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ADVANCED STRUCTURAL GEOMETRY STUDIES

Part I - Polyhedral Subdivision Concepts for Structural Applications

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for

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16. Abstract

A study leading to the formulation of computer-oriented mathematical models pertaining to methods of subdividing polyhedra into triangulated spherical space frames. The models perform the truncations and transformations of the polyhedral forms and calculate the geometrical properties of the generated space frames (spheres, hemispheres, and domes).

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FOREWORD

This final report was prepared by the School of Technology at Southern Illinois University, Carbondale, Illinois under NASA Contract NGR 14-008-002. The contract was administered by the NASA Office of Advanced Research and Technology.

Personnel participating in the research included: Julian H. Lauchner, principal investigator; Joseph D. Clinton, prime investigator; R. Buckminster Fuller, research consultant; Wayne Booth, Ann C. Garrison, Michael Keeling, Allen Kilty, Mark B. Mabee, and Richard M. Moeller, computer programmers.

PART I

POLYHEDRAL SUBDIVISION CONCEPTS FOR STRUCTURAL APPLICATIONS

Computer Software Management
and Information Center

The documentation and program developed for the advanced structural geometry studies will be made available to the public through COSMIC.

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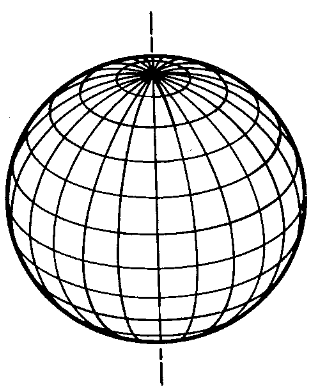
1.1 INTRODUCTION

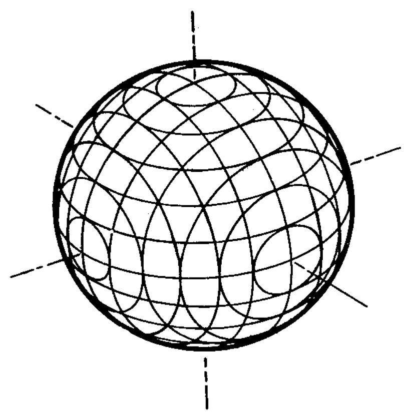
One of the most economical structural systems in contemporary use has been based on the spherical form. Designs for such structures are influenced primarily by the ultimate purpose of the structure, spacial environment to which it will be subjected, and the materials of fabrication.

Two basic systems are used for subdividing the spherical form for structural application: The bi-polar system and the multi-polar system. Figures 1.1, 1.2.



Bi-polar System

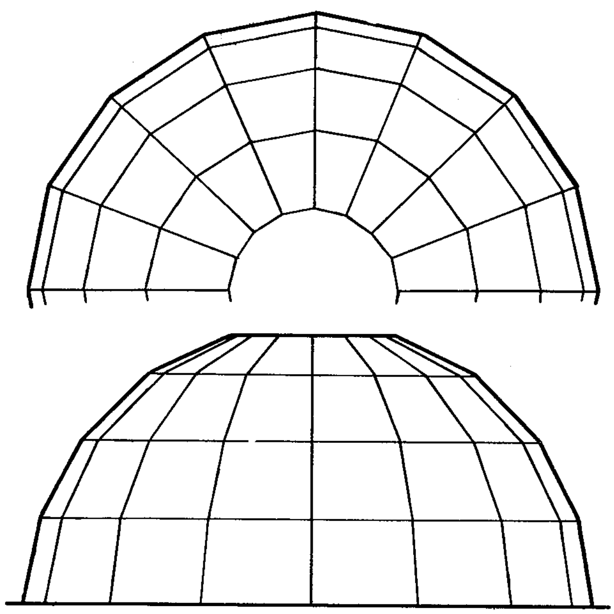
Figure 1.1



Multi-polar System

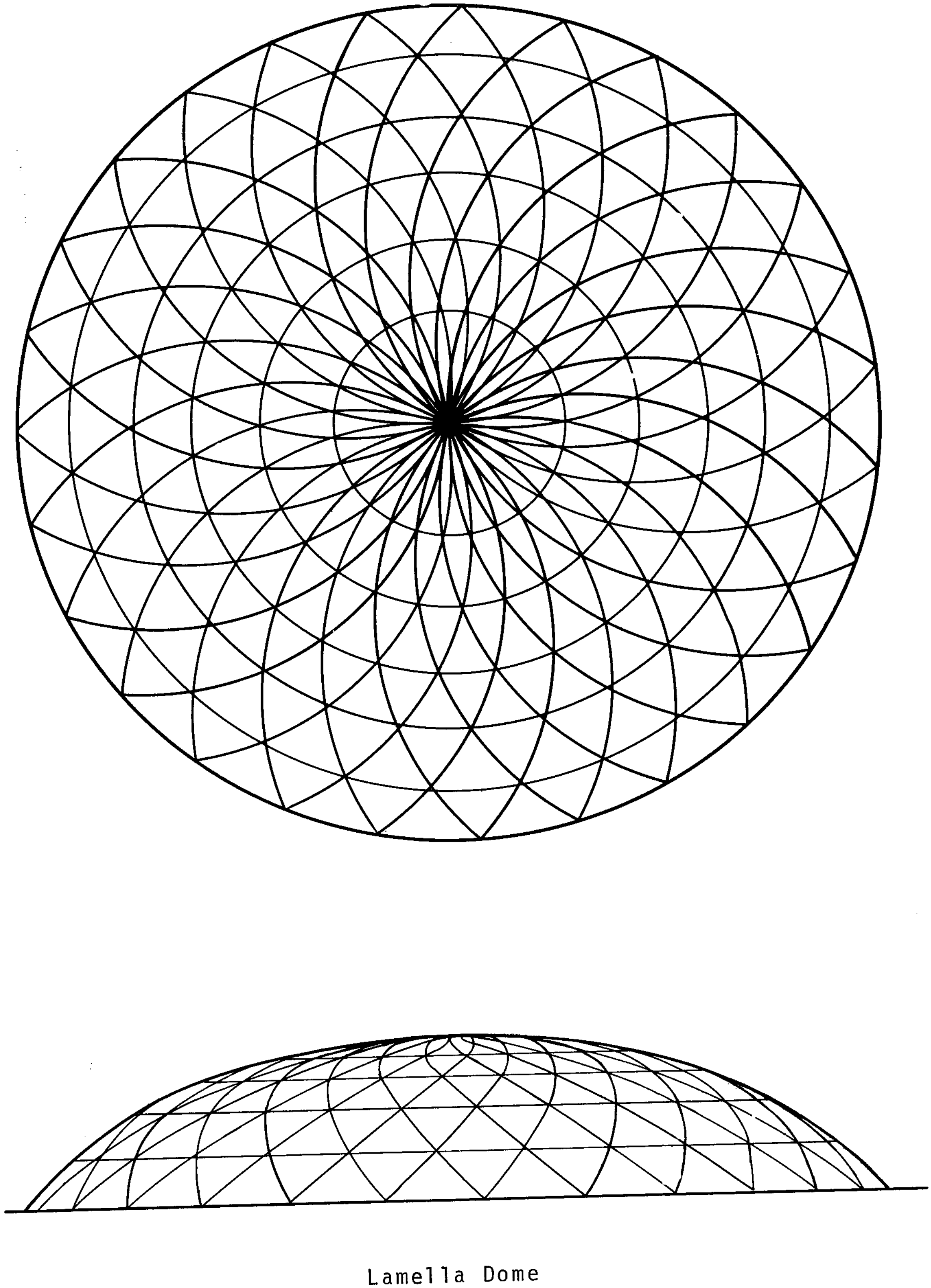
Figure 1.2

The bi-polar system is related to the familiar latitude-longitude approach to subdividing a sphere. Two common examples of this system are the ribbed dome (Figure 1.3) and the lamella dome (Figure 1.4).



Ribbed Dome

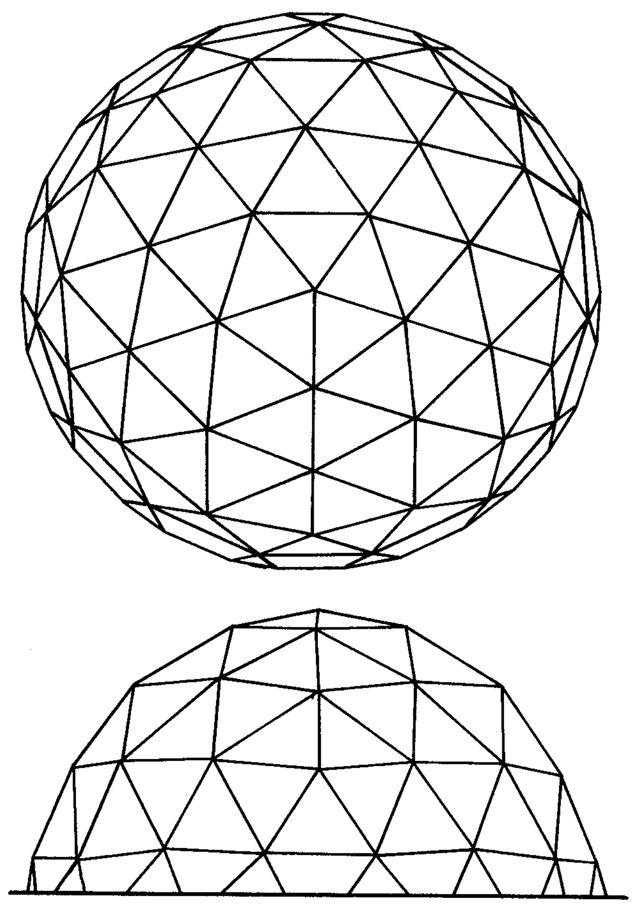
Figure 1.3



Lamella Dome

Figure 1.4

The multi-polar system is related to the spherical form of polyhedra. Perhaps the most familiar example of this system is the geodesic dome discoverd by R. Buckminster Fuller. Figure 1.5



Geodesic Dome

Figure 1.5

A typical design problem with the bi-polar system is with the geometrical relationships of the frame configuration. Computer aids in the design investigation may be initiated to handle the great number of variables in the determination of lengths, frequency of members, total number of joints, number of members intersecting at a single joint, relationship of members to each other, etc.

This portion of the report will concern itself with a mathematical and computer model for determining the geometrical properties of the multi-polar system for subdividing a spherical form for structural applications. The model has been limited to the three polyhedral forms made up completely of regular triangles: the tetrachedron, octahedron and icosahedron.

1.2 POLYHEDRON

Hoppe, in 1882, coined the word polytope: a geometrical figure bounded by portions of lines, planes, or hyperplanes; in two dimensions it is a polygon; in three a polyhedron.* However, the Greeks studied polyhedra over two thousand years ago with the findings of Euclid. Others such as Klein, Schiäfli and Coxeter, introduced much to the study and concepts of the polytope. In this section of the report, the structural configurations discussed are based on several of the polyhedral forms, specifically the regular (Platonic) polyhedra; the Tetrahedron, Octahedron, and Icosahedron.

In Euclid's writings, The Elements, explanation and definition is give to the five regular solids as known to the ancient world. The convex polyhedra are said to be regular if each have regular and equal faces, if they are congruent, and if they are of regular polyhedral angles. Table 1.1 list the properties of the Tetrahedron, Octahedron, and Icosahedron which are considered as three of the five regular polyhedral forms.

*Coxeter, N. S. M. 1

Table 1.1

Properties of the Basic Polyhedra
Tetrahedron, Octahedron, Icosahedron

Tetrahedron 3^3

V=4, F=4, E=6

Dihedral angle $\beta = 2 \sin \sqrt{3}/3 = 70° \ 31' \ 44''$

Angle subtended by an edge at center of polyhedron $\delta = \cos(-1/3) = 109°28'16''$

Vertices	$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ $(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$ $(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$ $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$	
Center to mid edge	$1/\sqrt{3}$	0.57735
center to center of face	$1/3$	0.33333
center to vertex	1	1
edge	$2\sqrt{2}/\sqrt{3}$	1.63299
mid edge to center of face	$\sqrt{2}/3$	0.47140
mid edge to vertex	$\sqrt{2}$	1.41421
mid edge to opposite mid edge	$2/\sqrt{3}$	1.15470
height (vertex to center of opposite face)	$4/3$	1.33333
area of face	$2/\sqrt{3}$	1.15470
volume	$\frac{8\sqrt{3}}{27}$	0.51320

Table 1.1 (cont.)

Octahedron 3^4

V=6, F=8, E=12

Dihedral Angle β= tan $2\sqrt{2}$ = 109° 28' 16"

Angle substended by
an edge at center
of polyhedron $\delta = 90°$

Vertices		$(\pm 1, 0, 0)$
		$(0, \pm 1, 0)$
		$(0, 0, \pm 1)$

edge	$\sqrt{2}$	1.41421
center to vertex	1	1
center to mid edge	$1/\sqrt{2}$	0.70711
center to center of face	$1/\sqrt{3}$	0.57735
Mid edge to near vertex	$\sqrt{3/2}$	1.22474
Mid edge to distant vertex	$\sqrt{5/2}$	1.58114
area of face	$\sqrt{3}/2$	0.86603
Volume	4/3	1.33333

Table 1.1 (cont.)

