(\frac{1}{\nu}, 0) = \frac{\sqrt{2}}{\pi\sqrt{1/\nu}} \; k_\epsilon \left(\sqrt{\frac{1 + 1/\nu}{\frac{2}{\nu}}}\right) \tag{6.32}$$

where k_ϵ represents the complete integral of the first kind.

6.4 Third Representation

As stated in Section (6.4), the number of natural frequencies for a thin shell of revolution is expressed as

$$N(\omega) = \frac{a_1 a_2}{2\pi^2} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \int_{\theta_1}^{\theta_2} [\omega^2 - \Omega_R^2 (X \cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}} \, d\theta \,. \tag{6.33}$$

However for a thin cylindrical shell in particular

$a_1 = \ell$ = length of cylinder

$a_2 = a\pi$ = half the circumference of cylinder

$X = 0$.

Then equation (6.33) reduces to

$$N(\omega) = \frac{\ell a}{2\pi} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \int_{\theta_1}^{\theta_2} [\omega^2 - \Omega_R^2 \sin^4\theta]^{\frac{1}{2}} \, d\theta \tag{6.34}$$

where the upper and lower limits of the integral are taken in the first quadrant ($0 \leq \theta \leq \frac{\pi}{2}$) in such a way so as to keep the integrand real and positive.

Rewriting equation (6.34) as a function of dimensionless frequency, it reduces to

$$N(\nu) = \frac{\ell\sqrt{3}}{\pi h} \int_0^{\sin^{-1}\sqrt{\nu}} [\nu^2 - \sin^4\theta]^{\frac{1}{2}} \, d\theta \, . \qquad (6.35)$$

The upper limit on the integration holds for $\nu < 1$. For ν greater than or equal to one, the upper limit $\frac{\pi}{2}$ is used.

Differentiating equation (6.35) with respect to ν gives

$$n(\omega) = \frac{\ell\sqrt{3}}{2h} \; \frac{2}{\pi} \int_0^{\sin^{-1}\sqrt{\nu}} \frac{d\theta}{[1 - \frac{1}{\nu^2} \sin^4\theta]^{\frac{1}{2}}} \, . \qquad (6.36)$$

Again the upper limit must be $\frac{\pi}{2}$ when ν is equal to or greater than one.

Equation (6.36) is the expression for modal density of thin walled circular cylinders and is referred to as the modified Bolotin's result. It can be evaluated numerically by means of Simpson's rule using a digital computer.

6.5 Graphical Results and Discussion

Figure 6.2 shows the variation of the modal density for the number of natural frequencies for three different representations. Above the ring frequency ($\nu > 1$), all three representations give the identical results as ν becomes very large. However below the ring

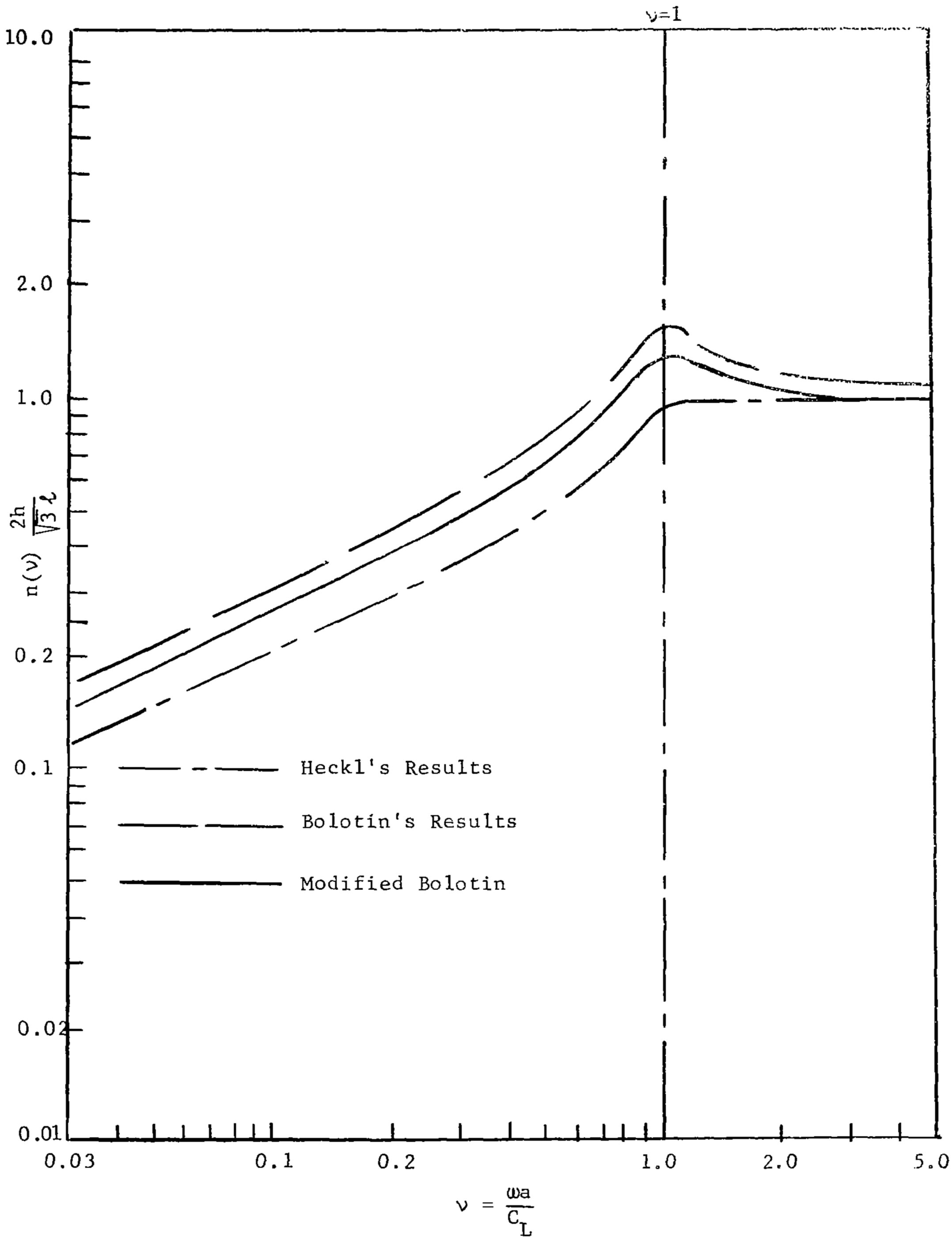


Figure 6.2 Normalized modal density versus dimensionless frequency for thin cylindrical shells

frequency the differences are quite noticeable. The results of Bolotin and the modified Bolotin analysis are slightly different for the modal density. The reason for this slight difference in the modal density curves is due to an approximation made by Bolotin in order to express the modal density in elliptical integral form. The results of Heckl below the ring frequency are lower than other results, but since it was shown earlier that Heckl's results would be on the conservative side, it is reasonable to assume that this is the reason for the difference. Comparison of expressions for modal densities with that of a plate shows that modal density of a thin cylindrical shell above the ring frequency ($\nu > 1$) is equal to one-half the modal density of a flat plate with the same surface area.

7. MODAL DENSITY OF THIN SPHERICAL SHELLS

7.1 Introduction

In this chapter, the thin walled spherical shell is discussed and the expression for modal density is developed in two different ways following the second and third representation used in deriving the expression for model density of thin circular cylinders. In the first representation, the general shell of revolution is considered and then applied to the case of the thin spherical shell, whereas in the second representation the spherical shell is considered in particular.

The frequency equation derived for the thin shell of revolution is in general for the shell with simply supported edges; however, the expressions for the number of modes and modal density hold good for all the boundary conditions, since it is examined in detail that for a spherical shell, edge effects never dominate the mode shapes (Bolotin, 1960).

The expression for modal density is obtained in terms of dimensionless frequency and a graph of dimensionless modal density versus frequency is plotted.

7.2 First Representation

As stated in the previous chapter, the expression for the number of resonant modes for free transverse vibrations of a thin shell of revolution is given as:

$$N(\omega) = \frac{a_1 a_2}{2\pi^2} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \int_{\theta_1(\omega)}^{\theta_2(\omega)} [\omega^2 - \Omega_R^2 (X \cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}} \, d\theta \qquad (7.1)$$

where

a_1 and a_2 are the general curvilinear co-ordinates

R_1 and R_2 are the principal radii of curvature

$\chi = \frac{R_1}{R_2}$

ρ = density of material

h = thickness of the shell

E = Young's modulus

g_c = gravitational constant

ω = natural frequency of vibration

$\Omega_R = \frac{1}{R_1}\left(\frac{Eg_c}{\rho}\right)^{\frac{1}{2}}$.

Differentiating expression (7.1) with respect to frequency ω under the integral sign using Leibnitz's rule gives

$$n(\omega) = \frac{a_1 a_2}{2\pi^2}\left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{[\omega^2 - \Omega_R^2(\chi\cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}}} \quad . \tag{7.2}$$

This is an expression for the modal density of a thin shell of revolution. These expressions (7.1) and (7.2) as written by Bolotin can be represented in the form:

$$N(\omega) = \frac{a_1 a_2}{4\pi}\left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} \omega\, H\left(\frac{1}{\nu}, \chi\right) \tag{7.3}$$

$$n(\omega) = \frac{a_1 a_2}{4\pi}\left(\frac{\rho h}{Dg_c}\right)^{\frac{1}{2}} H_1\left(\frac{1}{\nu}, \chi\right) \tag{7.4}$$

where

$$H(\frac{1}{\nu}, \chi) = \frac{2}{\pi} \int_{\theta_1(\omega)}^{\theta_2(\omega)} [1 - \frac{1}{\nu^2} (\chi \cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}} \, d\theta$$

$$H_1(\frac{1}{\nu}, \chi) = \frac{2}{\pi} \int_{\theta_1}^{\theta_2} \frac{d\theta}{[1 - \frac{1}{\nu^2} (\chi \cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}}}$$

$$\nu = \frac{\omega R_1}{C_L} = \frac{\omega}{\Omega_R} = \text{dimensionless frequency} .$$

For a spherical shell of radius a

$$\chi = 1$$

$$a_1 = a_2 = \pi a = \text{half the circumference of sphere} .$$

The reason for taking only half of a sphere into account is that the sphere is a closed surface and that the vibrational modes are limited to one-half by that fact.

Hence equations (7.3) and (7.4) take the form

$$N(\omega) = \frac{\pi a^2}{4} (\frac{\rho h}{D g_c})^{\frac{1}{2}} \omega H(\frac{1}{\nu}, 1) \tag{7.5}$$

and

$$n(\omega) = \frac{\pi a^2}{4} (\frac{\rho h}{D g_c})^{\frac{1}{2}} H_1(\frac{1}{\nu}, 1) . \tag{7.6}$$

Again, rewriting the expressions as a function of the dimensionless frequency ν gives

$$N(\nu) = \frac{\pi a\sqrt{3}}{2h}\,\nu\, H(\frac{1}{\nu}, 1) \tag{7.7}$$

and

$$n(\nu) = \frac{\sqrt{3}\pi a}{2h} H_1(\frac{1}{\nu}, 1) \tag{7.8}$$

Now

$$H_1(\frac{1}{\nu}, x) = \frac{2}{\pi}\int_{\theta_1}^{\theta_2} \frac{d\theta}{[1 - \frac{1}{\nu^2}(\chi\cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}}} \ . \tag{7.9}$$

The integration with respect to θ is carried out over that part of the quadrant $0 \leq \theta \leq \frac{\pi}{2}$ in which the integrand is positive and real. For a spherical shell $\chi = 1$ and the integral (7.9) can be expressed as

$$H_1(\frac{1}{\nu}, 1) = 0 \qquad (\nu < 1) \tag{7.10}$$

$$H_1(\frac{1}{\nu}, 1) = \frac{\nu}{\sqrt{\nu^2 - 1}} \qquad (\nu > 1) \ . \tag{7.11}$$

Hence equation (7.8) is written as

$$n(\nu) = \frac{\sqrt{3}\pi a}{2h}\frac{\nu}{\sqrt{\nu^2 - 1}} \qquad (\nu > 1) \tag{7.12}$$

$$n(\nu) = 0 \qquad (\nu < 1) \ . \tag{7.13}$$

Equations (7.12) and (7.13) represent the modal density for a thin walled spherical shell.