Icosahedron 3^5

V=12, F=20, E=30

Dihedral angle $\beta = \pi - (1 \sin^{-1})^{2/2}$ = 138°11'22"

Angle subtended by an edge at center of polyhedron $\delta = \cos(\sqrt{5}/5)$ = 63°26'05.818"

Vertices $\tau = \frac{1+\sqrt{5}}{2} = 1.61803$ $\left(0, \pm\frac{\sqrt{\tau}}{5^{1/4}}, \pm\frac{1}{5^{1/4}\sqrt{\tau}}\right)$

$\frac{1}{\tau} = \tau - 1$ $\left(\pm\frac{1}{5^{1/4}\sqrt{\tau}}, 0, \pm\frac{\sqrt{\tau}}{5^{1/4}}\right)$

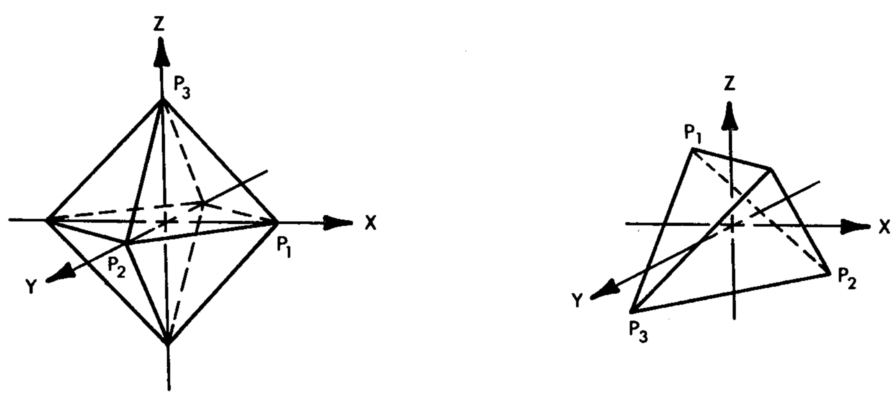
$\tau^2 = \tau + 1$

$\tau\sqrt{5} = \tau + 2$ $\left(\pm\frac{\sqrt{\tau}}{5^{1/4}}, \pm\frac{1}{5^{1/4}\sqrt{\tau}}, 0\right)$

edge	$2/5^{1/4}\sqrt{\tau}$	1.05146
center to vertex	1	1
center to mid edge	$\sqrt{\tau}/5^{1/4}$	0.85065
center to center of face	$\tau^{3/2}/5^{1/4}\sqrt{3}$	0.79465
area of face	$1/\tau\sqrt{3/5}$	0.47873
volume	$(4(5^{1/4})\sqrt{\tau})/3$	2.53615

1.3 STRUCTURAL ORIENTATION

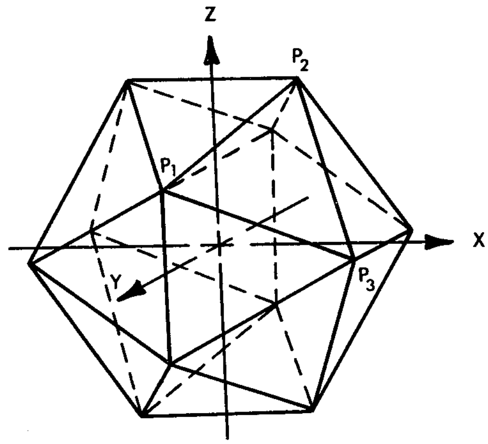
The structural configuration desired is acquired through a three-way gridding of the faces of the polyhedral form chosen from one of the three triangular faced regular polyhedra. The grid is then tr..nslated onto the surface of a circumscribed sphere. A three dimensional rectangular coordinate system was chosen for the basis of the computations. Due to the symmetries existing in the polyhedral forms only one face of the polyhedron is used in calculation of the geometrical properties of the structure. Figure 1.6 shows the orientation of the polyhedral form with respect to the x, y, z axis with the vertices of the face chosen for the geometrical computations. The intersection of the x, y, z axis is located at the origin (0,0,0) of the polyhedron, this point being the center of the circumscribed sphere. Table 1.2 list the coordinates of the vertices of the faces chosen as the PPT.



Octahedron

Tetrahedron

Figure 1.6



Icosahedron

Figure 1.6 (cont.)

Table 1.2

Coordinates of the Principal Polyhedral Triangles

Tetrahedron

$P_1 = (-1/\sqrt{3},\ -1/\sqrt{3},\ 1/\sqrt{3}) = (-.57735027, -.57735027,\ .57735027)$

$P_2 = (1/\sqrt{3},\ -1/\sqrt{3},\ -1/\sqrt{3}) = (.57735027, -.57735027, -.57735027)$

$P_3 = (-1/\sqrt{3},\ 1/\sqrt{3},\ -1/\sqrt{3}) = (-.57735027,\ .57735027, -.57735027)$

Octahedron

$P_1 = (1,\ 0,\ 0)$

$P_2 = (0,\ 1,\ 0)$

$P_3 = (0,\ 0,\ 1)$

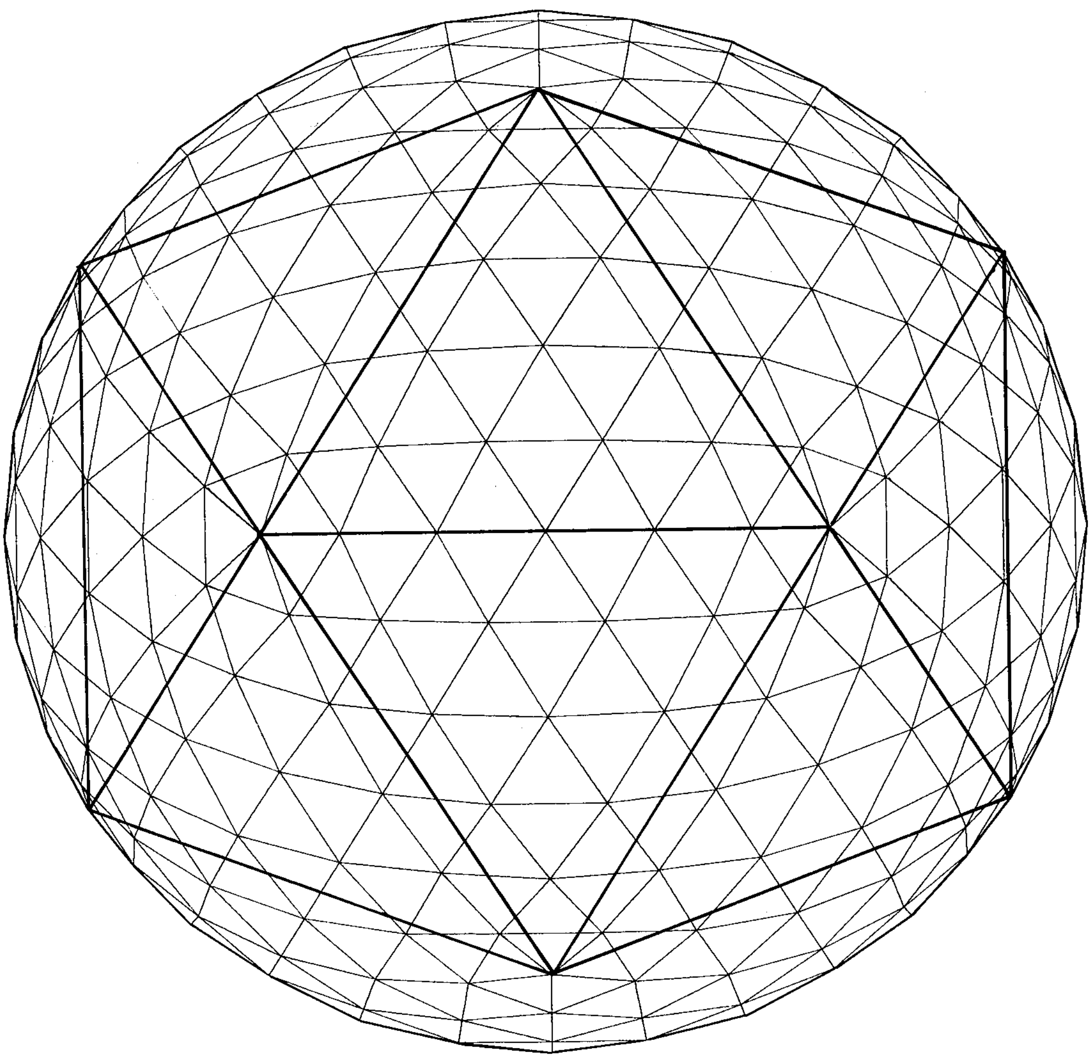
Icosahedron

$P_1 = (0, \sqrt{\tau}/5^{1/4},\ 1/5^{1/4}\sqrt{\tau}) = (0,\ .85065081,\ .52573111)$

$P_2 = (1/5^{1/4}\sqrt{\tau}\ 0,\ \sqrt{\tau}/5^{1/4}) - (.52573111,\ 0,\ .85065081)$

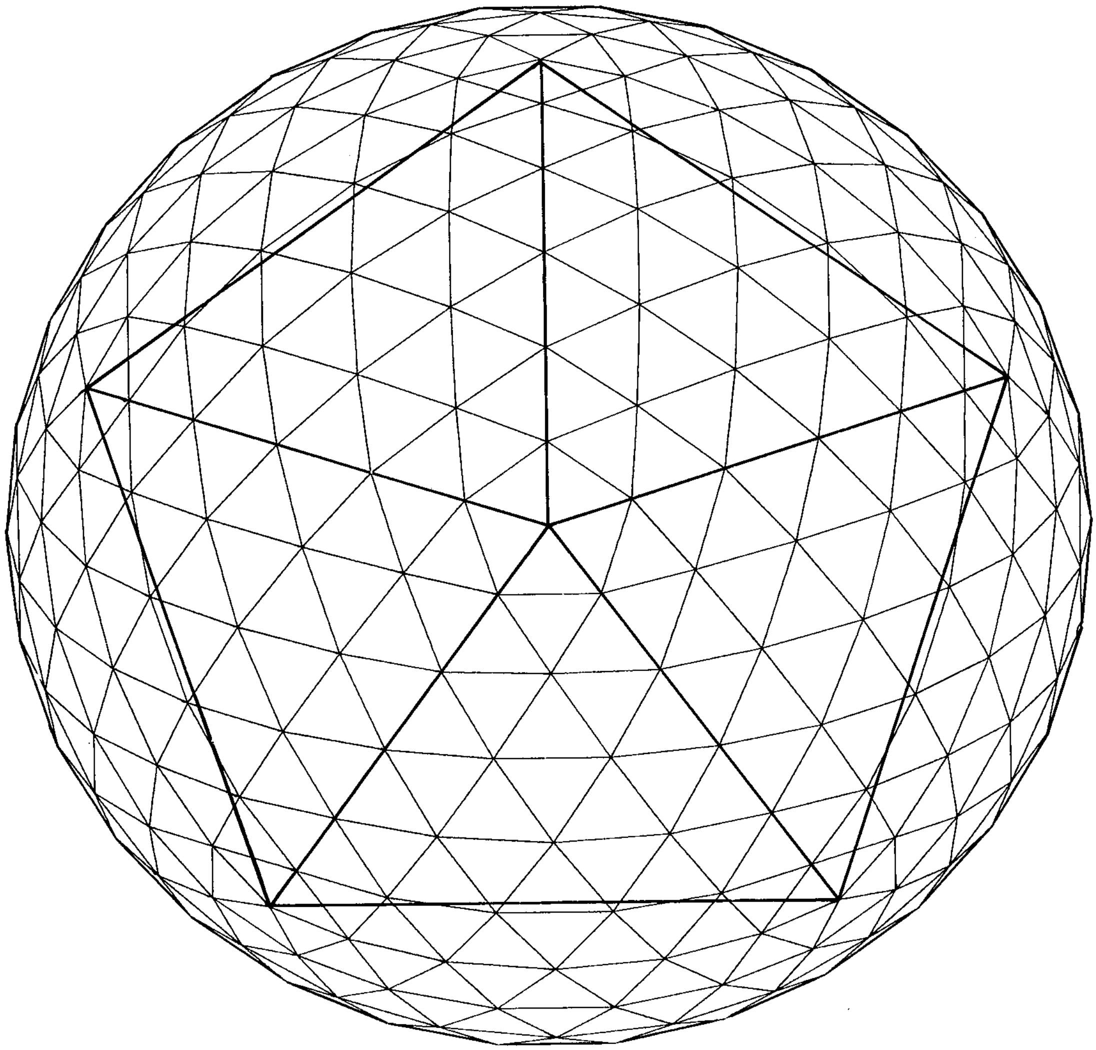
$P_3 = (\sqrt{\tau}/5^{1/4},\ 1/5^{1/4}\sqrt{\tau},\ 0) = (.85065081,\ .52573111,\ 0)$

Throughout the discussion of the methods of subdivision of the polyhedral forms the Icosahedron will be used with examples of computer maps of the spherical forms derived using the three traditional orientations: edge, vertex, and face. Figure 1.7.



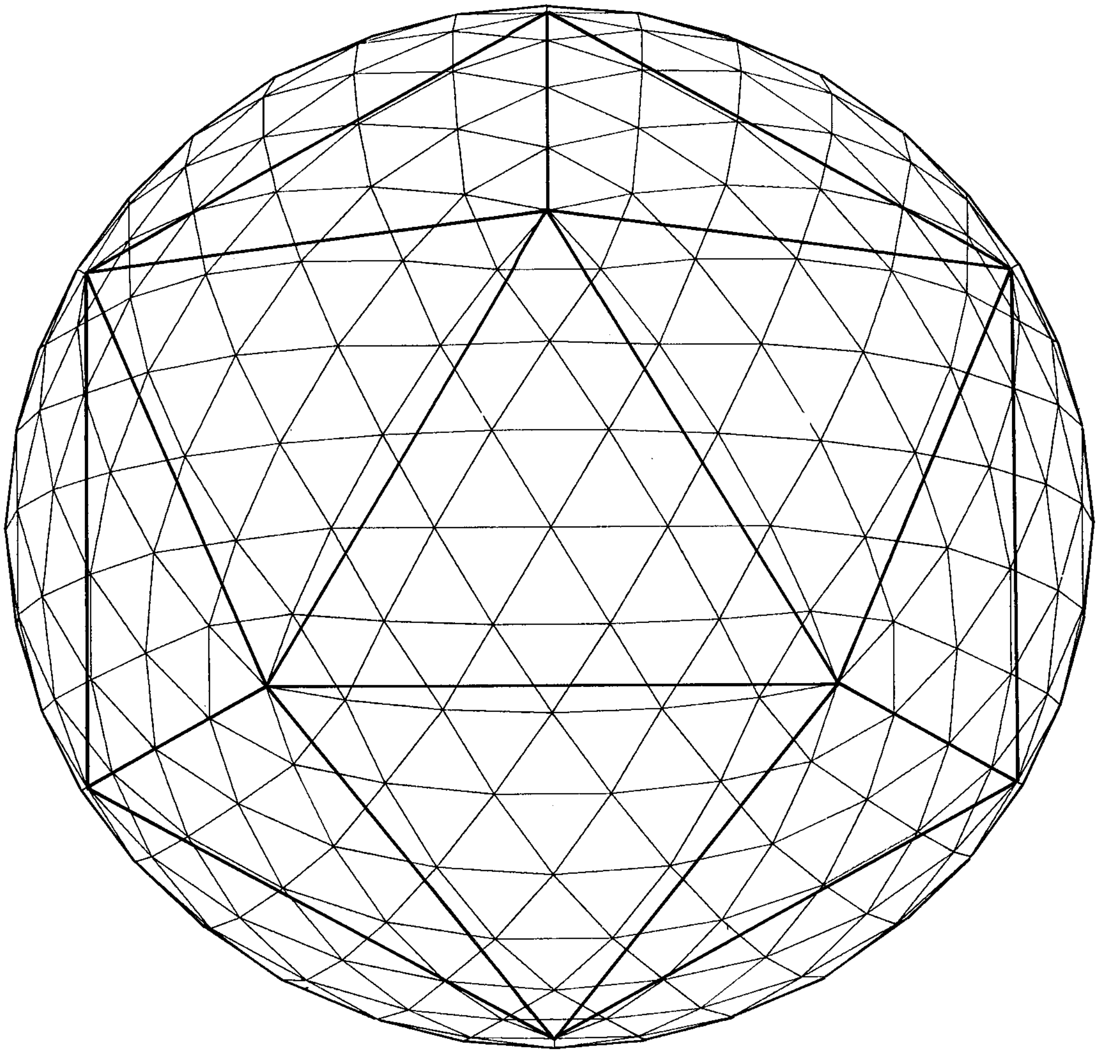
Edge Orientation

Figure 1.7



Vertex Orientation

Figure 1.7 (cont.)



Face Orientation

Figure 1.7
(cont.)

1.4 DEFINITIONS

AXIAL ANGLE (Ω) = an angle formed by an element and a radius from the center of the polyhedron meeting in a common point and the vertex of the axial angle sharing a vertex of the polyhedron.

CENTRAL ANGLE (δ) = an angle formed by two radii of the polyhedron passing through the end points of a principal side.

CHORD FACTOR (cf) = the element lengths calculated based upon a radius of a non-dimensional unit 1 for the spherical form. The length of any element for larger structures may be found by the equation:

$$cf \times r = l$$

where cf = chord factor

r = the radius of the desired structural forms

l = the length of the element sought

DIHEDRAL ANGLE (β) = an angle formed by two planes meeting in a common line. The planes themselves are the faces of the dihedral angle, and the common line is the element. To measure the dihedral angle measure the angle whose vertex is on the element of the dihedral angle and whose sides are perpendicular to the element and lie one in each face of the dihedral angle.

FACE ANGLE (α) - an angle formed by two elements meeting in a common point and lying in a plane that is one of the faces of the polyhedron.

FACES = the triangles making up the "exploded" structural form.

FREQUENCY (ν) = the number of parts or segments into which a principle side is subdivided.

PRINCIPLE POLYHEDRAL TRIANGLE (PPT) = any one of the equal equilaterial triangles which forms the face of the regular Polyhedron.

PRINCIPLE SIDE (PS) = any one of the three sides of the principle polyhedral triangle.

1.5 METHODS OF SUBDIVISION

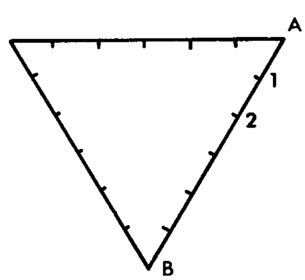
Upon using the spherical form as a structural unit, it is readily apparent that the basic polyhedral form, in its pure state, can not satisfy the range of conditions that must be geometrically and structurally met. Seven methods will be discussed for reducing the basic polyhedral form into a larger number of components from which the geometrical properties may be made to remain within the structural fabrication and erection limits for a desired configuration.

Due to the symmetrical characteristics of the basic polyhedral form only one face of the polyhedron is used for calculating the geometrical properties of the structural configuration. The remaining faces may be found by rotations or reflections of the principal polyhedral triangle and its transformations.

Attention is given here to the seven methods of subdividing the PPT in a broad sense and will be treated in detail in the following section.

Method 1:

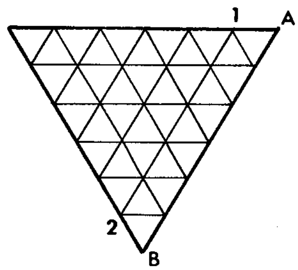
The PPT is subdivided into n frequency, with the parts chosen as equal divisions along the three principal sides. Figure 1.8



NOTE: $\overline{A1} = \overline{12}$

Figure 1.8

Each point of subdivision is then connected with a line segment parallel to their respective sides thereby giving a three-way grid so that a series of equilateral triangles are formed. Figure 1.9



Note: $\overline{AB}$ is parallel to $\overline{12}$

Figure 1.9

Each vertex on the PPT is then translated along a line passing through the origin (0,0,0) of the polyhedron and its respective vertex, onto the surface of the circumscribed sphere. The element connecting the translated vertices form the chords of a three-way great circular grid. Figure 1.10

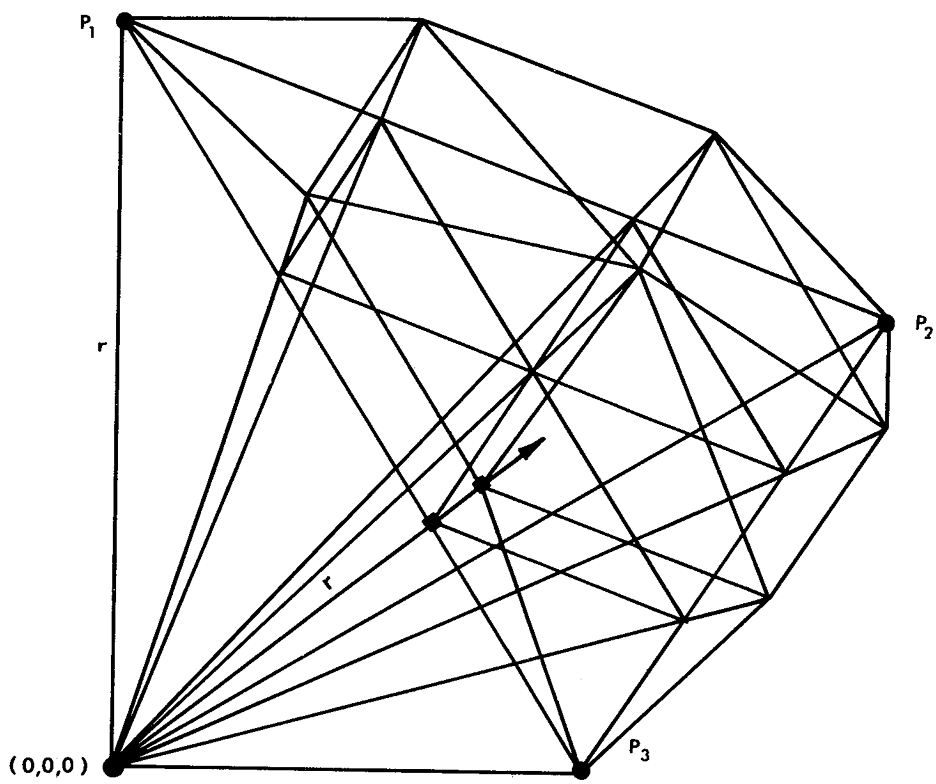
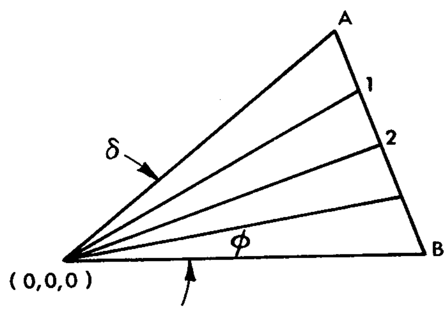


Figure 1.10

Methods 2 & 3:

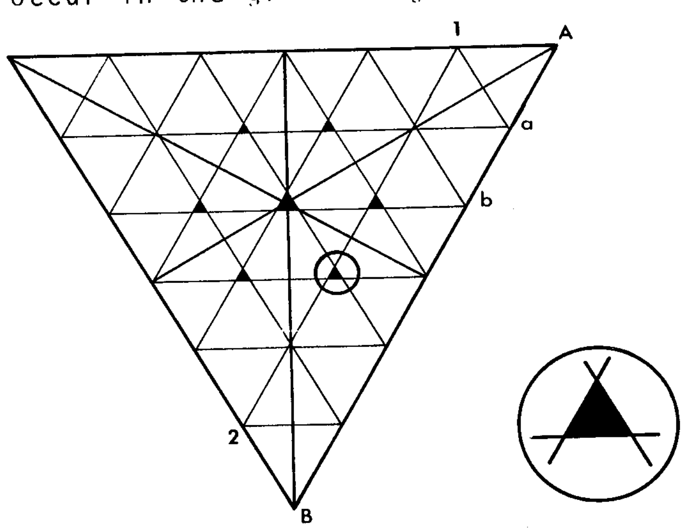
The PPT is subdivided into n frequency with the parts therein as equal arc divisions of the central angles of the polyhedron. Figure 1.11.



Note: $\overline{A1} \neq \overline{12}$

Figure 1.11

The points of subdivision on each principal side of the PPT are connected with line segments parallel to their respective sides. Each line segment intersects at a number of points which define a grid of subdivision. Due to the method of subdivision, small equilateral triangular "windows" occur in the grid. Figure 1.12.



NOTE: $\overline{AB}$ is parallel to $\overline{12}$

Aa ≠ ab

Windows are equilateral triangles

Figure 1.12

The center of these "windows" are found by one of two methods and are used as the vertices of the three-way grid for the PPT. They are then translated onto the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0,0,0) of the polyhedron. The element connecting the translated vertices form the chords of a three-way great circular grid. Figure 1.13.