7.3 Second Representation

The expression for modal density can also be obtained by finding the number of resonant frequencies for a spherical shell and then differentiating with respect to frequency, in the same way as the modified Bolotin results for cylindrical shells. From equation (7.1), the number of resonant frequencies for a thin shell of revolution is

$$N(\omega) = \frac{a_1 a_2}{2\pi^2} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \int_{\theta_1(\omega)}^{\theta_2(\omega)} [\omega^2 - \Omega_R^2(\chi \cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}} \, d\theta \; . \tag{7.14}$$

Now for the thin walled spherical shell

$a_1 = a_2 = \pi a$ = half the circumference

$\chi = \frac{R_1}{R_2} = 1$.

Hence expression (7.14) reduces to

$$N(\omega) = \frac{a^2}{2} \left(\frac{\rho h}{D g_c}\right)^{\frac{1}{2}} \int_{\theta_1}^{\theta_2} [\omega^2 - \Omega_R^2]^{\frac{1}{2}} \, d\theta \; . \tag{7.15}$$

Converting equation (7.15) in terms of the dimensionless frequency gives

$$N(\nu) = \frac{a\sqrt{3}}{h} \int_{\theta_1}^{\theta_2} \nu[1 - \frac{1}{\nu^2}]^{\frac{1}{2}} \, d\theta \; . \tag{7.16}$$

Here again the limits on the integral are taken in the quadrant $0 \leq \theta \leq \frac{\pi}{2}$ such that integrans is real and positive.

Hence the following integral expression for the number of resonant frequencies is obtained:

$$N(\nu) = \frac{a\sqrt{3}}{h} \int_0^{\sin^{-1}\sqrt{\nu}} [\nu^2 - 1]^{\frac{1}{2}} \, d\theta \ . \tag{7.17}$$

The upper limit on the integrations holds for $\nu < 1$. For $\nu \geq 1$ the upper limit $\frac{\pi}{2}$ is used.

Differentiating with respect to ν

$$n(\nu) = \frac{a\sqrt{3}}{h} \int_0^{\sin^{-1}\sqrt{\nu}} \frac{\nu}{[\nu^2-1]^{\frac{1}{2}}} \, d\theta \ . \tag{7.18}$$

Now for $\nu < 1$ the integrand is negative, hence

$$n(\nu) = 0 \qquad (\nu < 1) \tag{7.19}$$

and for $\nu > 1$, using the upper limit $\frac{\pi}{2}$, it gives

$$n(\nu) = \frac{\sqrt{3}\pi a}{2h} \frac{\nu}{(\nu^2-1)^{\frac{1}{2}}} \qquad (\nu > 1) \ . \tag{7.20}$$

Equations (7.19) and (7.20) represent the modal density for a spherical shell.

7.4 Graphical Results and Discussion

Expressions obtained for modal density following two different approaches gives identical results. In plotting the graph, the expression is normalized so as to make it independent of geometry.

Figure 7.1 shows that modal density has a singularity at $\nu = 1$, below which the modal density is zero. For $\nu > 1$, (above the ring frequency) the modal density of the shell decreases monotonically and asymptotically approaches that of the flat plate as ν becomes very large.

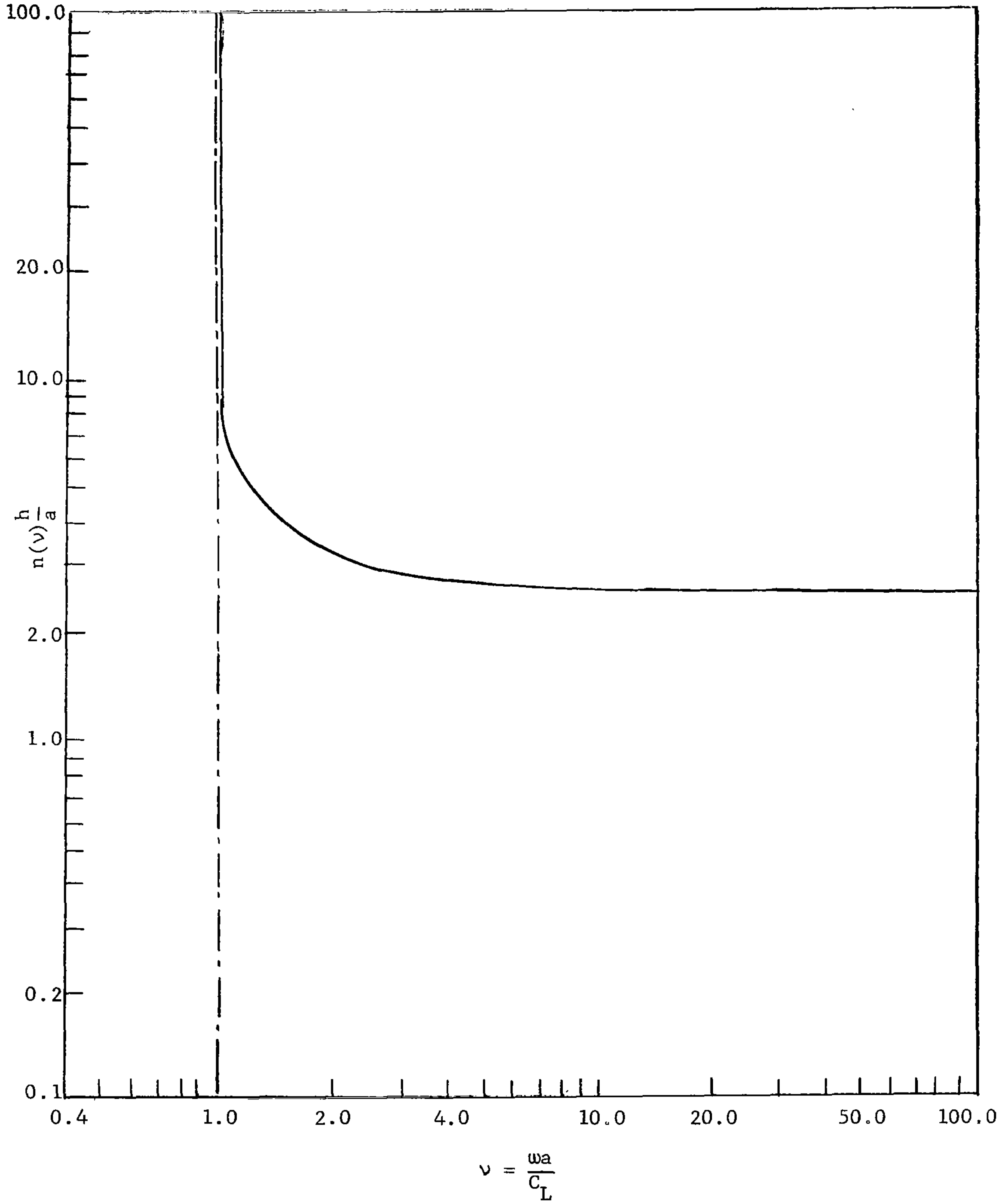


Figure 7.1 Normalized modal density versus dimensionless frequency for thin spherical shells

8. MODAL DENSITY OF THIN CONICAL SHELLS

8.1 Introduction

In this chapter, the thin walled conical shell is discussed and the expressions are developed for the number of resonant modes and modal density. These expressions are applicable to a wide range of geometry and are valid over a frequency range sufficiently wide to be of engineering value.

The expressions for the number of resonant modes are developed from two separate frequency equations for thin conical shells using the k-space integration technique for frequencies below the lower ring frequency and a different approach for frequencies above the upper ring frequency. The expressions obtained for modal density for two cases are normalized and plotted versus the dimensionless frequency.

The problem of conical shells was readily available in the literature and is reproduced for graphical representation (Miller, 1969).

8.2 Frequency Equation One

The frequency equation for a thin conical shell is given as

$$\omega^2 = \left[\frac{g_c E}{\rho L^2}\right] \frac{\frac{D}{EL^2 h}\left\{\frac{a_n^4}{10}(1-\alpha_1^5) + a_n\left(1+\frac{2m^2}{\sin^2\psi}\right)\left[\frac{a_n}{6}(1+\alpha_1^3) - \frac{1}{2a_n}(1-\alpha_1)\right] + \right.}{\left[\frac{a_n^4}{10}(1-\alpha_1^5) + a_n\left(1+\frac{2m^2}{\sin^2\psi}\right)\left[\frac{a_n}{6}(1+\alpha_1^3) - \frac{1}{2a_n}(1-\alpha_1)\right] + \right.}$$

$$\frac{\left.+\left(\frac{m^4}{\sin^4\psi} - \frac{4m^2}{\sin^2\psi}\right)\frac{1}{2}(1-\alpha_1)\right\}^2 + \frac{a_n^4}{\tan^2\psi}\left\{\frac{1}{8}(1-\alpha_1^4) - \frac{3}{8a_n^2}(1-\alpha_1^2)\right\}^2}{\left.+\left(\frac{m^4}{\sin^4\psi} - \frac{4m^2}{\sin^2\psi}\right)\frac{1}{2}(1-\alpha_1)\right\}^2 \left\{\frac{1}{10}(1-\alpha_1^5) - \frac{1}{2a_n^2}(1-\alpha_1^3) + \frac{3}{4a_n^4}(1-\alpha_1)\right\}} \tag{8.1}$$

where

ω = natural frequency of vibration

E = modulus of elasticity

g_c = gravitational constant

ρ = density of shell material

D = stiffness of shell material = $\dfrac{Eh^3}{12(1-\mu^2)}$

μ = Poisson's ratio

h = the thickness of shell wall

ψ = one-half cone angle at apex

α_1 = truncation ratio = $\dfrac{L_T}{L}$

L_T = length of cone truncation, apex to top slant length

L = length of cone, apex to base slant length

m = number of circumferential waves

n = number of one-half longitudinal waves

$a_n = \dfrac{n\pi}{1-\alpha_1}$.

In deriving the above frequency equation, the following assumptions are made:

(1) The circumferential modes are independent of the longitudinal modes.

(2) The modes in the circumferential direction are sinusoidal and uniform over the length of the cone.

Equation (8.1) may be written in dimensionless form by defining a dimensionless frequency ratio and longitudinal and circumferential wave numbers in the following manner:

$$\nu^2 = \frac{\omega^2 \rho L^2}{g_c E} \; ; \quad k_1 = a_n \; ; \quad k_2 = \frac{m}{\sin\psi} \qquad (8.2)$$

Substituting equations (8.2) into equation (8.2) and rearranging terms results in the following expression for the dimensionless frequency:

$$\nu^2 = \frac{\frac{h^2}{12L^2(1-\nu^2)} k_1^4\{k_1^4 \frac{1-\alpha_1^5}{10} + (1+2k_2^2)(k_1^2 \frac{1-\alpha_1^3}{6} - \frac{1-\alpha_1}{2}) + (k_2^4 - 4k_2^2) \frac{1-\alpha_1}{2}\}^2 +}{\{k_1^4 \frac{1-\alpha_1^5}{10} + (1+2k_2^2)(k_1^2 \frac{1-\alpha_1^3}{6} - \frac{1-\alpha_1}{2}) + (k_2^4 - 4k_2^2) \frac{1-\alpha_1}{2}\} \quad x}$$

$$\frac{+ \cot^2\psi[k_1^4 \frac{1-\alpha_1^4}{8} - k_1^2 \frac{3(1-\alpha_1^2)}{8}]^2}{x\ [k_1^4 \frac{1-\alpha_1^5}{10} - k_1^2 \frac{1-\alpha_1^3}{2} + \frac{3(1-\alpha_1)}{4}]} \quad . \tag{8.3}$$

Now the number of resonant modes for a thin conical shell is given by the double integral

$$N(\nu) = \frac{1}{\Delta k_1 \Delta k_2} \int\int dk_1 dk_2 \tag{8.4}$$

where the integral is to be taken over that portion of the first quadrant where the k-space exists.