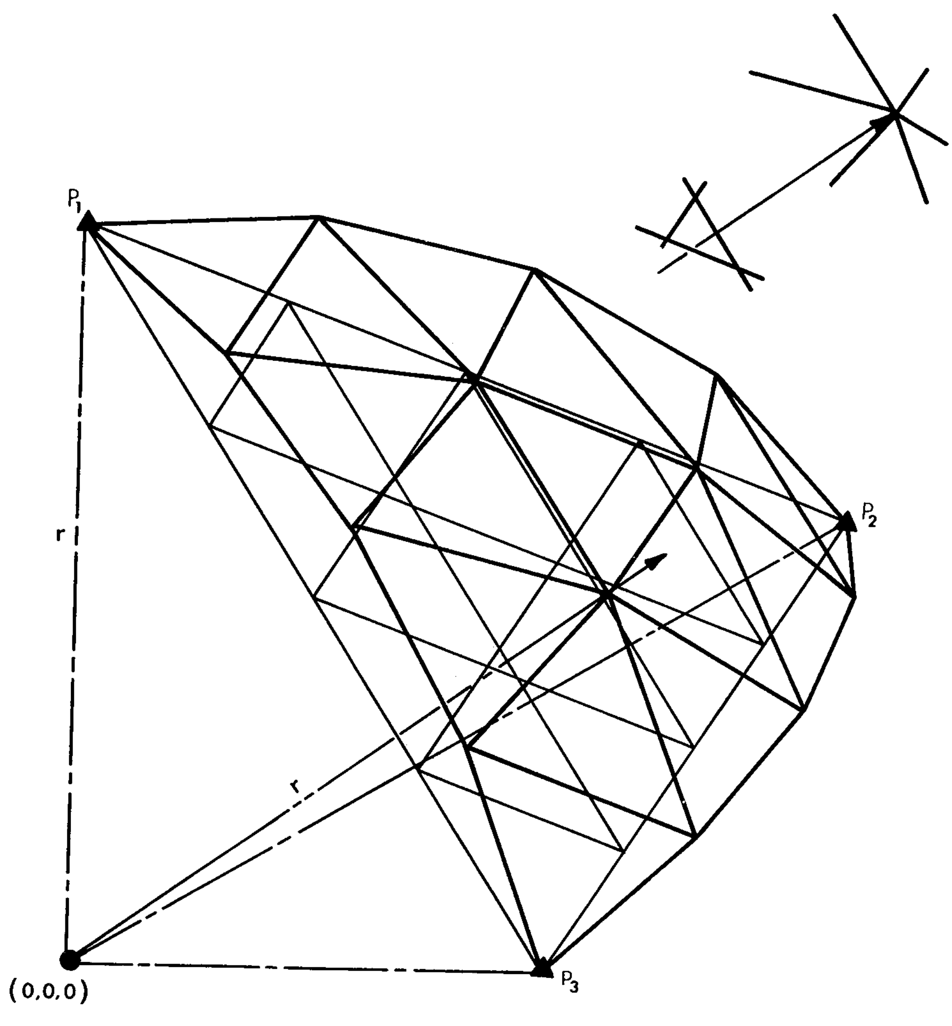
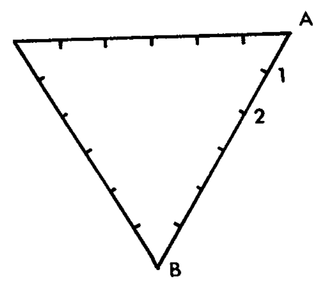


Figure 1.13

Method 4:

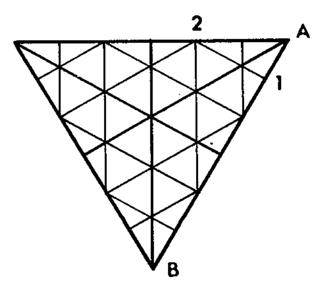
The PPT is subdivided into n frequency, with the parts chosen as equal divisions along the three principal sides. Figure 1.14.



Note: $\overline{A1} = \overline{12}$

Figure 1.14

Each point of subdivisions is then connected with line segments perpendicular to their respective principal side thus giving a three-way grid comprised of equilateral and right triangles. Figure 1.15.



Note: $\overline{AB} \perp \overline{12}$

Figure 1.15

Each vertex on the PPT is then translated onto the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0,0,0) of the polyhedron. The elements connecting the translated vertex form the chords of a three-way great circular grid. Figure 1.16

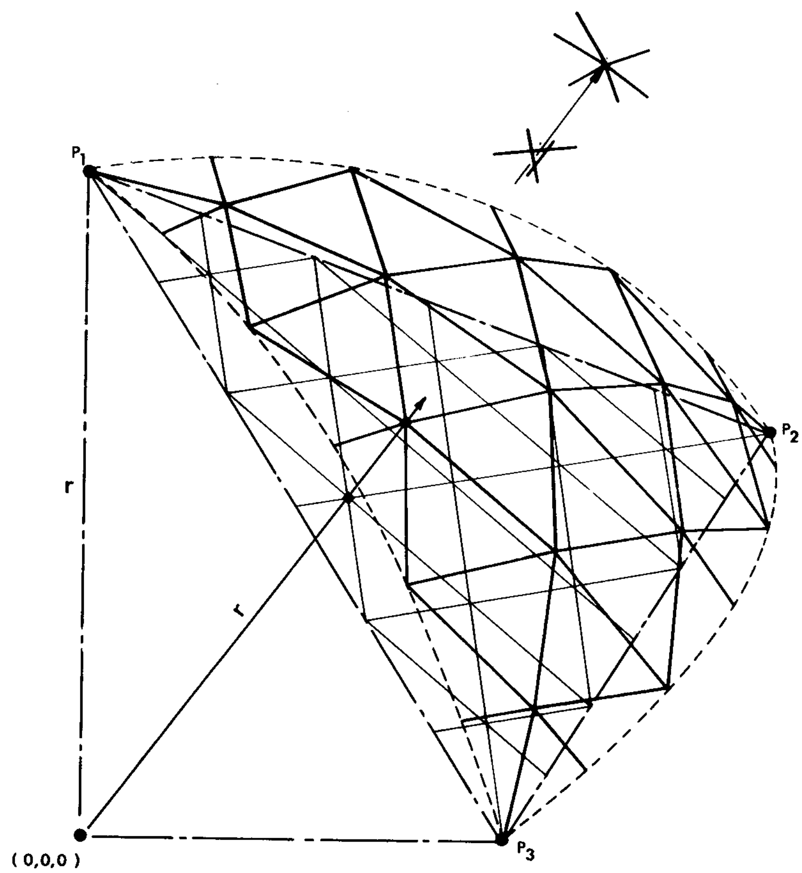
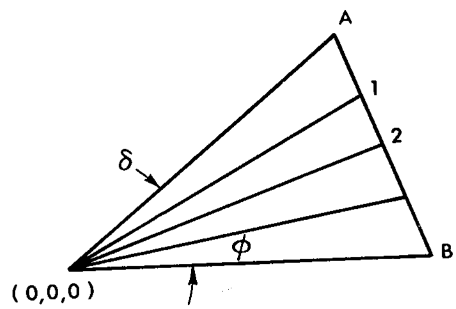


Figure 1.16

Method 5

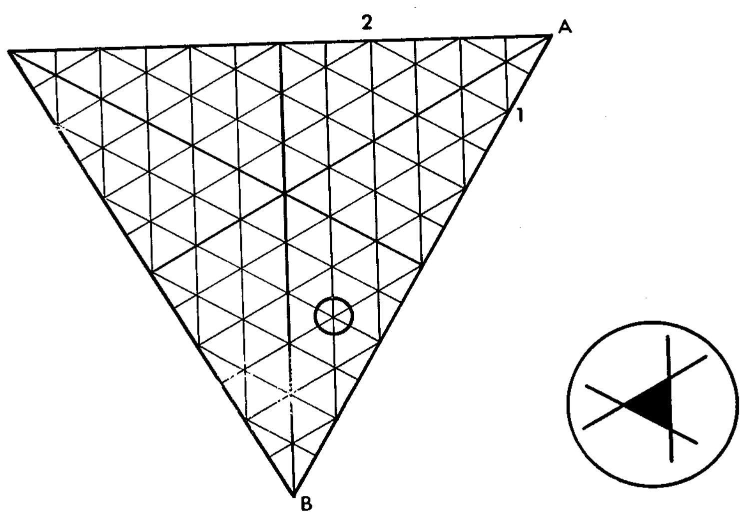
The PPT is subdivided into n frequency with the parts chosen as equal arc divisions of the central angle of the polyhedron. Figure 1.17.



NOTE: $\overline{A1} \neq \overline{12}$

Figure 1.17

The points of subdivision on each principal side of the PPT are connected with line segments similar to Method 4. However, the line segments are not perpendicular to their respective sides. Upon completion of the connections a grid is created. Due to the method of subdivision, small triangular "windows" occur in the grid. Figure 1.18.



NOTE: $\overline{AB} \not\angle \overline{12}$

Small triangular windows occur

Figure 1.18

The centers of these "windows" are found and are used as the vertices of a three-way grid for the PPT. The vertices are then translated onto the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0,0,0) of the polyhedron. The elements joining the translated vertices form the chords of a three-way great circle grid. Figure 1.19.

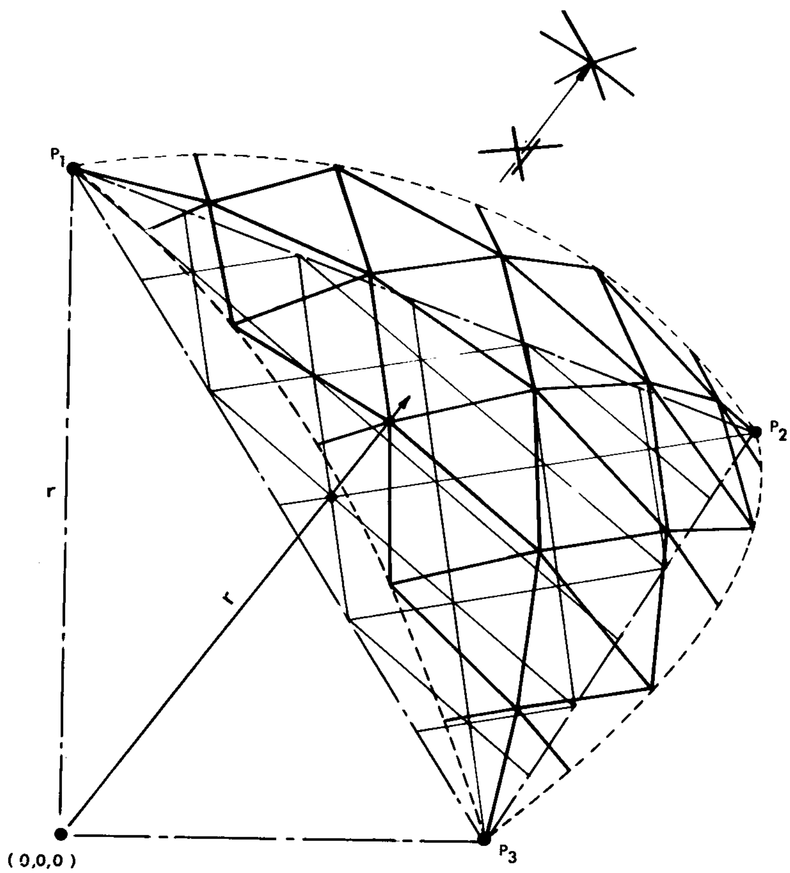
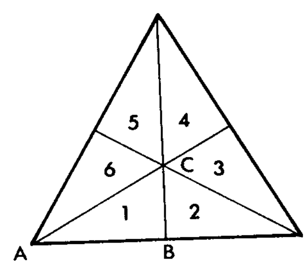


Figure 1.19

Method 6:

The PPT may be described as six right triangles each being a reflection or rotation of the other. Figure 1.20.



Note: ABC is a right triangle

Figure 1.20

In this method of subdivision we shall treat only triangle ABC. The remaining section of the PPT may be found through rotations and reflections of this basic unit. The Line AB is subdivided into parts chosen as equal arc divisions of the central angle of the polyhedron. Figure 1.21.

Note: $\overline{A1} \neq \overline{12}$

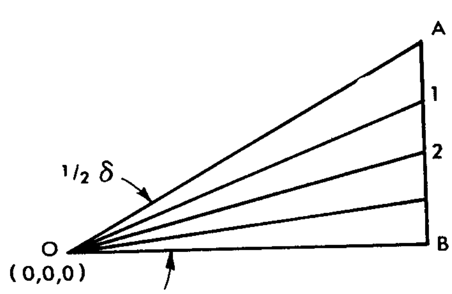
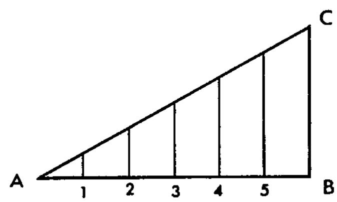


Figure 1.21

Once the subdivisions are found they are used to find the points of division on side $\overline{AC}$ and $\overline{CB}$. Perpendiculars through the points of division on side $\overline{AB}$ are extended to side $\overline{AC}$, this giving the points of subdivision on side $\overline{AC}$. Figure 1.22.



Note: $\overline{24} \perp \overline{AB}$

$\overline{A1} \neq \overline{23}$

Figure 1.22

The points of division on the side $\overline{CB}$ were formed by extending a line through the points of subdivision on side AC perpindicular to side $\overline{CB}$. Figure 1.23.

Note: $\overline{54} \perp \overline{CB}$

$\overline{12} \neq \overline{34}$

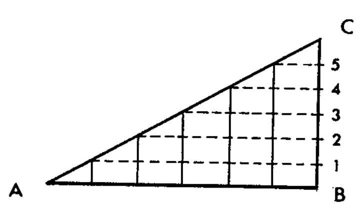


Figure 1.23

Having acquired the points of subdivision along the three sides of the triangle, diagonals are drawn from each point on side $\overline{AC}$ to alternate points of sides $\overline{AB}$ and $\overline{BC}$. Figure 1.24.