The integration of the equation (8.4) to obtain an expression for the cumulative number of eigenvalues or the resonants modes in the usual way is a somewhat impractical approach to the problem and hence the number of eigenvalues above some selected frequency is obtained utilizing a different approach.

It will be of some value to first define upper and lower ring frequencies. The upper ring frequency is defined as the frequency at which the longitudinal wave length is equal to the circumference of the small end of the cone. In dimensionless form the upper ring frequency would be given by

$$\nu \text{ (upper ring)} = \frac{1}{\alpha_1 \sin\psi} \quad . \tag{8.5}$$

This is equivalent to ω times the small radius of the cone divided by the longitudinal wave velocity equal to unity.

Similarly the lower ring frequency is defined as the frequency at which the longitudinal wave length is equal to the circumference of the large end of the cone. In dimensionless form it is

$$\nu \text{ (lower ring)} = \frac{1}{\sin\psi} \quad . \tag{8.6}$$

This is equivalent to ω times the large radius of the cone divided by the longitudinal wave velocity equal to unity.

The frequency equation (8.3) shows that dimensionless frequency is affected by these geometric parameters assuming that Poisson's ratio is constant and equal to 0.3. These are the cone angle (ψ), the thickness over length ratio ($\frac{h}{L}$) and truncation ratio (α_1).

Using frequency equation (8.3), the eigenvalues may be computed for different values of m and n, the circumferential and longitudinal wave numbers respectively. In this manner for different cone geometries, the number of eigenvalues occurring above certain specified dimensionless frequencies may be obtained using digital computer and results can be plotted in a graphical form.

Now the results obtained by normalizing with respect to cone geometry in the frequency range above the upper ring frequency (Miller,

1969) can be expressed in the form

$$N(\nu) \frac{\pi}{\sin\psi(1-\alpha_1)} [\frac{h}{L} (1-\alpha_1)^{1/5}] = F(\nu) \quad . \tag{8.7}$$

The graphs of $F(\nu)$ versus dimensionless frequency can be plotted varying various geometry parameters and it can be found that the value of $F(\nu)$ is independent of the cone geometry above the upper ring frequency (Miller, 1969) and is given as

$$F(\nu) = 2.0\nu \qquad \text{for} \qquad \nu > \frac{1}{\alpha_1 \sin\psi} \quad . \tag{8.8}$$

The number of resonant modes is given by

$$N(\nu) = 2.0 \left[\frac{L \sin\psi \ (1-\alpha_1)^{4/5}}{\pi h}\right] \nu \quad . \tag{8.9}$$

Differentiating equation (8.9) with respect to ν gives

$$n(\nu) = 2.0 \left[\frac{L \sin\psi \ (1-\alpha_1)^{4/5}}{\pi h}\right] \quad . \tag{8.10}$$

This is the expression for modal density for thin conical shells above the upper ring frequency.

8.3 Frequency Equation Two

The second frequency equation for a thin conical shell is given as

$$\omega^2 = \left[\frac{g_c E}{\rho L^2}\right] \left[\frac{\frac{D}{EL^2 h} \left(\frac{m^4}{\sin^4\psi} x \frac{1-\alpha_1}{2}\right)^2 + \frac{a_n^4}{\tan^2\psi} \left[\frac{1-\alpha_1^4}{8} - \frac{3(1-\alpha_1)^2}{8a_n^2}\right]^2}{\left(\frac{m^4}{\sin^4\psi} \frac{1-\alpha_1}{2}\right) \left(\frac{1-\alpha_1^5}{10} - \frac{1-\alpha_1^3}{2a_n^2} + \frac{3(1-\alpha_1)}{4a_n^4}\right)} \right] . \tag{8.11}$$

The notations used in the above equation are the same as those for equation (8.1). In deriving the frequency equation (8.1) it is assumed that:

1. Mode shapes are axially symmetric, and
2. Longitudinal bending is small when compared with circumferential bending.

Defining the dimensionless frequency parameter, the circumferential wave number and the longitudinal wave number in exactly the same way as in Section 8.2, equation (8.11), in the dimensionless form can be written as

$$\nu^2 = \frac{\frac{D}{EL^2h}\left(k_2^4\,\frac{1-\alpha_1}{2}\right)^2 + \cot^2\psi\; k_1^4\left[\frac{1-\alpha_1^4}{8} - \frac{3(1-\alpha_1^2)^2}{8k_1^2}\right]}{k_2^4\left(\frac{1-\alpha_1}{2}\right)\left(\frac{1-\alpha_1^5}{10} - \frac{1-\alpha_1^3}{2k_1^2} + \frac{3(1-\alpha_1)}{4k_1^4}\right)} \quad . \tag{8.12}$$

Now the number of resonant modes or eigenvalues for thin conical shells is expressed in the double integral form as

$$N(\nu) = \frac{1}{\Delta k_1 \Delta k_2} \int\int dk_1 dk_2 \tag{8.13}$$

where the integral is to be taken over that portion of the first quadrant where the k-space exists.

The region over which the integral (8.13) is to be evaluated is bounded by an upper and lower value of k_1, referred to as $\underline{b}$ and $\underline{a}$ respectively. The region is also bounded by upper and lower curves which will be referred to as $[k_2]_u$ and $[k_2]_\ell$ respectively. Equation (8.13) therefore can be written in the following form:

$$N(\nu) = \frac{1}{\Delta k_1 \Delta k_2} \int_{\underline{a}}^{\underline{b}} ([k_2]_u - [k_2]_\ell)\, dk_1 \tag{8.14}$$

where $\underline{a}$ and $\underline{b}$ are functions of ν and the cone geometry parameters and $[k_2]_u$ and $[k_2]_\ell$ are functions of ν, k_1 and cone geometry parameters.

In equation (8.14) Δk_1 and Δk_2 are the changes in the longitudinal and circumferential wave numbers respectively from one mode to the next and are given as

$$\Delta k_1 = \frac{\pi}{1-\alpha_1} \; ; \qquad \Delta k_2 = \frac{1}{\sin\psi} \quad . \tag{8.15}$$

Substituting equation (8.15) into (8.14) and rearranging the terms gives

$$N(\nu) \left[\frac{\pi}{(1-\alpha_1)\sin\psi}\right] = \int_a^b ([k_2]_u - [k_2]_\ell)\, dk_1 \; . \tag{8.16}$$

The upper and the lower bounds of the space $[k_2]_u$ and $[k_2]_\ell$ may be obtained from the solution of the equation derived from frequency equation (8.12) and may be expressed in the following form:

$$[k_2]_u = 4\sqrt{\frac{-B \pm \sqrt{B^2-4AC}}{2A}} \tag{8.17}$$

and

$$[k_2]_\ell = 4\sqrt{\frac{-B - \sqrt{B^2-4AC}}{2A}} \tag{8.18}$$

where

$$A = \frac{D}{EhL^2} 80k_1^4$$

$$B = \nu^2[-16k_1^4(1+\alpha_1+\alpha_1^2+\alpha_1^3+\alpha_1^4) + 8k_1^2(1+\alpha_1+\alpha_1^2) - 120]$$

$$C = 5\cot^2\psi[k_1^4(1+\alpha_1+\alpha_1^2+\alpha_1^3) - 3k_1^2(1+\alpha_1)]^2 \quad .$$

Therefore equation (8.16) may be expressed in the following form:

$$N(\nu)\left[\frac{\pi}{(1-\alpha_1)\sin\psi}\right] \approx \int_a^b 4\sqrt{\frac{-B+\sqrt{B^2-4AC}}{2A}}\,dk_1$$

$$- \int_a^b 4\sqrt{\frac{-B-\sqrt{B^2-4AC}}{2A}}\,dk_1 \quad . \qquad (8.19)$$

The upper and lower limits of the space, b and a, can be obtained by equating the upper and lower bounds of the space given by equation (8.17) and (8.18). The expression resulting from this process is given as follows:

$$k_1^6 + Sk_1^4 + Tk_1^2 + U = 0 \ , \qquad (8.20)$$

where

$$S = [-3(1-\alpha_1^2)/(1-\alpha_1^4)] - [4\nu(1-\alpha_1^5)L\sqrt{3(1-\mu^2)}\,/\,5h\cot\psi(1-\alpha_1^4)]$$

$$T = 4\nu(1-\alpha_1^3)\,L\sqrt{3(1-\mu^2)}\,/\,h\cot\psi(1-\alpha_1^4)$$

$$U = -6\nu(1-\alpha_1)\,L\sqrt{3(1-\mu^2)}\,/\,h\cot\psi(1-\alpha_1^4) \quad .$$

For the truncated cone, three real roots are obtained; the largest root is denoted as $\underline{b}$ and the second largest root is denoted by $\underline{a}$. For the closed cone, one real and two imaginary roots are obtained. The real root is denoted as $\underline{b}$ and the real part of either of the imaginary roots is denoted as $\bar{a}$.

The evaluation of the equation (8.19) for the number of eigenvalues and also upper and lower limits of the k-space is handled by numerical procedures on an IBM 360 model 75 digital computer and graphs are plotted.

Now from the graphical results (Miller, 1969) it can be concluded that the number of modes ($N \frac{\pi}{\sin\psi(1-\alpha_1)}$) varies with changes in cone angle directly as $(\tan\psi)^{\frac{1}{2}}$, with changes in thickness inversely as $\frac{h}{L}$, and with changes in truncation inversely as $(1-\alpha_1)^{1/4}$. Hence the number of modes may be normalized in the following manner:

$$N(\nu) \frac{\pi}{\sin\psi(1-\alpha_1)} \left[\frac{h(1-\alpha_1)^{1/4}}{L(\tan\psi)^{1/2}} \right] = G(\nu) \quad . \tag{8.21}$$

In deriving frequency equation two, it was assumed that the contribution to the differential equation due to longitudinal bending is small in comparison with the contribution due to circumferential bending which limits equation two to the lower frequencies. Hence, the results of the analysis based on frequency equation two are valid below the lower ring frequency.

The graphical representation reported by Miller (1969) indicates that $G(\nu)$ is independent of the geometry of the conical shell except the variations in the vicinity of values associated with $N(\nu) = 1$ and is a function of only the dimensionless frequency ν. Moreover, the normalized number of eigenvalue curves are straight lines on log-log paper except near the values associated with the first few resonant frequencies. Hence equation (8.21) may be approximated as

$$N(\nu) \frac{\pi}{\sin\psi(1-\alpha_1)} \left[\frac{h(1-\alpha_1)^{1/4}}{L(\tan\psi)^{1/2}} \right] = 0.876\nu^{3/2} . \tag{8.22}$$

This equation is valid for the frequencies below the lower ring frequency only.

Hence the number of resonant modes for a conical shell below the lower ring frequency is given as

$$N(\nu) = 0.876 \left[\frac{\pi}{\sin\psi} \times \frac{h(1-\alpha_1)^{3/4}}{L(\tan\psi)^{1/2}} \right] \nu^{3/2} . \tag{8.23}$$

Differentiating equation (8.23) with respect to ν gives

$$n(\nu) = 1.31 \left[\frac{\pi}{\sin\psi} \times \frac{h(1-\alpha_1)^{3/4}}{L(\tan\psi)^{1/2}} \right] \nu^{1/2} . \tag{8.24}$$

This is an expression for modal density of thin conical shells below the lower ring frequency.

8.4 Graphical Results and Discussion

Expressions (8.10) and (8.24) represent the modal density for a thin conical shell and variation in modal density is shown in Figure 8.1.

In plotting the graphs the expressions are normalized so as to make them independent of geometry. For the variations below the lower

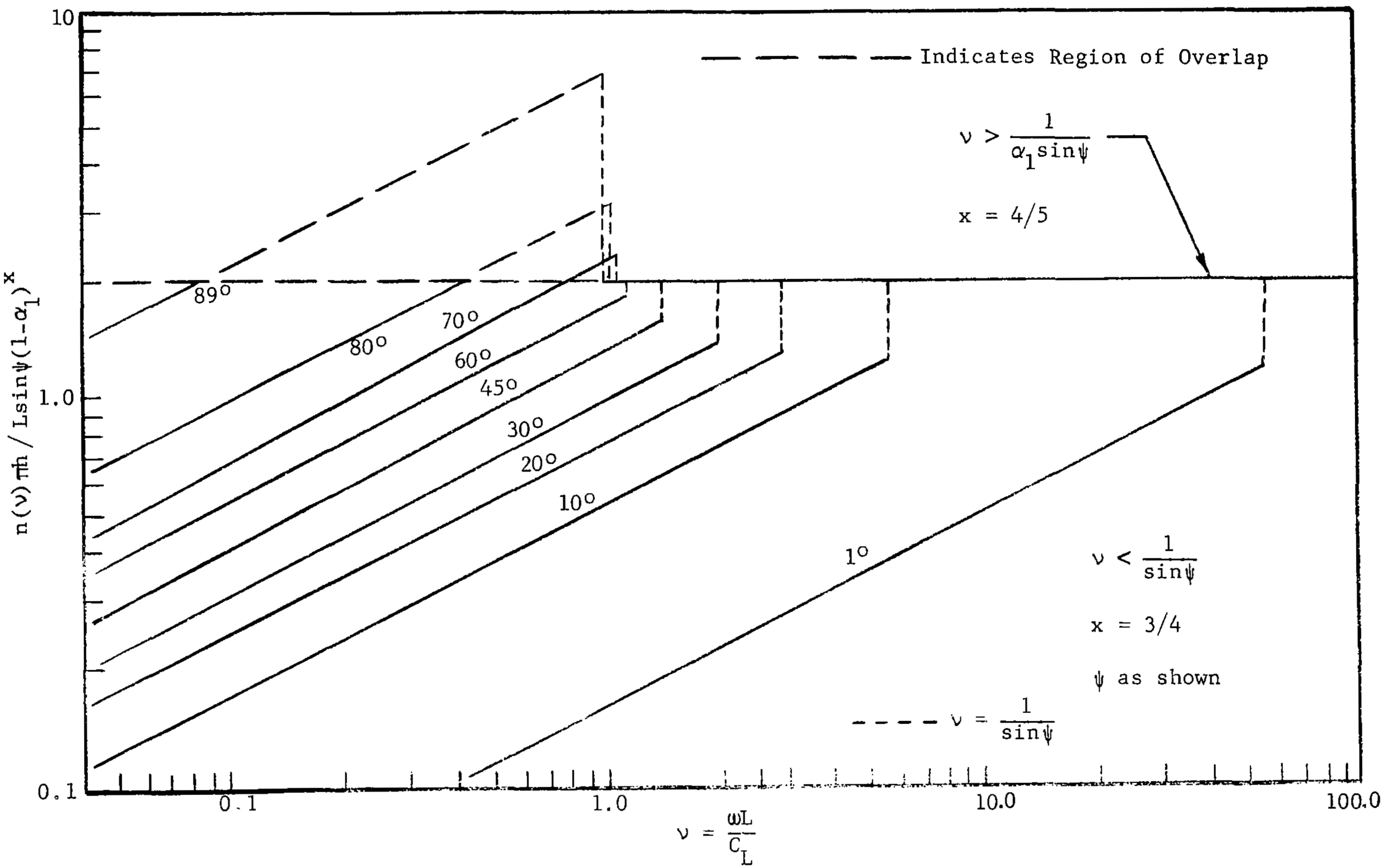


Figure 8.1 Normalized modal density versus dimensionless frequency for thin conical shells

ring frequency cone angle appears as a parameter as the lower ring frequency depends on the cone angle. The upper ring frequency has not been indicated in the figure since it is a function of the truncation ratio of the cone and may vary anywhere from the lower ring frequency for a completely truncated cone to infinity for a closed cone.

The graph and expressions are invalid for cones with large cone angles as well as with little or no truncation at all as the shell in these goes into a so called plate mode (above upper ring frequency solution). However it can be concluded that expressions are valid over a wide range of cone geometries and frequency ranges of practical interest.

9. MODAL DENSITY OF COMPOSITE STRUCTURES

9.1 Introduction

In this chapter the additive property of modal density for composite structures is discussed. Modal density of certain basic structures such as rods, beams, plates, cylinders and spheres have been already discussed in the previous chapters. However these basic structures rarely occur in a real application in engineering as separate elements. Therefore the modal density of composite structures must be considered.

The composite structure analyzed consists of two beams joined at right angles to form an L-shaped frame. The case of composite structure was readily available in literature (Hart and Desai, 1967).

9.2 Composite Structures

A composite structure is composed of a number of substructures which may be taken as the sum of the basic structural elements. Assuming that the modal density of each substructure is known, it is postulated that the modal density of a composite structure is equal to the sum of the modal densities of its components.

If the j^{th} component of the composite structure exhibits N_j modes within the frequency interval $\Delta\omega$, then its modal density at the center of the band $\Delta\omega$ is defined as

$$n_j(\omega) = \frac{N_j}{\Delta\omega} \quad . \tag{9.1}$$

Thus the modal density of the composite at ω would be given by

$$n(\omega) = \frac{1}{\Delta\omega} \sum_{J=1}^{m} N_J \qquad (9.2)$$

where the summation extends over the total number of elements, m, that give rise to the composite structure.

To demonstrate analytically, a composite structure consisting of two beams joined at right angles to form an L-shape is considered (Hart and Desai, 1967). The subsystems may then be supposed to be two beams as illustrated in Figure 9.2.

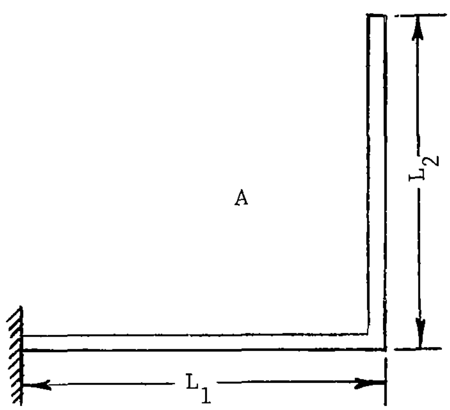


Figure 9.1 Composite structure

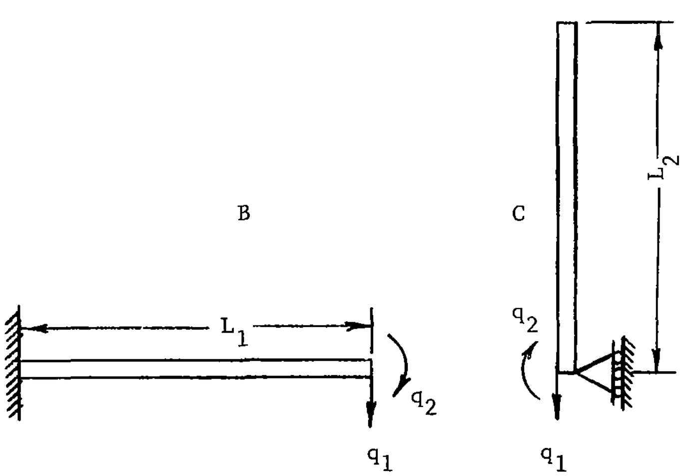


Figure 9.2 Two substructures

It is assumed that $L_1 = L_2 = \ell$ and that both members are of the same material.

If the additive postulate for modal density holds good, then the total numbers of resonant modes for the composite structure are given as follows:

$$N_A(\lambda\ell) = N_B(\lambda\ell) + N_C(\lambda\ell) \quad . \tag{9.3}$$

By considering the frequency equation for the composite structure, a graph of $N_A(\lambda\ell)$ against $\lambda\ell$ can be plotted. The relationship derived from this graph (Hart and Desai, 1967) can be expressed as follows:

$$N_A(\lambda\ell) = \frac{16}{25}(\lambda\ell) \quad . \tag{9.4}$$

Now the resonant frequency for a composite structure derived from the frequency equation is given by the expression

$$\omega = \frac{(\lambda\ell)^2}{\ell^2} C_L K \tag{9.5}$$

where

C_L = longitudinal wave velocity

K = radius of gyration.

Equation (9.5) gives

$$(\lambda\ell) = \sqrt{\frac{\omega\ell^2}{C_L K}} \quad . \tag{9.6}$$

Defining dimensionless frequency ν

$$\nu = \frac{\omega\ell}{C_L} \quad . \tag{9.7}$$

Equation (9.6) can be written as

$$\lambda \ell = \sqrt{\frac{\nu \ell}{K}} \quad . \tag{9.8}$$

Substituting (9.8) into (9.4) gives

$$N_A(\nu) = \frac{16}{25} \sqrt{\frac{\nu \ell}{K}} \quad . \tag{9.9}$$

Differentiating equation (9.9) with respect to dimensionless frequency ν gives

$$n_A(\nu) = \frac{8}{25} \sqrt{\frac{\ell}{K\nu}} \quad . \tag{9.10}$$

Equation (9.10) gives the expression for modal density of a composite structure.

The modal density for a beam, irrespective of the boundary conditions, can be written as

$$n(\nu) = \frac{1}{2\pi} \sqrt{\frac{\ell}{K\nu}} \quad . \tag{9.11}$$

Since the composite structure is constructed of two identical beams, the sum of the modal densities of the substructures is

$$n_B(\nu) + n_C(\nu) = \frac{1}{\pi} \sqrt{\frac{\ell}{K\nu}} \quad . \tag{9.12}$$

Comparison of equations (9.11) and (9.12) proves that additive property of modal density holds good for this particular composite system.

9.3 Graphical Results and Discussion

The additive property of modal densities for composite structures holds for the composite structure composed of two identical beams welded at right angles as verified analytically.

Figure 9.3 shows that modal density of a composite structure varies along a straight line on a log-log scale and variation with respect to frequency is proportional to the beam except that the magnitude of modal density is doubled.