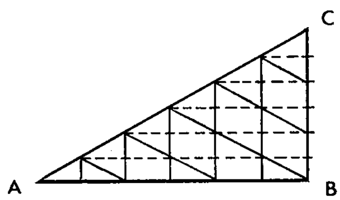


Figure 1.24

To complete the three-way grid connect alternate points of subdivision of side $\overline{AB}$ to alternate points of division of side $\overline{BC}$. Figure 1.25

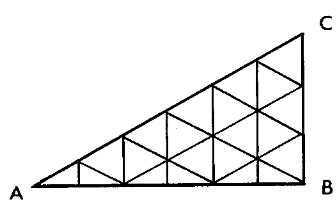


Figure 1.25

Through rotations and reflections of the basic unit and its subdivisions, the entire three-way gridding of the PPT may be found. Figure 1.26

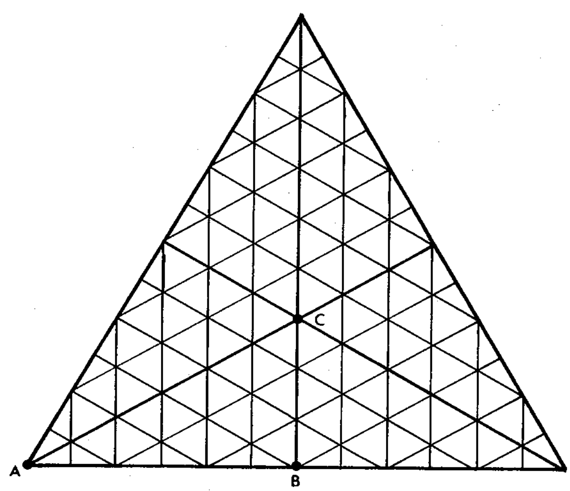


Figure 1.26

The vertices of the three-way grid are then translated to the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0,0,0) of the polyhedron. The element joining the translated vertices form the chords of a three-way great circle grid. Figure 1.27.

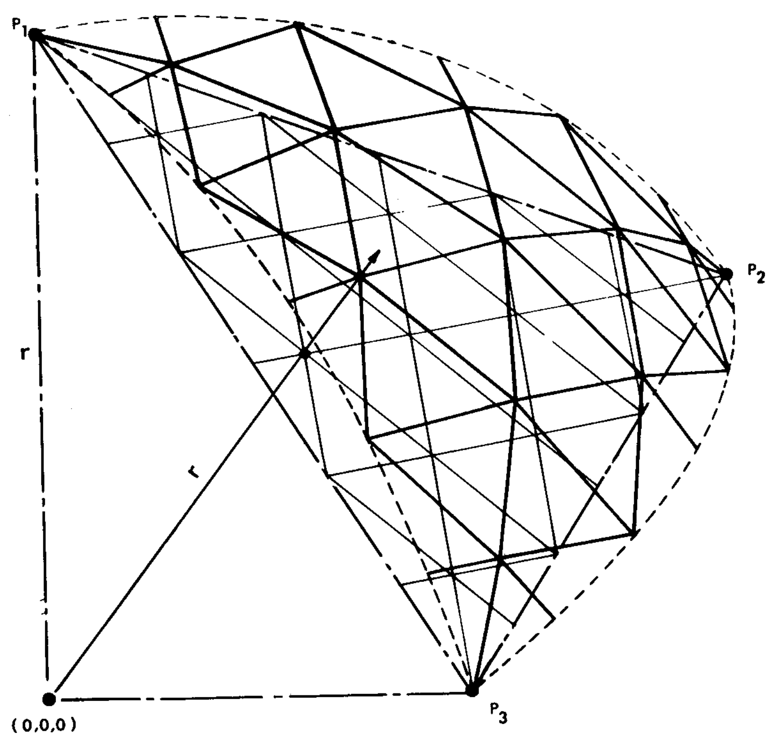
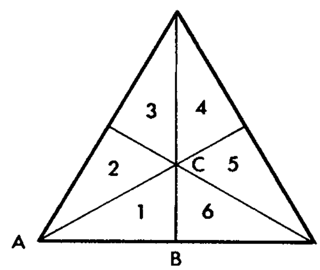


Figure 1.27

Method 7

The PPT is described as six right triangles each being a reflection or rotation of the other. Figure 1.28.



NOTE: ABC is a right triangle

Figure 1.28

In this method of subdivision only one of the right triangles will be treated as the basic unit for subdivision into a three-way grid. The line $\overline{AC}$ is subdivided into parts chosen as equal arc division of an angle made up of the triangle AC and the origin of the polyhedron with the origin (0,0,0) bieng the center of the triangle of subdivision. Figure 1.29

NOTE: $\overline{A1} \neq \overline{23}$

$\overline{OC} \perp \overline{AC}$

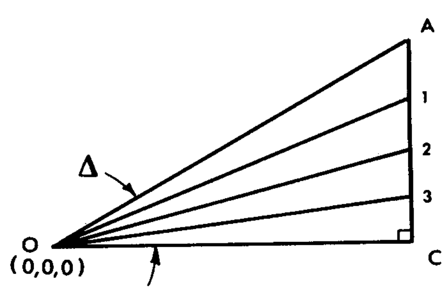
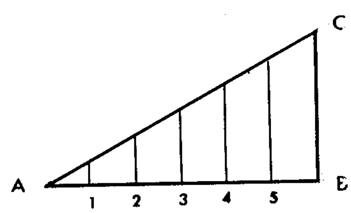


Figure 1.29

Once the subdivisions are found on line $\overline{AC}$ they are used to find the points of division on side $\overline{AB}$ and $\overline{CB}$. The lines through the points of sibudvision on side $\overline{AC}$ are taken perpendicular to side $\overline{AB}$, this giving the points of division on side $\overline{AB}$. Figure 1.30.

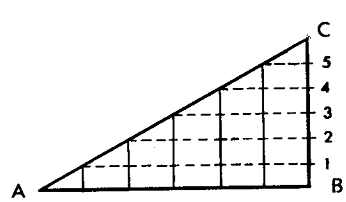


Note: $\overline{25} \perp \overline{AB}$

$\overline{12} \neq \overline{34}$

Figure 1.30

The points of division on $\overline{CB}$ are found by extending a line through the points of subdivision on $\overline{AC}$ perpendicular to $\overline{CB}$. Figure 1.31.



Note: $\overline{54} \perp \overline{CB}$

$\overline{12} \neq \overline{54}$

Figure 1.31

Having acquired the points of subdivision along the three sides of the triangle, diagonals are drawn from each point on $\overline{AC}$ to alternate points on $\overline{AB}$ and $\overline{BC}$. Figure 1.32.

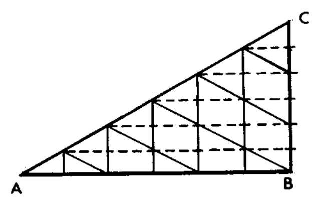


Figure 1.32

To complete the three-way grid connect alternate points on $\overline{AB}$ to alternate points on $\overline{BC}$. Figure 1.33.

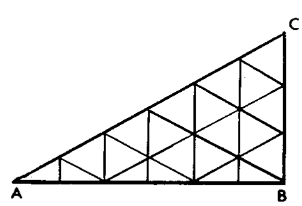


Figure 1.33

Through rotations and reflections of the basic unit found and it's subdivison, the entire three-way grid of the PPT may be found. Figure 1.34.

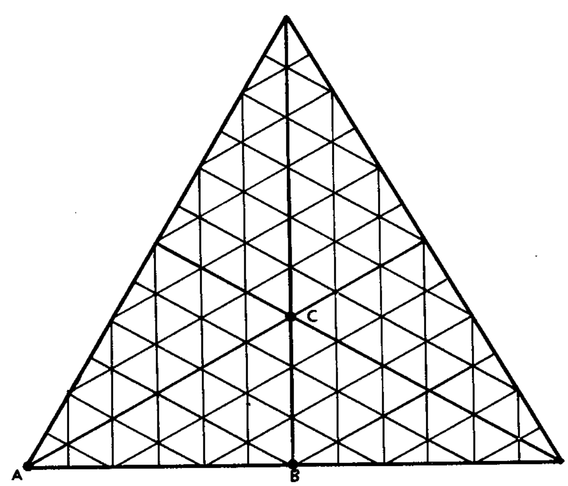


Figure 1.34

The vertices of the three-way grid are then translated to the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0,0,0) of the polyhedron. The elements joining the translated vertices form the chords of a three-way great circle grid. Figure 1.35.

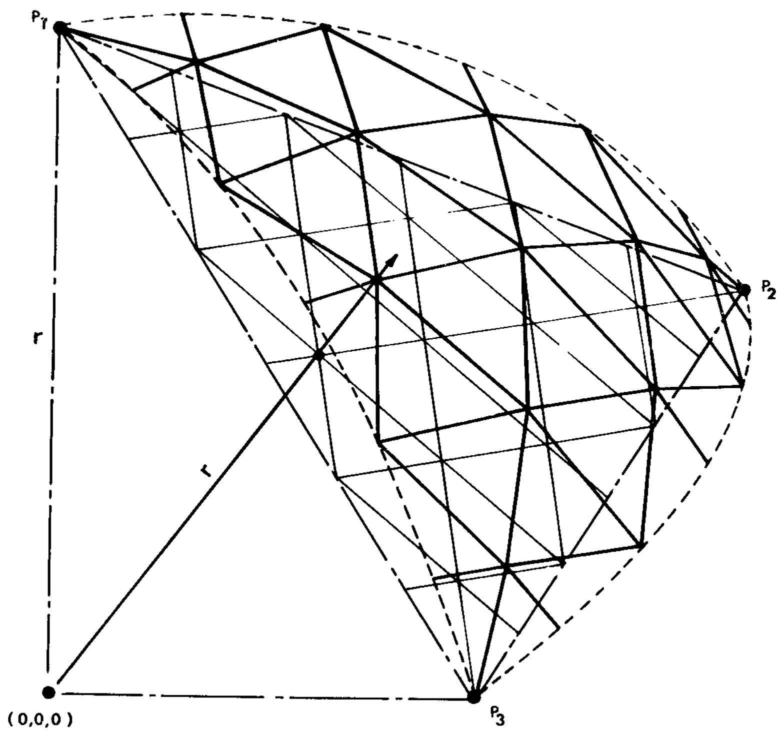


Figure 1.35

1.6 METHOD 1

The mathematical and computer model was developed for subdividing a tetrahedron, octahedron, or an icosahedron circumscribed by a unit sphere. The icosahedron was chosen as an example to illustrate the geometry of the model.

The icosahedron is oriented in a three dimensional rectangular coordinate system so that the vertices of one PPT are:

$$(X_1, Y_1, Z_1) = \left(0, \frac{\sqrt{\tau}}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}\sqrt{\tau}}\right)$$

$$\simeq (0, .850651, .525731)$$

$$(X_2, Y_2, Z_2) = \left(\frac{1}{\sqrt[4]{5}\sqrt{\tau}}, 0, \frac{\sqrt{\tau}}{\sqrt[4]{5}}\right)$$

$$\simeq (.525731, 0, .850651)$$

$$(X_3, Y_3, Z_3) = \left(\frac{\sqrt{\tau}}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}\sqrt{\tau}}, 0\right)$$

$$\simeq (.850651, .525731, 0)$$

$$\text{where } \tau = \frac{1+\sqrt{5}}{2}$$

with the intersection of the axis X, Y, Z located at the origin (0,0,0) of the icosahedron. Figure 1.36.