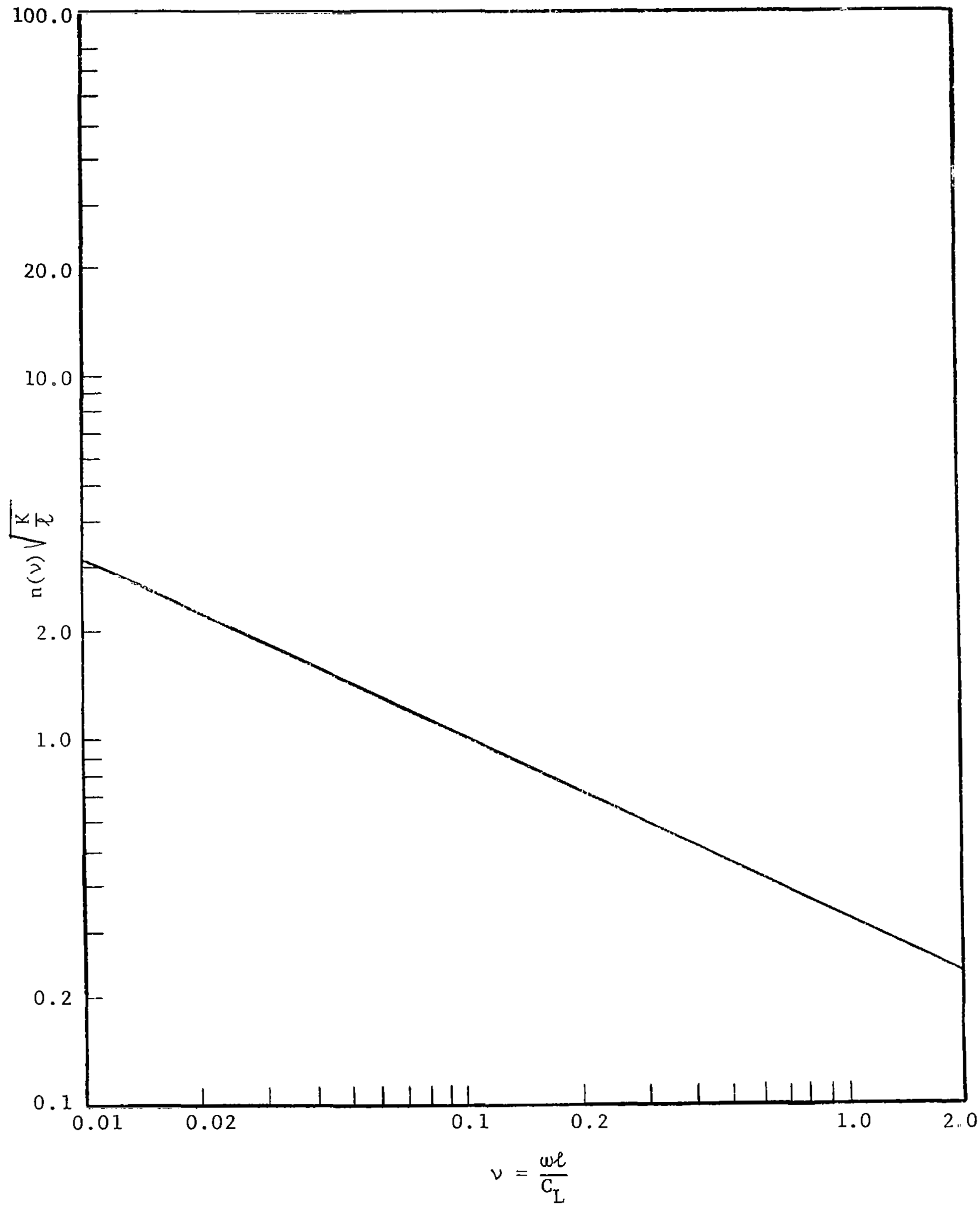


Figure 9.3 Normalized modal density versus dimensionless frequency for composite structure

10. MODAL DENSITY OF SHALLOW STRUCTURAL ELEMENTS

10.1 Introduction

In this chapter the expressions for the modal densities of some shallow sandwich shells, orthrotropic plates, pretwisted plates, plates subjected to in-plane forces and shells on an elastic foundation are presented.

The expressions are developed on the basis of shallow-shell theory and neglecting the effect of longitudinal inertia and hence only the frequencies of transverse vibrations are considered. The coupled longitudinal modes cannot be obtained from these expressions. However the effect of the longitudinal modes on the modal density of a shallow element is negligible as these modes occur only at widely spaced intervals over the frequency spectrum. All the expressions are strictly valid only for simply supported structural elements, however for large values of ω it is reasonable to suppose that asymptotic relations for modal density are relatively independent of the boundary conditions (Courant, 1953).

All the structural elements considered are discussed in detail (Wilkinson, 1967) and they are reproduced here for the graphical representation. The graphs for modal density versus frequency are plotted for various elements for specified dimensions and properties.

10.2 Sandwich Shells

The governing equation of motion of the shallow sandwich element of constant curvature is given as follows:

$$\left[\frac{h_1^4 \nabla^4}{1 - \mu^2 - \frac{h_1^2 \nabla^2}{S}} + \frac{h_1^2 \nabla_R^4}{\nabla^4} - \Omega^2 \right] w = 0 \tag{10.1}$$

where

$S \quad = \dfrac{Gh_1}{Eh_2}$

$\Omega^2 \quad = \dfrac{4\pi^2 (\rho_1 h_1 + \rho_2 h_2) \omega^2 h_1^2}{Eh_2}$

$\nabla^2 \quad = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

$\nabla_R^2 \quad = \dfrac{1}{R_2} \dfrac{\partial^2}{\partial x^2} + \dfrac{1}{R_1} \dfrac{\partial^2}{\partial y^2}$

h_1 = half thickness of core

h_2 = thickness of facing sheets

R_1, R_2 = radii of curvature

G = shear modulus of core

E = Young's modulus

ρ_1 = density of core

ρ_2 = density of facing sheets .

Equation (10.1) is based on the following assumptions:

(1) The facing layers have the same material properties, are of equal thickness, and are much thinner than the core. They carry only direct stress and have no flexural rigidity about their own middle surfaces.

(2) The core layer contributes negligibly to the moment resultants and membrane stress resultants of the composite shell. However resistance of the core to the transverse shear is quite considerable.

Now the frequency equation for a sandwich element in terms of the wave numbers is given as

$$\omega_{mn} = \frac{1}{2\pi}\left(\frac{Eh_2}{\rho_1 h_1 + \rho_2 h_2}\right)^{\frac{1}{2}} \times \left[\frac{h_1^2(k_1^2+k_2^2)^2}{1-\mu^2+h_1^2(k_1^2+k_2^2)/S} + \frac{(\chi k_1^2+k_2^2)^2}{R_1^2(k_1^2+k_2^2)^2}\right]^{\frac{1}{2}} \tag{10.2}$$

where

$$\chi = \frac{R_1}{R_2} \quad .$$

Using the transformation $k_1 = r\cos\theta$ and $k_2 = r\sin\theta$ and solving (10.2) for r, it gives

$$r^2_{max}(\omega,\theta) = \frac{1}{2Sh_1^2}\left[f_1 + [f_1^2 + 4(1-\mu^2)S^2 f_1]^{\frac{1}{2}}\right] \tag{10.3}$$

where

$$f_1(\omega,\theta) = \Omega^2 - h_1^2(\chi\cos^2\theta + \sin^2\theta)^2/R_1^2 \; .$$

Here it is assumed that $|\chi| \leq 1$.

Now the number of resonant modes is given as

$$N(\omega) = \frac{\ell_1\ell_2}{2\pi^2}\int_{\theta_1(\omega)}^{\theta_2(\omega)} r^2_{max}(\omega,\theta)\,d\theta \tag{10.4}$$

where the integration is to be carried out for values of θ over that part of the quadrant $0 < \theta \leq \frac{\pi}{2}$ in which the integrand is real and positive.

Hence from equation (10.4) the number of resonant modes is given by

$$N(\omega) \approx \frac{\ell_1 \ell_2}{4\pi^2 Sh_1^2} \int_{\theta_1(\omega)}^{\theta_2(\omega)} \{f_1 + [f_1^2 + 4(1-\mu^2)S^2 f_1]^{\frac{1}{2}}\}\, d\theta \,. \qquad (10.5)$$

Differentiating equation (10.5) with respect to ω gives

$$n(\omega) = \frac{\ell_1 \ell_2}{2\pi^2 Sh_1^2} \{ \frac{\Omega^2}{\omega} (\theta_2 - \theta_1) + \frac{\Omega^2}{\omega} \int_{\theta_1(\omega)}^{\theta_2(\omega)} \frac{[f_1 + 2(1-\mu^2)S^2]}{[f_1^2 + 4(1-\mu^2)S^2 f_1]^{\frac{1}{2}}} d\theta$$

$$+ Sh_1^2 [r_{max}^2(\omega, \theta_2) \frac{d\theta_2}{d\omega} - r_{max}^2(\omega, \theta_1) \frac{d\theta_1}{d\omega}]\} \,. \qquad (10.6)$$

Utilizing the transformation

$$y = \chi \cos^2\theta + \sin^2\theta \qquad (10.7)$$

equation (10.6) can be written as follows

$$n(\omega) = \frac{\ell_1 \ell_2}{2\pi^2 Sh_1^2} \{\frac{\Omega^2}{\omega} (\theta_2 - \theta_1) + \frac{\Omega^2}{\omega} \int_{y_1}^{y_2} \frac{((\frac{\omega}{\omega_s})^2 + \frac{c^2}{2-y^2})\, dy}{[(y-\chi)(1-y)((\frac{\omega}{\omega_s})^2 - y^2)(\frac{\omega_0^2}{\omega_s^2} - y^2)]^{\frac{1}{2}}}$$

(10.8)

$$+ Sh_1^2 [r_{max}^2(\omega, \theta_2) \frac{d\theta_2}{d\omega} - r_{max}^2(\omega, \theta_1) \frac{d\theta_1}{d\omega}] \}$$

where

$$\omega_s^2 = \frac{Eh_2}{[4\pi^2(\rho_1 h_1 + \rho_2 h_2) R_1^2]}$$

$$\omega_0^2 = \omega^2 + \omega_s^2 c^2$$

$$c^2 = 4(1-\mu^2) \frac{S^2 R_1^2}{h_1^2}$$

$$y_1 = \chi + (1-\chi)\sin^2\theta_1$$

and
$$y_2 = \chi + (1-\chi)\sin^2\theta_2 \ .$$

Equation (10.8) cannot be expressed immediately in a simpler form as it contains hyperelliptical integral. However it can be used to obtain the expressions for special cases.

For a spherical cap

$$R_1 = R_2, \qquad \chi = 1$$

and for a flat sandwich plate

$$R_1 = R_2 = \infty \ .$$

Hence the expressions for modal density for a spherical cap and flat sandwich plate are given by

$$n(\omega) = 0 \qquad\qquad \omega < \omega_s$$

$$n(\omega) \approx \left(\frac{\ell_1 \ell_2}{4\pi S h_1^2}\right)\left(\frac{\Omega^2}{\omega}\right)\{1 + [\rho_2 + 2(1-\mu^2)S^2][f_2^2 + 4(1-\mu^2)S^2 f_2]^{-\frac{1}{2}}\}$$

$$\text{for} \quad \omega > \omega_s \ . \tag{10.9}$$

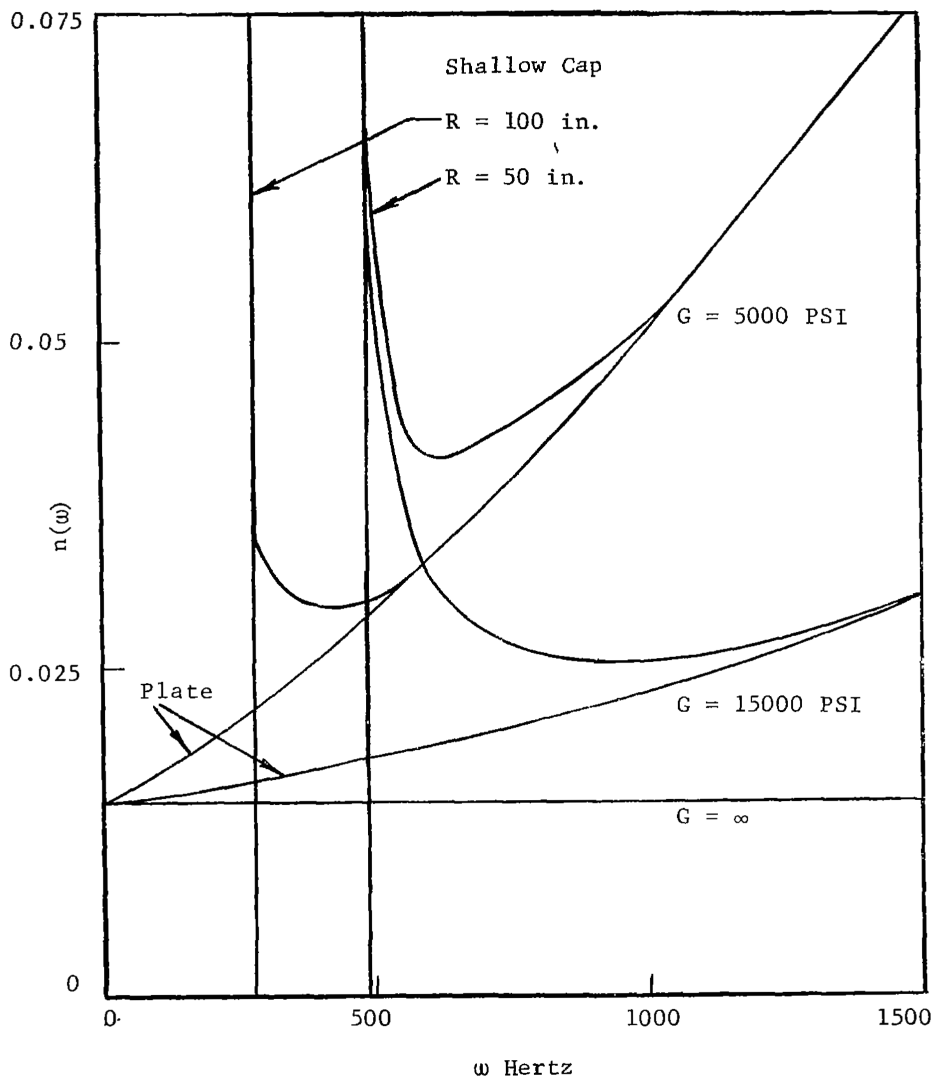


Figure 10.1 Modal density versus frequency for a sandwich spherical cap and flat sandwich plate

The variation of modal density versus frequency for spherical cap and a flat sandwich plate is illustrated in Figure 4.1 by considering the plate and spherical cap having the following material and geometric properties: $\ell_1 = 60$ in., $\ell_2 = 36$ in., $E = 10 \times 10^6$ PSI, $\mu = 0.3$, $h_1 = 0.5$ in., $h_2 = 0.02$ in., $\rho_1 = 5.5$ lb/cu ft, $\rho_2 = 170$ lb/cu ft.

10.3 Orthrotropic Plates

An orthrotropic plate is characterized by five elastic constants E_x, E_y, G_{xy}, μ_{xy}, μ_{yx} where E_x and E_y are Young's moduli in the x and y direction, G_{xy} is the shear modulus and μ_{xy} and μ_{yx} represent Poisson's ratio.

The bending stiffnesses of plate in x and y directions are given by

$$D_x = \frac{E_x h^2}{12\mu_0} \tag{10.10}$$

$$D_y = \frac{E_y h^2}{12\mu_0} \tag{10.11}$$

and

$$H = (G_{xy} h^3/6) + \left(\frac{\mu_{xy} E_x h^3}{12\mu_0}\right) . \tag{10.12}$$

The natural frequencies of free vibrations of a simply supported rectangular orthrotropic plate are given by

$$\omega_{mn} = \left(\frac{4\pi^2 \rho h}{g_c}\right)^{\frac{1}{2}} [k_1^4 D_x + 2Hk_1^2 k_2^2 + k_2^4 D_y]^{\frac{1}{2}} \tag{10.13}$$

where k_1 and k_2 are the wave numbers.

Setting

$$D_x^{\frac{1}{4}} k_1 = r \cos \theta \qquad (10.14)$$

and

$$D_y^{\frac{1}{4}} k_2 = r \sin \theta \qquad (10.15)$$

then the number of resonant modes are given by

$$N(\omega) \approx \frac{\ell_1 \ell_2}{\pi^2 (D_x D_y)^{1/4}} \int_{\theta_1(\omega)}^{\theta_2(\omega)} \int_0^{r_{max}} r dr d\theta \ . \qquad (10.16)$$

where ℓ_1 and ℓ_2 are the dimensions of the orthrotropic plate in x and y directions respectively.

The integration of equation (10.6) is carried out over the values of θ in the quadrant $0 \leq \theta \leq \frac{\pi}{2}$ for which the integrand is real and positive.

Combining equations (10.13), (10.14) and (10.15) and solving for r gives

$$r_{max}^2 = 2\pi\left(\frac{\rho h}{g_c}\right)^{\frac{1}{2}} \omega(1 - \gamma_1^2 \sin^2 2\theta)^{-\frac{1}{2}} \qquad (10.17)$$

where

$$2\gamma_1^2 = 1 - H(D_x D_y)^{-\frac{1}{2}} \ .$$

For most of the materials

$$0 < \gamma_1 < \frac{1}{2} \ .$$

Therefore equation (10.16) reduces to

$$N(\omega) \approx \frac{\ell_1 \ell_2}{\pi} \left(\frac{\rho h}{D_x g_c}\right)^{\frac{1}{2}} \left(\frac{D_x}{D_y}\right)^{\frac{1}{4}} \omega \int_0^{\frac{\pi}{2}} (1 - \gamma_1^2 \sin^2 2\theta)^{-\frac{1}{2}} d\theta \ . \qquad (10.18)$$

Expressing equation (10.18) in terms of the complete integral of the first kind, it becomes

$$N(\omega) \approx \frac{\ell_1 \ell_2}{\pi} \left(\frac{\rho h}{D_x g_c}\right)^{\frac{1}{2}} \left(\frac{D_x}{D_y}\right)^{\frac{1}{4}} F\left(\frac{\pi}{2}, \gamma_1\right) \qquad (10.19)$$

where

$$F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \theta)^{-\frac{1}{2}} d\theta \ .$$

Differentiating expression (10.19) with respect to ω gives

$$n(\omega) = \frac{\ell_1 \ell_2}{\pi} \left(\frac{\rho h}{D_x g_c}\right)^{\frac{1}{2}} \left(\frac{D_x}{D_y}\right)^{\frac{1}{4}} F\left(\frac{\pi}{2}, \gamma_1\right) \ . \qquad (10.20)$$

Equation (10.20) gives the expression for modal density of the orthrotropic plate.

10.4 Pretwisted Plates

A pretwisted plate has a middle surface Z defined by

$$Z = \phi xy$$

where ϕ is a pretwist constant and x and y are the Cartesian plate co-ordinates. The plate may be considered as hyperbolic paraboloidal shell. The plate is shallow if $\phi \ell_1$ and $\phi \ell_2$ are small in comparison with unity.

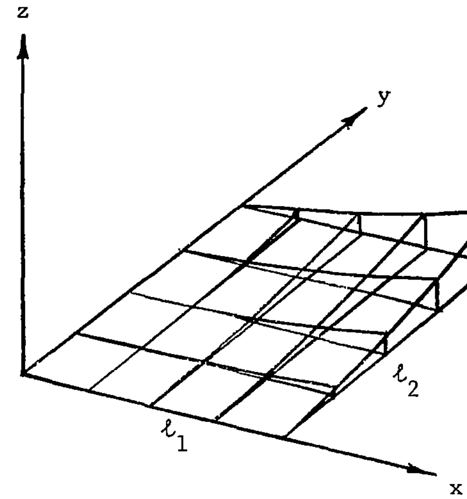


Figure 10.2 Pretwisted plate

The natural frequencies of a simply supported shallow pretwisted rectangular plate are given by

$$\omega_{mn} = \frac{1}{2\pi} \left(\frac{Eg_c}{\rho}\right)^{\frac{1}{2}} \left[\frac{h^2 (k_1^2 + k_2^2)^2}{12(1-\mu^2)} + \frac{4\phi^2 k_1^2 k_2^2}{(k_1^2+k_2^2)^2}\right] \tag{10.21}$$

where

k_1 and k_2 are the wave numbers

ϕ is a pretwist constant, and

h is a plate thickness.

Using the transformation

$$k_1 = r\cos\theta \qquad \text{and} \qquad k_2 = r\sin\theta \tag{10.22}$$

it can be found that

$$r_{max}^2 = 4\pi\sqrt{3}(1-\mu^2)^{\frac{1}{2}} \frac{\omega}{h} (\frac{\rho}{Eg_c})^{\frac{1}{2}} (1 - \beta_1^2 \sin^2 2\theta)^{\frac{1}{2}} \qquad (10.23)$$

where

$$\beta_1^2 = \frac{\phi^2 Eg_c}{4\pi^2 \rho\omega^2} .$$

Hence, as described in the previous section, the number of resonant modes is given as

$$N(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)^{\frac{1}{2}}}{\pi} \frac{\ell_1 \ell_2 \omega}{h} (\frac{\rho}{Eg_c})^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} (1 - \beta_1^2 \sin^2 2\theta)^{\frac{1}{2}} d\theta \qquad \beta_1^2 < 1 .$$

$$(10.24)$$

Changing equation (10.24) to the standard form of an elliptical integral as

$$N(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)^{\frac{1}{2}}}{\pi} \frac{\ell_1 \ell_2}{h} \omega (\frac{\rho}{Eg_c})^{\frac{1}{2}} E(\frac{\pi}{2} , \beta_1) \qquad (10.25)$$

where $E(\frac{\pi}{2} , k)$ is a complete elliptical integral of second kind and is expressed as $E(\frac{\pi}{2} , k) = \int_0^{\pi/2} (1 - k^2 \sin^2 2\theta)^{-\frac{1}{2}} d\theta$.

Differentiating (10.15) with respect to ω,

$$n(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)^{\frac{1}{2}}}{\pi} \frac{\ell_1 \ell_2}{h} (\frac{\rho}{Eg_c})^{\frac{1}{2}} F(\frac{\pi}{2} , \beta_1) \qquad \omega^2 > \frac{\phi^2 Eg_c}{4\pi^2 \rho} \qquad (10.26)$$

$$n(\omega) = 0 \qquad \omega^2 < \frac{\phi^2 Eg_c}{4\pi^2 \rho} . \qquad (10.27)$$

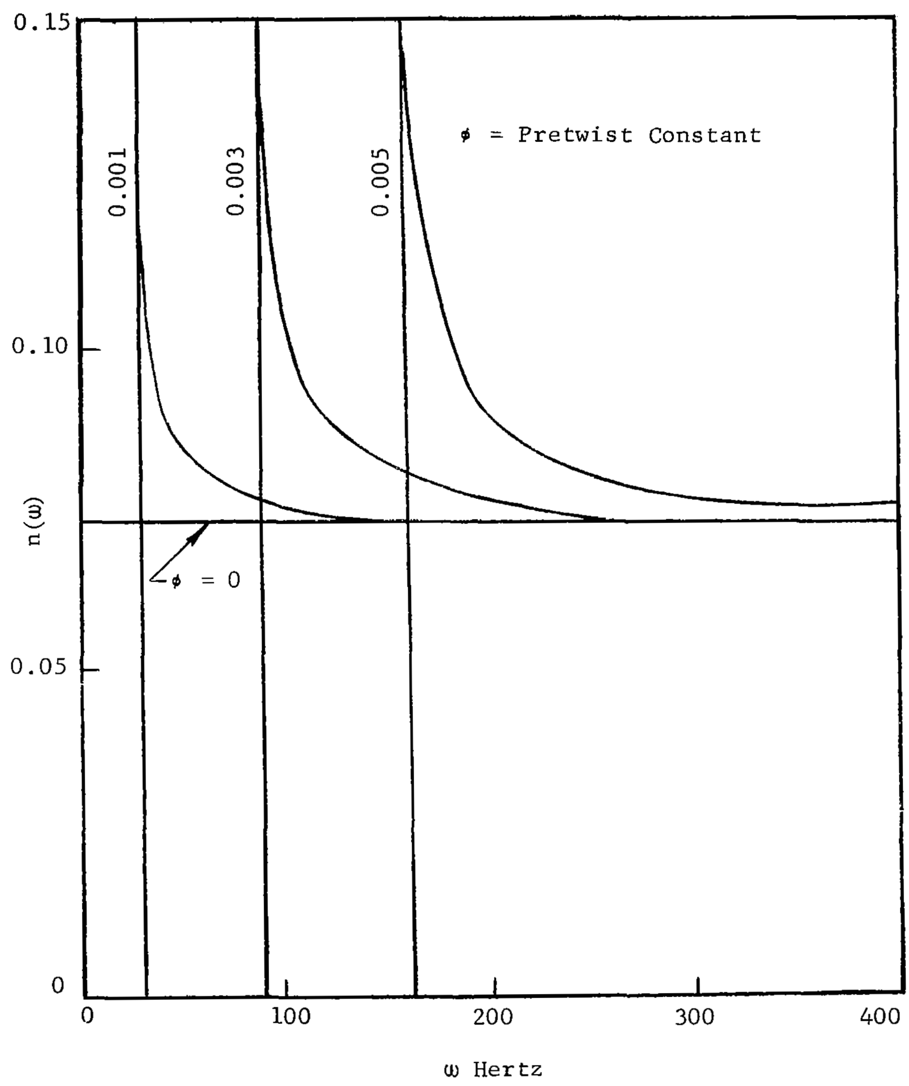


Figure 10.3 Modal density versus frequency for pretwisted plate with pretwist constant as parameter

The expressions (10.26) and (10.27) represent the modal density for the pretwisted plate.