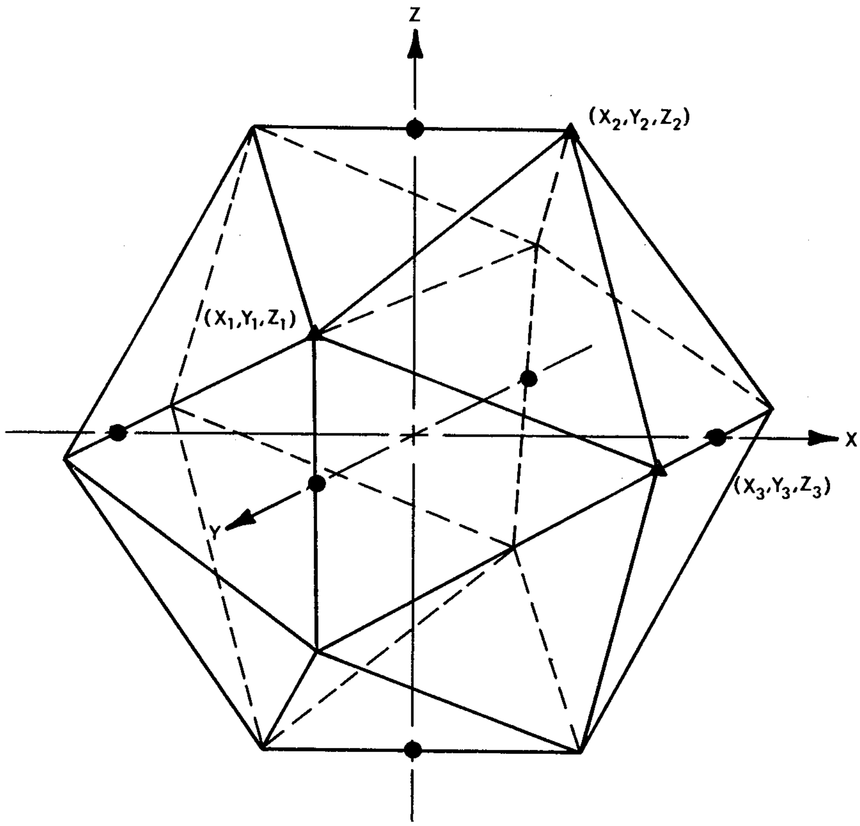


Figure 1.36

This PPT is divided into smaller equilateral triangles where the vertices of the triangles are of the form

$$\left(X_1 + I\,\frac{X_2 - X_1}{N} + J\,\frac{X_3 - X_2}{N} \quad Y_1 + I\,\frac{Y_2 - Y_1}{N} + J\,\frac{Y_3 - Y_2}{N} \right.$$

$$\left. Z_1 + I\,\frac{Z_2 - Z_1}{N} + J\,\frac{Z_3 - Z_2}{N} \right) \qquad 1.1$$

where N is the frequency of the structure and I and J are integers such that $0 \leq J \leq I \leq N$. The values of I and J are unique for each vertex and are used to identify each vertex as shown in Figure 1.37.

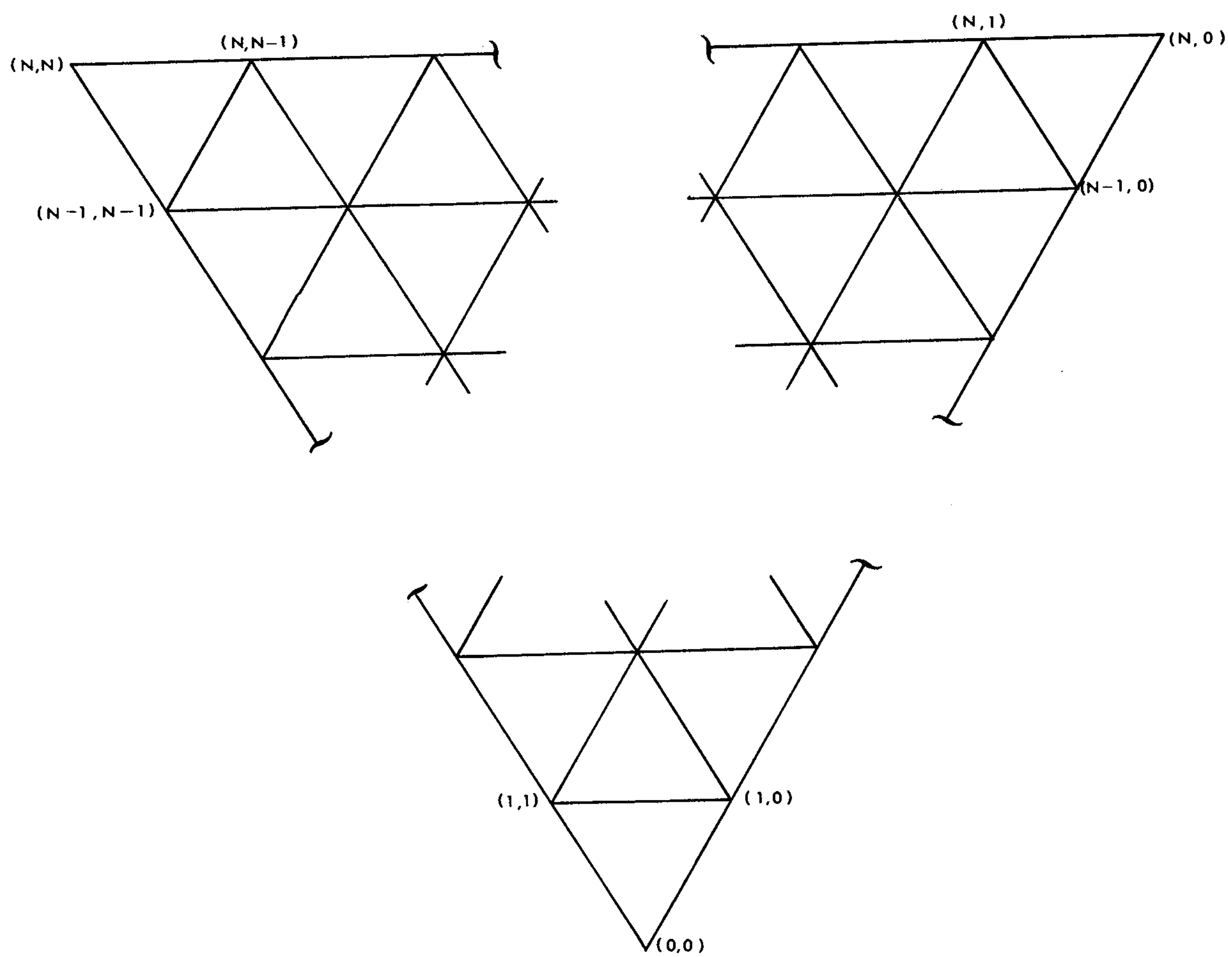


Figure 1.37

To find the projection of each vertex of the PPT onto the unit sphere along a line segment through the vertex of the PPT and the origin each coordinate of each vertex, PPT, is divided by the distance between the vertex PPT and the origin. Figure 1.38.

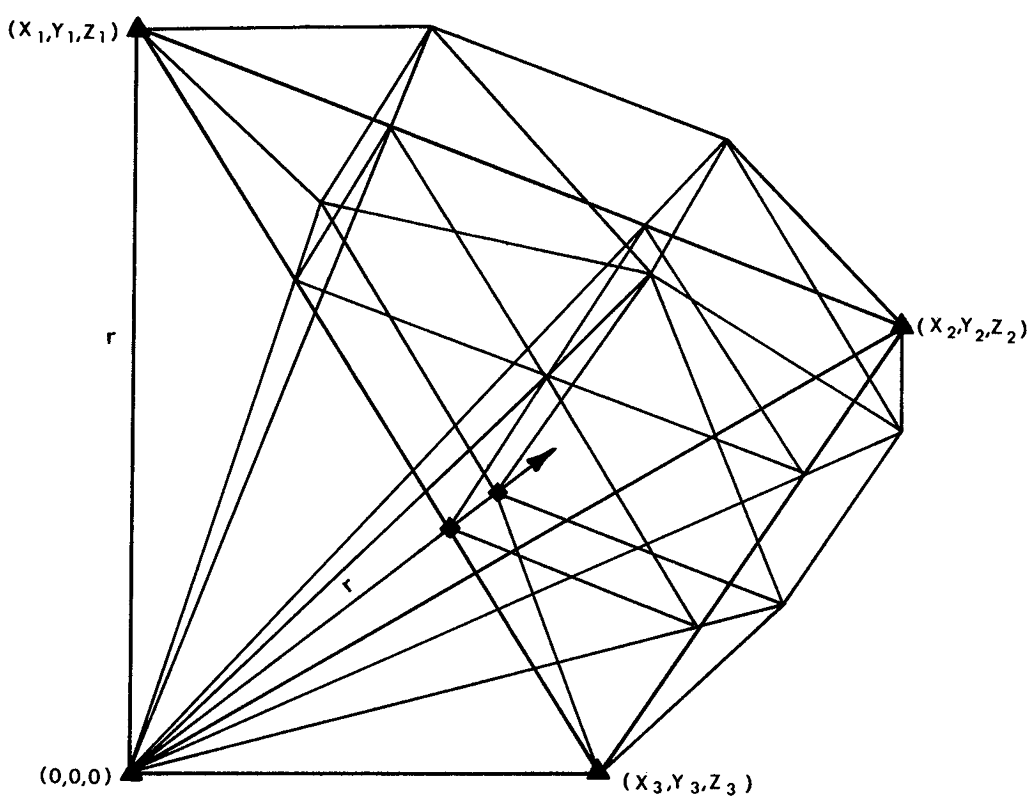


Figure 1.38

Using the coordinates, this program finds the lengths of the elements of the structure (ℓ), the angle between pairs of elements (face angle α), the angle between the elements and a radius from the origin to an endpoint of the element (axial angle Ω), and the angle between adjacent faces of the structure (dihedral angle β). Figure 1.39

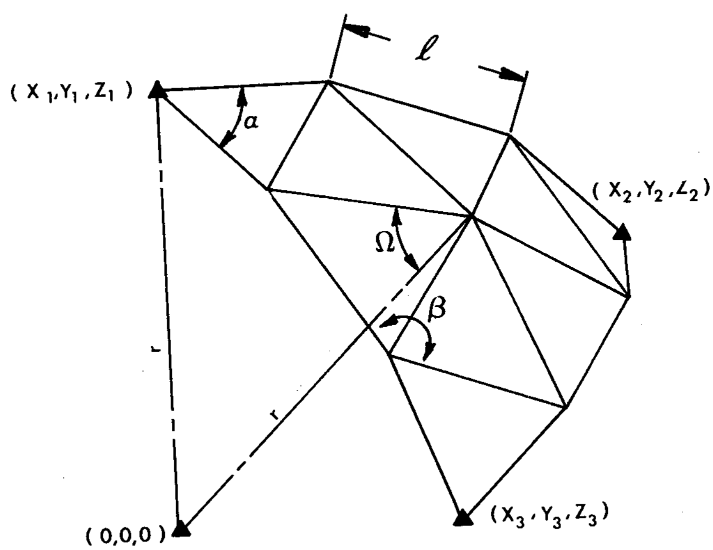


Figure 1.39

To find the angle between elements the face ∢ α, we use the coordinates of their endpoints. The vertex of the angle is a common endpoint to each element and is translated to the origin. The other two endpoints P_1 and P_2 are translated in the same manner. Letting (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) be the points resulting from the translations of the endpoints P_1 and P_2 ,

$$\cos \alpha = \left| \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{d_1d_2} \right| \qquad 1.2$$

$$\text{where } d_1 = \sqrt{X_1^2 + Y_1^2 + Z_1^2}$$

$$\text{and } d_2 = \sqrt{X_2^2 + Y_2^2 + Z_2^2}$$

α is the desired angle.

To find axial angles the above method is used except that the vertex is established at one end of an element and the origin is used with the other endpoint to define the angle. The desired angle is Ω.

The angle between two adjacent faces, the dihedral $\measuredangle\ \beta$, is found using

$$\cos \beta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{A_1^2 + B_1^2 + C_1^2 \quad A_2^2 + B_2^2 + C_2^2} \qquad 1.3$$

where

β is the desired angle.

$A_1X + B_1Y + C_1Z + D_1 = 0$ defines the plane containing one face and $A_2X + B_2Y + C_2Z + D_2 = 0$ defines the plane containing the other face.

The negative sign is used because the obtuse angle is desired.

The A, B, and C fcr each plane are computed as

$$A = \begin{vmatrix} Y_1 & Z_1 & 1 \\ Y_2 & Z_2 & 1 \\ Y_3 & Z_3 & 1 \end{vmatrix}$$

$$B = \begin{vmatrix} X_1 & Z_1 & 1 \\ X_2 & Z_2 & 1 \\ X_3 & Z_3 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} \qquad 1.4$$

where (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), and (X_3, Y_3, Z_3) lie in the plane. In particular the three vertices of each face are used. For the special case where the two faces used lie above separate faces of the polyhedron, the assumption is made that the plane containing the element common to each face and the origin bisects the angle. This angle is found in the same manner and doubled. This method is used because the structural face above the polyhedral face is not generated properly.

The length of the elements, ℓ, are found by using the general equation

$$\ell = \sqrt{(P_{x_1} - P_{x_2})^2 + (P_{y_1} - P_{y_2})^2 + (P_{z_1} - P_{z_2})^2} \qquad 1.5$$

ℓ is the desired length.

To reduce total output, this program takes into account certain symmetries and outputs only a part of the total angles and lengths. The rest of the values are the same as at least one outputed value and can easily be found using the following symmetries. Figure 1.40.

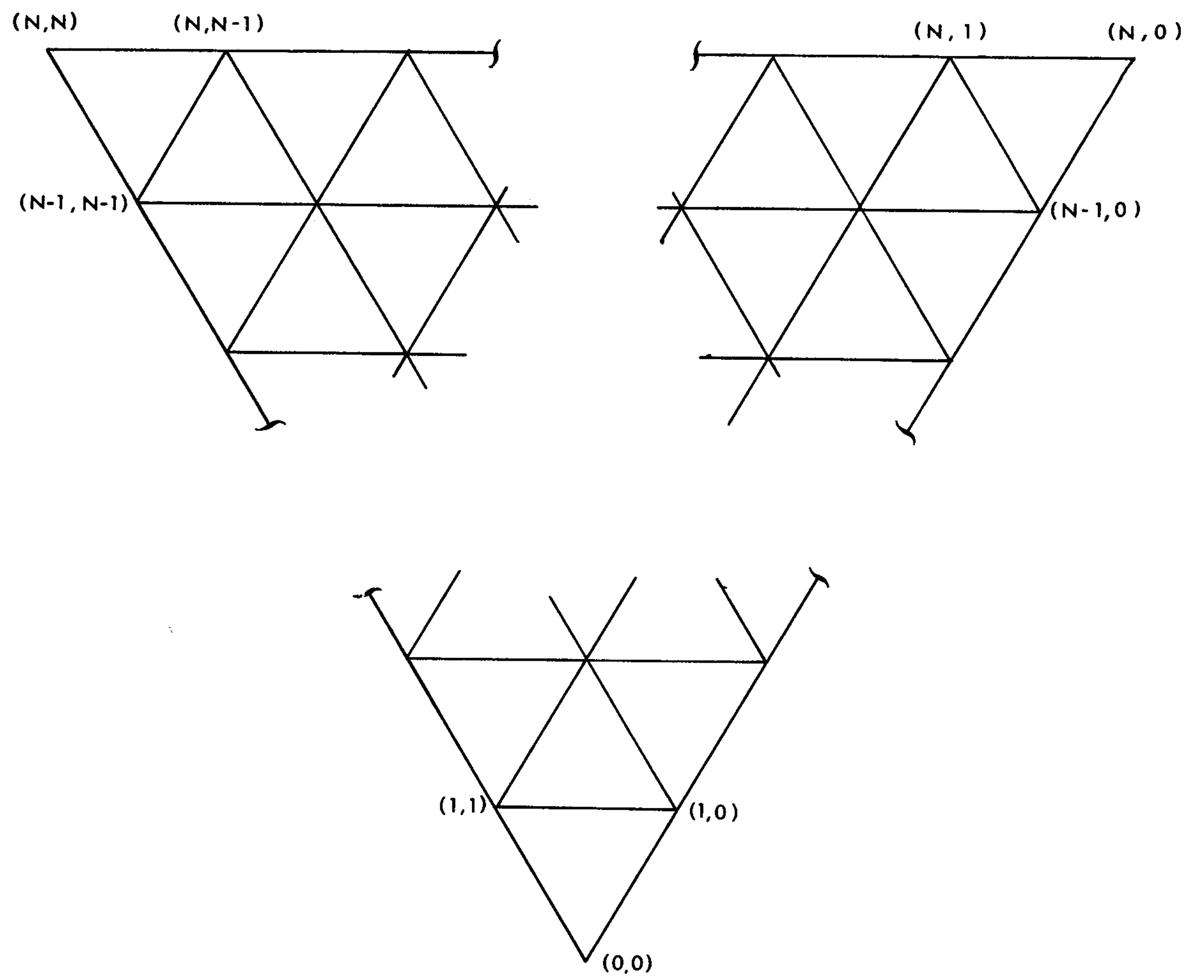


Figure 1.40

FACE ANGLES

For every face angle opening directly towards (or away from) the point (0,0), there are equal angles opening towards (or away from) the point (N,0) and (N,N). For example, the angle (1,1), (0,0), (1,0) with vertex at (0,0) is equal to the angle (N-1,0), (N,0), (N,1) and the angle (N,N-1), (N,N), (N-1, N-1). Thus, only the face angles facing directly towards or away from (0,0) are computed. If the vertex is to lie at (I,J), the angle will be either (I + 1, J + 1), (I,J), (I +1, J) or (I-1, J-1), (I,J), (I-1,J). Also, only the face angles falling on the right of or on a line passing through (X_1, Y_1, Z_1,) and the mid-point of the opposite side are computed.

The elements of the structure can be put into one-to-one correspondence with the lengths and dihedral angles. The dihedral angle associated with an element is the angle between the two faces containing the element. For each element, there are two axial angles, one at each end, but since the element is a cord of the circle, the two angles are equal and may be considered one. In this case, we have a one-to-one correspondence between elements and axial angles. This program will only compute values around elements parallel to the side opposite (X_1, Y_1, Z_1) and on the right side of a line through (X_1, Y_1, Z_1) and the midpoint of the opposite side. All other lengths and angles are symmetric to one of the lengths and angles computed in this manner.

THE COMPUTER PROGRAM DESCRIBED ON

PAGES I-45 to I-75

IS AVAILABLE FROM COSMIC

1.7 METHODS 2 & 3

This program works with a tetrahedron, octahedron, or icosahedron circumscribed by a unit sphere. The icosahedron was chosen as an example to illustrate the geometry of the program. The icosahedron is oriented in a three dimensional rectangular coordinate system so that the vertices of one PPT are

$$(x_1, y_1, z_1) = \left(0, \frac{\sqrt{\tau}}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}\sqrt{\tau}}\right)$$

$$\simeq (0, .850651, .525731)$$

$$(x_2, y_2, z_2) = \left(\frac{1}{\sqrt[4]{5}\sqrt{\tau}}, 0, \frac{\sqrt{\tau}}{\sqrt[4]{5}}\right)$$

$$\simeq (.525731, 0, .850651)$$

$$(x_3, y_3, z_3) = \left(\frac{\sqrt{\tau}}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}\sqrt{\tau}}, 0\right)$$

$$\simeq (850651, .525731, 0)$$

$$\text{where } \tau = \frac{1 + \sqrt{5}}{2}$$

with the intersections of the axis X, Y, Z, located at the origin (0,0,0) of the icosahedron, Figure 1.43.

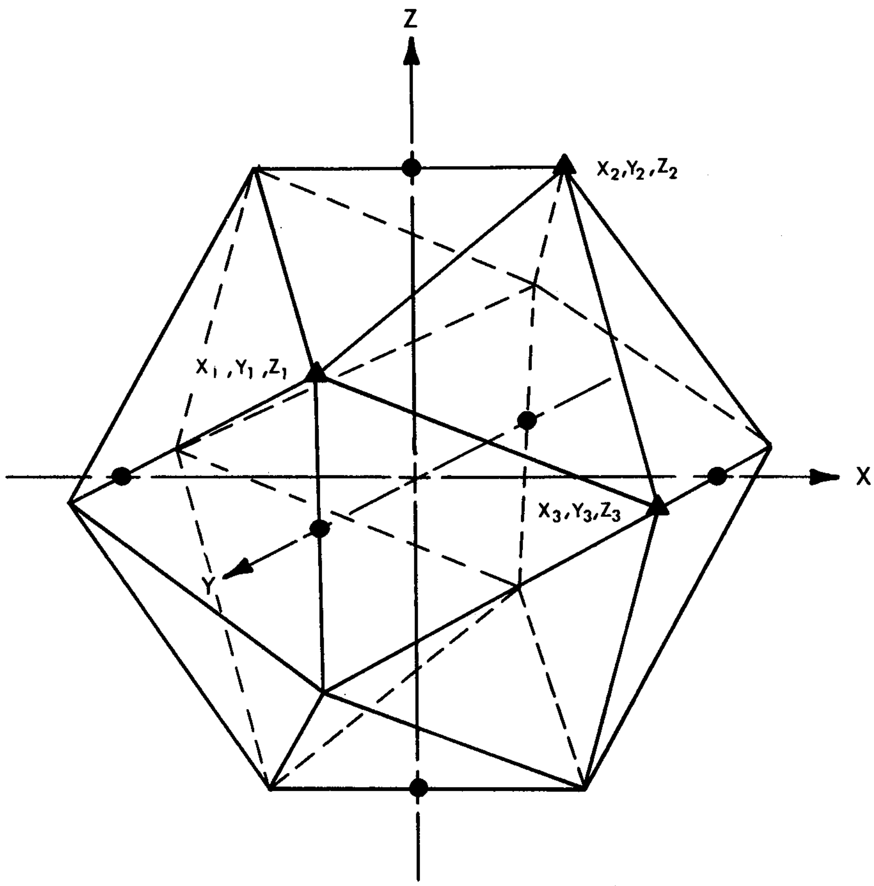


Figure 1.43

This PPT is divided into smaller triangluar units which are translated onto the surface of a sphere constituting the desirable space form.

Using the following formula the planes consisting of the edges of the PPT and the origin (X_1, Y_1, Z_1) (X_2, Y_2, Z_2) (X_3, Y_3, Z_3) are rotated from 3-space into 2-space, Figure 1.44.

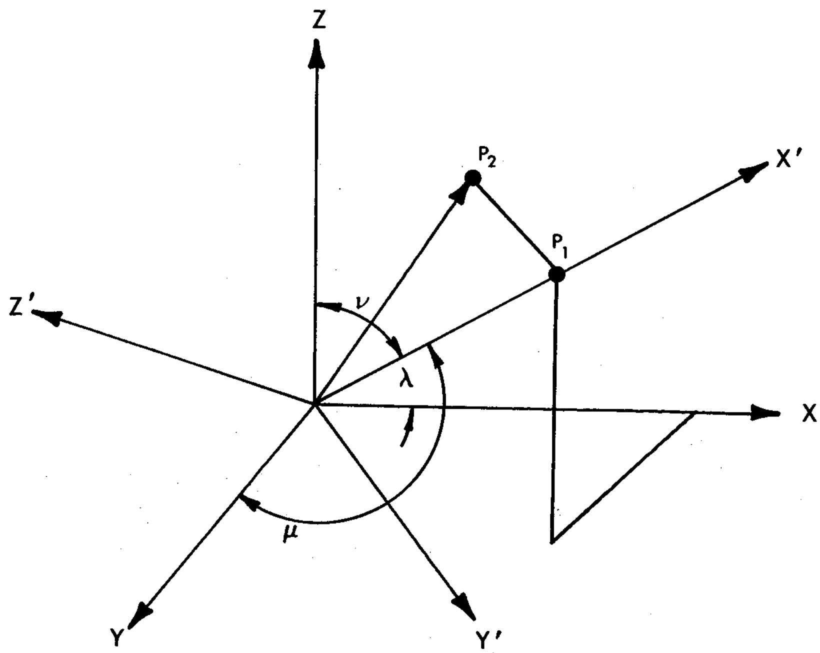


Figure 1.44

$$x' = \lambda_1 x + \mu_1 y + \nu_1 z \qquad 1.6$$
$$y' = \lambda_2 x + \mu_2 y + \nu_2 z$$
$$z' = \lambda_3 x + \mu_3 y + \nu_3 z$$

Where λ, μ, ν are direction cosines of the X'-axis, and Y'-axis, and Z'-axis respectively with respect to the old axis and are found by:

$$\lambda_1 = x_1 / \sqrt{x_1^2 + y_1^2 + z_1^2}$$
$$\mu_1 = y_1 / \sqrt{x_1^2 + y_1^2 + z_1^2}$$
$$\nu_1 = z_1 / \sqrt{x_1^2 + y_1^2 + z_1^2}$$

λ_2, λ_3; μ_2,μ_3; and ν_2,ν_3 are found similarly.

The edge of the PPT is subdivided into units by the following method, Figures 1.45 and 1.46.

FIND: the angle ϕ contained within the rotated triangle consisting of $\overline{P_1P_2}$, and the origin with the vertex located at the origin.

$$\phi = \text{Arctan}\left(\frac{P_{y^2}}{P_{x^2}}\right) \quad r \qquad 1.7$$

where r = 1 and is considered constant

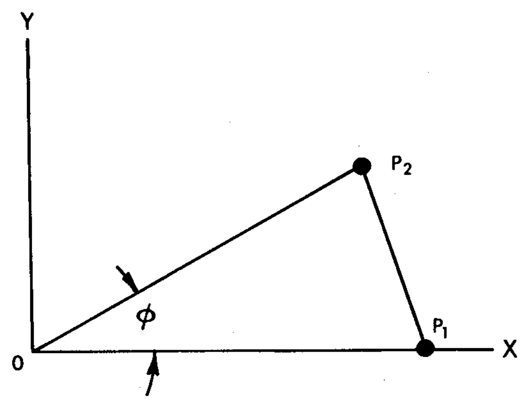


Figure 1.45

THEN: subdivide the angle ϕ into N angles θ

$$\theta = \frac{\phi}{N} \cdot T \qquad 1.8$$

Where T = Increment 1 to N

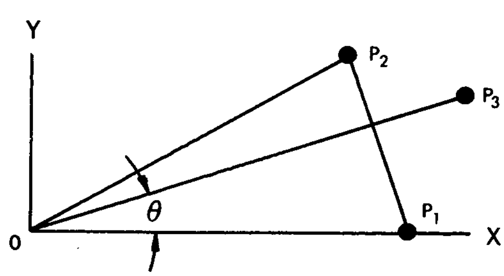


Figure 1.46

The points of intersection of $\overline{OP_3}$ and $\overline{P_1P_2}$ are found:

$$\overline{P_1P_2} \text{ is } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad 1.9$$

$$\overline{OP_3} \text{ is } \frac{y - 0}{x - 0} = \frac{y_3 - 0}{x_3 - 0}$$

The equation takes the following form:

$$\overline{P_1P_2} \text{ is } x(y_2 - y_1) + y(x_1 - x_2) = y_1(x_1 - x_2) + x_1(y_2 - y_1)$$

let

$$(y_2 - y_1) = a_1$$
$$(x_2 - x_1) = b_1$$
$$y_1(x_1 - x_2) + x_1(y_2 - y_1) = C_1$$

$$\overline{OP} \text{ is } xy_3 - yx_3 = 0$$

let

$$y_3 = a_2$$
$$-x_3 = b_2$$
$$0 = c_2$$

Solve the equations for the point of intersection

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad X = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} \qquad 1.10$$

$$Y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}$$

Rotate the points of intersection along the PPT edge from 2-spaces back to 3-spaces.

$$x = \lambda_1 x' + \mu_1 y' + \nu_1 z \qquad 1.11$$

$$y = \lambda_2 x' + \mu_2 y' + \nu_2 z$$

$$z = \lambda_3 x' + \mu_3 y' + \nu_3 z$$

where λ, μ, ν, are direction cosines of the X'-axis, Y'-axis, and Z'-axis with respect to the old axis and are found:

$$\lambda_1 = x_1 \Big/ \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\mu_1 = y_1 \Big/ \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\nu_1 = z_1 \Big/ \sqrt{x_1^2 + y_1^2 + z_1^2}$$

λ_2, λ_3; μ_2, μ_3; and ν_2, ν_3 are found similarly. Retain the co-ordinates along the edges S_1, S_2 and S_3 as shown in Figure 1.47.