The effect of pretwist on modal density is illustrated in Figure 10.4 by considering a plate of area 2160 sq in., having the following properties:

$$E = 10 \times 10^6 \text{ PSI}, \quad h = 0.25 \text{ in.}, \quad \mu = 0.3, \quad \rho = 170 \text{ lb/ft}^3.$$

10.5 Monocoque Plates under In-Plane Forces

The natural frequencies of monocoque plates under the action of the in-plane forces T_x and T_y (defined as positive in the outward direction) are given as follows:

$$\omega_{mn} = \frac{1}{2\pi} \left(\frac{Eg_c}{\rho}\right)^{\frac{1}{2}} \left[\frac{h^2(k_1^2 + k_2^2)^2}{12(1-\mu^2)} + \frac{T_y}{Eh} (\tau k_1^2 + k_2^2)\right]^{\frac{1}{2}} \tag{10.28}$$

where

$$\tau = \frac{T_x}{T_y}.$$

It is assumed that $|T_y| \geq |T_x|$.

Substituting the transformation $k_1 = r\cos\theta$ and $k_2 = r\sin\theta$ in equation (10.18) and solving for r, it gives

$$r_{max}^2 = 4\pi\sqrt{3}(1-\mu^2)^{\frac{1}{2}}\left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \frac{\omega}{h} \times \left(1 + \beta_2^2(\tau\cos^2\theta + \sin^2\theta)^2\right]^{\frac{1}{2}}$$

$$- \frac{6(1-\mu^2)T_y g_c}{Eh^3} (\tau\cos^2\theta + \sin^2\theta) \tag{10.29}$$

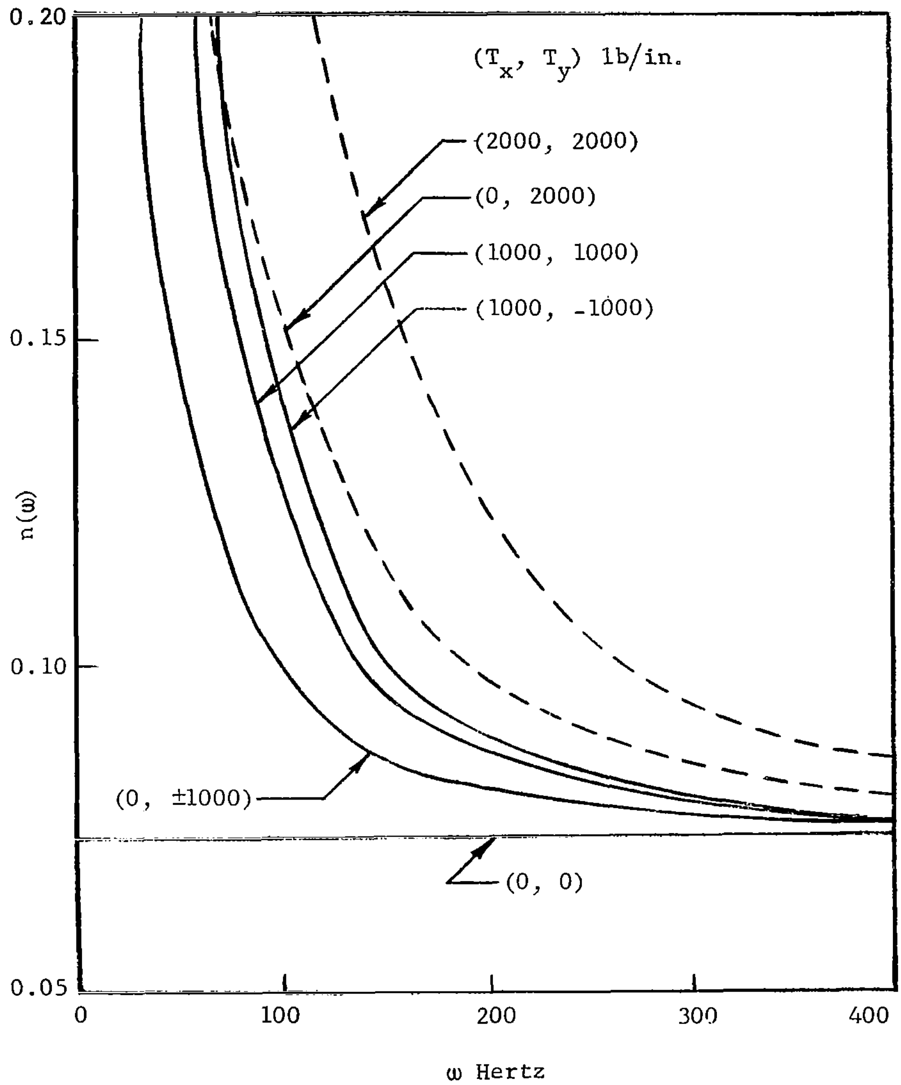


Figure 10.4 Modal density versus frequency for a rectangular plate under in-plane forces with in-plane forces as parameter

where

$$\beta_2^2 = \frac{3(1-\mu^2)T_y^2 g_c}{4\pi^2\omega^2\rho E h^4} \quad .$$

Again the number of resonant modes can be found as described in Section 10.2 and the expression is given as

$$N(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)^{\frac{1}{2}}}{\pi} \frac{\ell_1 \ell_2}{h} \left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \omega \int_0^{\frac{\pi}{2}} [1 + \beta_2^2(\tau \cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}} \, d\theta$$

$$- \frac{3\ell_1\ell_2}{4\pi E h^3} [(1-\mu^2) T_y (\tau+1)] \quad . \qquad (10.30)$$

Differentiating (10.30) with respect to ω,

$$n(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)^{\frac{1}{2}}}{\pi} \frac{\ell_1 \ell_2}{h} \left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} [1 + \beta_2^2(\tau \cos^2\theta + \sin^2\theta)^2]^{-\frac{1}{2}} \, d\theta .$$

(10.31)

Introducing the transformation

$$y = \tau \cos^2\theta + \sin^2\theta$$

the integral can be expressed in terms of the complete elliptic integral of the first kind, and consequently the modal density expression reduces to the form as shown:

$$n(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)^{\frac{1}{2}} \ell_1 \ell_2}{\pi h} \left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \frac{F(\frac{\pi}{2}, k)}{(\beta_2^2 + 1)^{1/4}(\beta_2^2\tau^2 + 1)^{1/4}} \qquad (10.32)$$

where

$$k^2 = \frac{\beta_2^2(1-\tau)^2 - [(\beta_2^2+1)^{\frac{1}{2}} - (\beta_2^2\tau^2+1)^{\frac{1}{2}}]^2}{4(\beta_2^2+1)^{\frac{1}{2}}(\beta_2^2\tau^2+1)^{\frac{1}{2}}} \quad .$$

The effects of the in-plane forces on the modal density of a rectangular plate is illustrated in Figure 10.3 by considering a plate of area 2160 sq in. and having the same material properties as the pretwisted plate of the previous section.

10.6 Monocoque Shells on an Elastic Foundation

When a rectangular isotropic monocoque shell lies on an elastic foundation of modulus K, its natural frequencies are approximately given by

$$\omega_{mn} = \frac{1}{2\pi(\frac{\rho h}{g_c})^{\frac{1}{2}}}\left[\frac{Eh^3(k_1^2+k_2^2)^2}{12(1-\mu^2)} + \frac{Eh(\chi k_1^2+k_2^2)^2}{R_1^2(k_1^2+k_2^2)^2} + K\right]^{\frac{1}{2}} \qquad (10.33)$$

where

$$\chi = \frac{R_1}{R_2} \quad .$$

It is assumed that $|R_1| \leq |R_2|$ or $|\chi| < 1$.

Expression (10.33) is similar to the expression obtained by Bolotin for the unsupported shell except the term K.

Hence the expression can be written as follows:

$$n(\omega) \approx \frac{2\sqrt{3}}{\pi} \frac{(1-\mu^2)^{\frac{1}{2}}}{(\omega^2-\omega_s^2)^{\frac{1}{2}}} \frac{\ell_1 \ell_2}{h} \left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \left\{ \omega \int_{\theta_1(\omega)}^{\theta_2(\omega)} \frac{d\theta}{f_3(\theta)} \right.$$

$$\left. + (\omega^2-\omega_s^2)\left[f_3(\theta_2) \frac{d\theta_2}{d\omega} - f_3(\theta_1) \frac{d\theta_1}{d\omega}\right] \right\} \quad \omega^2 > \omega_s^2 \qquad (10.34)$$

$$n(\omega) = 0 \qquad \omega^2 < \omega_s^2 \qquad (10.35)$$

where

$$f_3(\theta) = [1 - \beta_3^2(\chi\cos^2\theta + \sin^2\theta)^2]^{\frac{1}{2}}$$

$$\omega_s^2 = \frac{Kg_c}{4\pi^2 \rho h}$$

$$\beta_3^2 = E\left[\frac{4\pi^2 \rho R_1^2}{g_c} (\omega^2 - \omega_s^2)\right]^{-1} .$$

Depending on relative magnitudes of β_3, χ and ω, the integral in equation (10.34) has different values:

Considering only positive Gaussian curvature, there are then three subcases, each of which gives different modal densities within a certain frequency band. These expressions, derived by following the steps of Bolotin, are given as:

For $\omega < \omega_s^*$

$$n(\omega) = 0 . \qquad (10.36)$$

For $\omega_s^* < \omega < \omega^*$

$$n(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)}{\pi} \frac{\ell_1 \ell_2}{h} \left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \times \frac{\omega}{(\omega^2-\omega_s^2)^{\frac{1}{2}}} \times \frac{F(\frac{\pi}{2}, k)}{\beta_3^{\frac{1}{2}}(1-X)^{\frac{1}{2}}}$$

For $\omega^* < \omega$

$$n(\omega) \approx \frac{2\sqrt{3}(1-\mu^2)}{\pi} \frac{\ell_1 \ell_2}{h} \left(\frac{\rho}{Eg_c}\right)^{\frac{1}{2}} \times \frac{\omega}{(\omega^2-\omega_s^2)^{\frac{1}{2}}} \frac{F(\frac{\pi}{2}, \frac{1}{k})}{[(1+\beta_3)(1-\beta_3 X)]^{\frac{1}{2}}}$$

where

$$k^2 = (\beta_3 + 1)(1 - \beta_3 X)[2\beta_3(1-X)]^{-1}$$

$$\omega_s^* = [X^2 E(\frac{4\pi^2 \rho R_1^2}{g_c})^{-1} + \omega_s^2]^{\frac{1}{2}}$$

$$\omega^* = [E(\frac{4\pi^2 \rho R_1^2}{g_c})^{-1} + \omega_s^2]^{\frac{1}{2}} \quad .$$

The graph of modal density versus frequency is plotted with different elastic foundations for a cylindrical panel of area 2160 sq in. having the following properties: $E = 10 \times 10^6$ PSI, $\mu = 0.3$, $h = 0.25$ in., $\rho = 170$ lb/cu ft, $R_1 = 75$ in., $R_2 = \infty$.

10.7 Graphical Results and Discussion

Figure 10.1 shows that the modal density of a sandwich plate or spherical cap has a singularity at the frequency ω_s. Moreover when the shear modulus G of the core is large, the modal density of a plate approaches constant, which is the modal density of a monocoque plate of bending stiffness $[2Eh_1^2h_2 / (1-\mu^2)]$. The modal density of the spherical cap approaches asymptotically that of the plate as the frequency increases ($\omega >> \omega_s$). It can also be concluded that as ω becomes large, the modal density of any shallow shell approaches asymptotically that of the corresponding plate and above a certain frequency, the modal densities of all sandwich elements increase linearly with frequency.

As seen from expression (10.20) modal density for an orthrotropic plate is independent of the frequency ω, however it does depend on the geometry and material properties of the plate. For a plate having the same elastic properties in x and y co-ordinate direction expression for modal density is the same as that of the monocoque rectangular plate.

Figure 10.3 shows the effect of pretwist on the modal density of the plate. The pretwist introduces a singularity at the frequency $\omega = \frac{\phi}{2\pi} (\frac{Eg_c}{\rho})^{\frac{1}{2}}$ below which the modal density is zero. Above the frequency, the modal density asymptotically approaches that of an untwisted plate, which is constant.

The effect of in-plane forces on a modal density of a plate is illustrated in Figure 10.4. It shows that the introduction of in-plane forces produces a singularity in the modal density at zero frequency. Moreover the modal density of a monocoque plate with in-plane forces is greater than the modal density of the unloaded plate, regardless of the sign of forces, but the amount of increase is not simply related to the relative magnitude or sign of the forces. As seen from the graph, if both forces T_x and T_y in x and y co-ordinate directions are equal and of opposite sign, the modal density is higher than if they were of the same sign. In spite of loading the plate, as the frequency increases, the modal density of the loaded plate approaches asymptotically that of the unloaded plate.

For the monocoque shells of positive Gaussian curvature, placed on an elastic foundation, variation of the modal density is illustrated in Figure 10.5. It shows that the modal density is zero below the frequency ω_s^* at which it has a singularity. The modal density has second singularity at the frequency ω^*. However above ω^*, the modal density of the shell decreases monotonically and asymptotically approaches that of the corresponding plate as ω becomes very large. The foundation modulus k modifies the position of the singularities according to relative magnitudes of ω^* and ω_s^*.

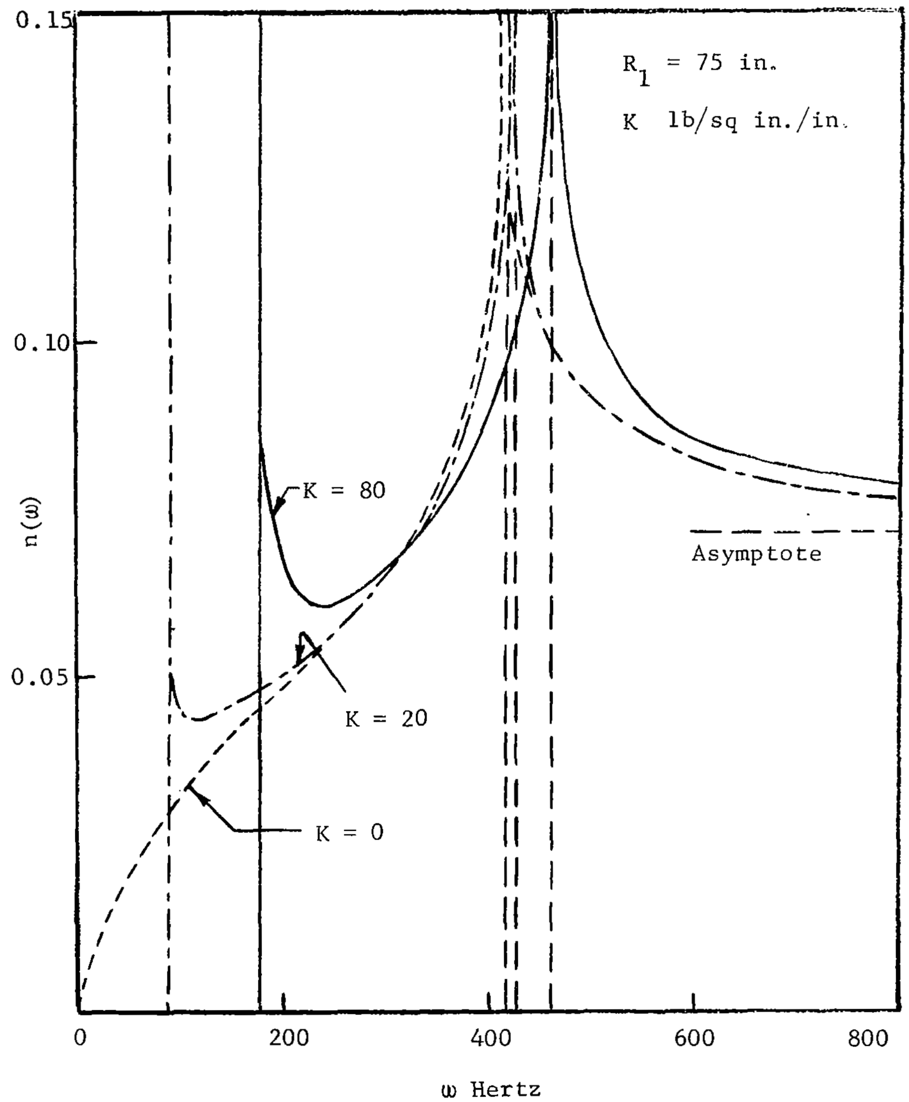


Figure 10.5 Modal density versus frequency for a cylindrical panel on an elastic foundation with modulus of elastic foundation as parameter

11. SUMMARY AND CONCLUSIONS

The modal density of a structural element is defined as the number of resonant modes within a unit frequency interval. In this thesis, the expressions and graphs are presented which can be used to estimate the average modal densities of structural elements like rods, beams, plates, thin cylindrical, spherical and conical shells, composite structures, shallow sandwich shells, orthrotropic plates, pretwisted plates, plates subject to in-plane forces and shells resting on elastic foundation. The expressions and graphs are valid for elements of arbitrary shape and having any prescribed boundary conditions.

In case of circular rods having uniform cross sections, modal density is independent of geometry of rod and frequency of vibrations for both longitudinal as well as torsional vibrations. However, modal density for torsional vibrations is about 1.5 times that of longitudinal vibrations.

The modal density of beams undergoing transverse vibrations depends both on geometry of beam and frequency of vibration. It decreases with the frequency and reaching asymptotically to zero value for large frequencies.

In case of solid flat rectangular and circular plates the modal density is constant for a given plate. Thus it is independent of frequency of vibration but not the geometry of the plate. For a given frequency, modal density of rectangular plate is approximately equal to that of the circular plate, both the plates having equal area, and thickness and of the same material.

Modal density of thin cylindrical shells decreases above the ring frequency, reaching a constant value asymptotically. However, below ring frequency there is a variation that is linear on a log-log plot.

The modal density for thin spherical shells has a singularity at the ring frequency, below which the modal density is zero. However, above ring frequency it decreases monotonically, approaching that of a plate asymptotically as frequency becomes large.

The thin conical shell shows that expressions presented for modal density are applicable over a wide range of the cone geometries and frequency ranges of practical interest.

The modal density of a composite structure, as derived analytically, is additive over its components.

For the shallow sandwich elements like the spherical cap modal density increases with increasing frequency, and for very large values of frequency modal density of the sandwich shells approaches asymptotically that of sandwich plates.

In case of the orthrotropic plates, modal density is frequency independent but it does depend on the geometry and material properties.

The modal density of pretwisted plate decreases monotonically after some value of frequency and asymptotically approaches that of the untwisted plate as frequency becomes large. By increasing the pretwist constant the value of frequency at which singularity occurs also increases.

The modal density of a monocoque plate with in-plane forces is greater than the modal density of an unloaded plate, regardless of the sign of forces and as the frequency increases, the modal density of the loaded plate approaches asymptotically that of the unloaded plate.

The modal density of shells on an elastic foundation decreases monotonically and asymptotically approaches that of the corresponding plate as frequency becomes large.

Thus in general modal density of all elements except plates and rods are frequency dependent. The modal density of the shells approaches asymptotically that of the corresponding plate. For the sandwich elements the modal density increases with increasing frequency, whereas the modal density of monocoque elements approaches a constant value.

The concept of modal density is very useful in solving the multimodal vibration problems and is of great value when the input force or excitation is random. In this type of analysis without the apparent knowledge of mode shapes and frequencies it is possible to give some insight into the response of the structures to the given excitation and some insight into the amount of energy which will be absorbed. Moreover modal density of structures is relatively independent of the boundary condition; it is a useful tool in estimating average response levels of multimodal vibration.

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13. APPENDIX. LIST OF SYMBOLS

a	radius of cylinder or circular plate, in.
a_1, a_2	principal dimensions of shell surface, in.
a_n	$n\pi / 1-\alpha_1$
C_L	longitudinal wave velocity $= \sqrt{\frac{Eg_c}{\rho}}$
C_T	torsional wave velocity $= \sqrt{\frac{Gg_c}{\rho}}$
D	stiffness modulus $= Eh^3 / (12(1-\mu^2)$
D_x, D_y	stiffness moduli in x and y co-ordinate direction
E	Young's modulus of elasticity
E_x, E_y	Young's moduli in x and y co-ordinate direction
G	shear modulus of elasticity
g_c	gravitational constant
h	thickness of shell wall or plate
h_1	half thickness of core in sandwich element
h_2	thickness of facing sheets in sandwich element
I	moment of inertia of a section
J_n	Bessel function of first kind of order n
K	modulus of elastic foundation
k	constant
k_1, k_2	wave number
ℓ	length of cylinder or beam
ℓ_1, ℓ_2	surface dimensions of the plate
L	length of cone, apex to base slant length

L_t	length of cone truncation, apex to top slant length
m	integer value or number of circumferential waves
$N(\omega)$	number of resonant modes
$N(\nu)$	number of resonant modes in terms of dimensionless frequency
$n(\omega)$	modal density
$n(\nu)$	modal density in terms of dimensionless frequency
n	integer value or number of one-half longitudinal waves
n_0	one-half circumferential modes
P_0	amplitude of excitation
R_1, R_2	radius of shell curvature
r	cylindrical co-ordinate
S	$\frac{Gh_1}{Eh_2}$
t	time
T_x, T_y	in-plane forces in x and y co-ordinate direction
V	radial velocity amplitude
V_a	axial velocity amplitude
V_t	tangential velocity amplitude
w	displacement normal to surface
X_1, X_2	generalized co-ordinates
X(x)	normal mode function
x, y	rectangular co-ordinates
Y_n	Bessel function of second kind of order n
α	stress coefficient
α_1	truncation ratio = L_T/L

β	$h/2\sqrt{3}a$
Δk_1	change in longitudinal wave number
Δk_2	change in circumferential wave number
θ	cylindrical co-ordinate
K	radius of gyration
μ	Poisson's ratio
ν	dimensionless frequency - $\frac{\omega a}{C_L} = \frac{\omega}{\Omega_R}$
π	3.14
ρ	density of material
ρ_1, ρ_2	density of core and facing sheets
σ	$m\pi a/\ell$
τ	$\frac{T_x}{T_y}$
ϕ	stress function or angular displacement
χ	R_1/R_2
ψ	one-half cone angle at apex
ω	angular frequency
$F(\frac{\pi}{2}, k)$	complete elliptic integral of first kind
$E(\frac{\pi}{2}, k)$	complete elliptic integral of second kind