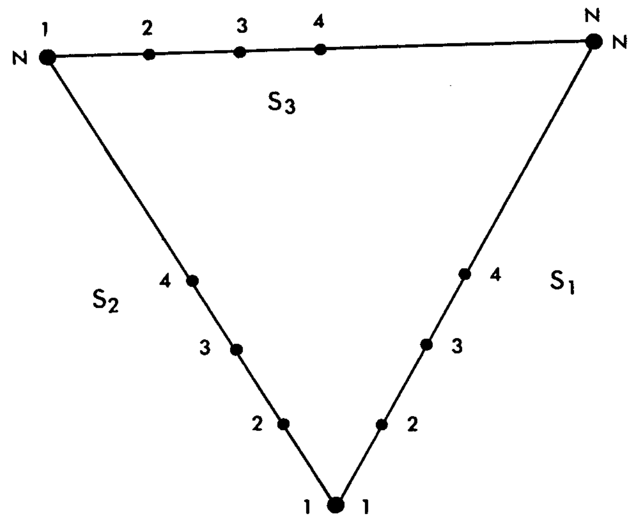


Figure 1.47

After finding the unit measurements along the edges of the PPT, they are connected through a grid determining a smaller grid network. Since the units along the PPT edge are not equal, the gridding will create "windows". The centers of these "windows" must be found to establish the final 3-way grid network. Figure 1.48.

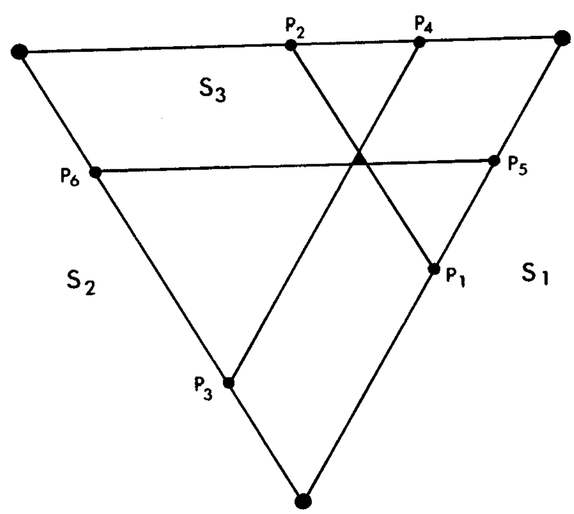


Figure 1.48

The gridding and windows are found by the following method: From the coordinates along the edges of S_1, S_2 and S_3, calculate coordinates of the window by finding the intersection of $\overline{P_1 P_2}$ with $\overline{P_3 P_4}$ and $\overline{P_1 P_2}$ with $\overline{P_5 P_6}$ and $\overline{P_3 P_4}$ with $\overline{P_5 P_6}$ by using the two point form of the equation of a line in three-space for the three lines and solve simultaneously for the points of intersection.

$$\overline{P_1P_2} \text{ is: } \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{1.12}$$

$$\overline{P_3P_4} \text{ is: } \quad \frac{x - x_3}{x_4 - x_3} = \frac{y - y_3}{y_4 - y_3} = \frac{z - z_3}{z_4 - z_3}$$

$$\overline{P_5P_6} \text{ is: } \quad \frac{x - x_5}{x_6 - x_5} = \frac{y - y_5}{y_6 - y_5} = \frac{z - z_5}{z_6 - z_5}$$

To find the intersection of $\overline{P_1P_2}$ with $\overline{P_3P_4}$ the equation takes the following form:

(1) $\overline{P_1P_2}$ is: $x(y_2-y_1) + y(x_1-x_2) = y_1(x_1-x_2) + x_1(y_2-y_1)$

(2) $\overline{P_1P_2}$ is: $y(z_2-z_1) + z(y_2-y_1) = z_1(y_1-y_2) + y_1(z_2-z_1)$

(3) $\overline{P_3P_4}$ is: $x(y_4-y_3) + y(x_3-x_4) = y_1(x_3-x_4) + x_1(y_4-y_3)$

(4) $\overline{P_3P_4}$ is: $y(z_4-z_3) + z(y_3-y_4) = z_3(y_3-y_4) + y_3(z_4-z_3)$

For $\overline{P_1P_2}$ let:

$$(y_2 - y_1) = a_1$$
$$(x_1 - x_2) = b_1$$
$$y_1(x_1 - x_2) + x_1(y_2 - y_1) = c_1$$

For $\overline{P_3P_4}$ let:

$$(y_4 - y_3) = a_2$$
$$(x_3 - x_4) = b_2$$
$$y_1(x_3 - x_4) + x_1(y_4 - y_3) = c_2$$

using the formula 1.10 solve for x and y coordinates of the intersections of $\overline{P_1P_2}$ with $\overline{P_3P_4}$.

Find the z coordinate:

For $\overline{P_1P_2}$ let:

$$(z_2 - z_1) = a_1$$
$$(y_1 - y_2) = b_1$$
$$z_1(y_1 - y_2) + y_1(z_2 - z_3) = c_1$$

For $\overline{P_3P_4}$ let:

$$(z_4 - z_3) = a_2$$
$$(y_3 - y_4) = b_2$$
$$z_3(y_3 - y_4) + y_3(z_4 - z_3) = c_2$$

The other two vertices of the window are found in a similar manner. Once the coordinates for the vertices of the window are determined, its center is found by one of the following two methods:

METHOD I:

On the PPT Plane the windows appear as equilateral triangles with vertices $P_1(x_1y_1z_1)$, $P_2(x_2y_2z_2)$, $P_3(x_3y_3z_3)$ as shown in Figure 1.49.

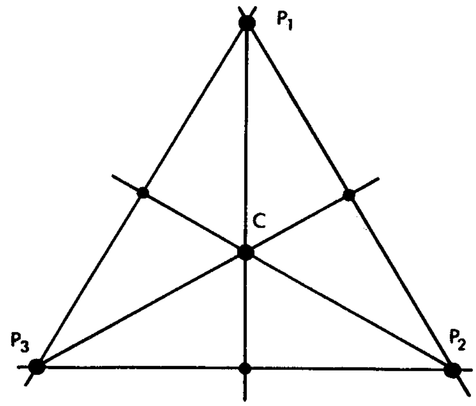


Figure 1.49

The center C(cx,cy,cz) is found with the following formula:

$$CX = \frac{x_1 + x_2 + x_3}{3} \qquad 1.13$$

$$CY = \frac{y_1 + y_2 + y_3}{3}$$

$$CZ = \frac{z_1 + z_2 + z_3}{3}$$

METHOD II:

The coordinates of the window found on the surface of the PPT are first "exploded" to the surface of the sphere. The center of the exploded window is then found by the intersection of angle bisectors. To find the projection of each vertex of the window onto the unit sphere, translate each vertex along a line through the vertex of the PpT and the origin; each coordinate of each vertex, PPT, is divided by the distance between the vertex PPT and the origin, Figure 1.50.

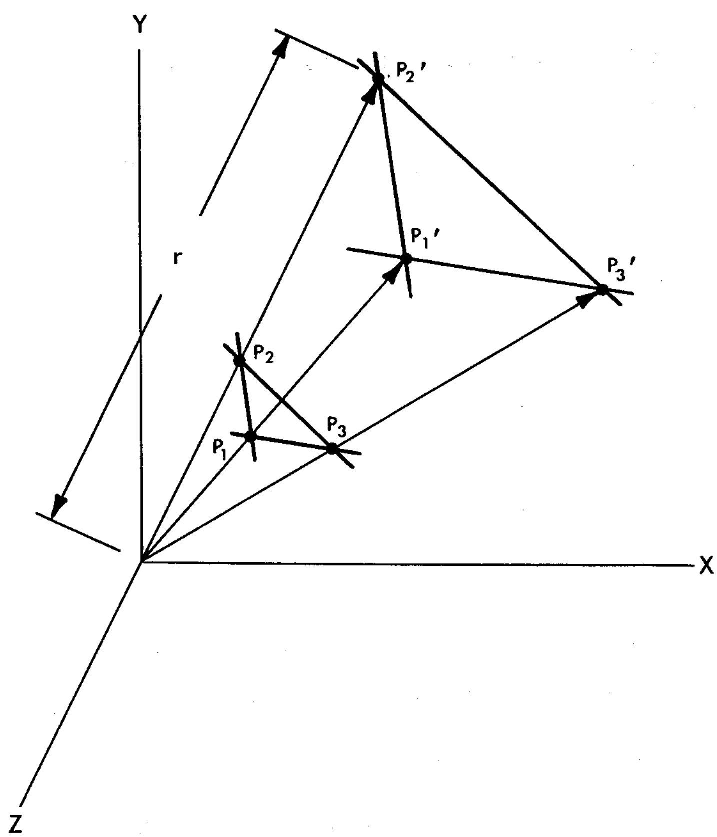


Figure 1.50

$$d = x_1^2 + y_1^2 + z_1^2$$ where d = distance from origin to P_1 1.14

$$r = 1$$ where r = radius of the sphere to be exploded upon and is considered constant

$$x'_1 = \frac{rx_1}{d}$$

$$y'_1 = \frac{ry_1}{d}$$ 1.15

$$z'_1 = \frac{rz_1}{d}$$

Translate "window" with vertice P_3 at the origin, Figure 1.51.

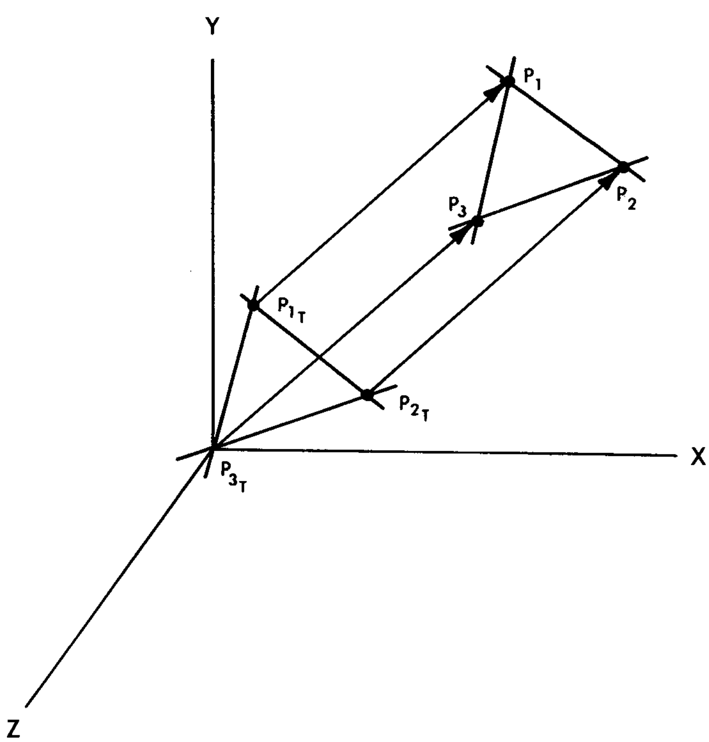


Figure 1.51

$$P1_{Tx} = P1_x - P3_x \qquad 1.16$$

$$P1_{Ty} = P1_y - P3_y$$

$$P1_{Tz} = P1_z - P3_z$$

$$P2_{Tx} = P2_x - P3_x$$

$$P2_{Ty} = P2_x - P3_x$$

$$P2_{Tz} = P2_z - P3_z$$

$$P3_{Tx_1},\ P3_{Ty_1},\ P3_{Tz1} = 0$$

Rotate plane P_1, P_2, P_3 so that $\overline{P_1P_3}$ will fall on the X-axis and P_3 is at the origin using equation 1.6

The center is found with the intersection of two angle bisectors of the triangular window $P_1P_2P_3$, Figure 1.52.

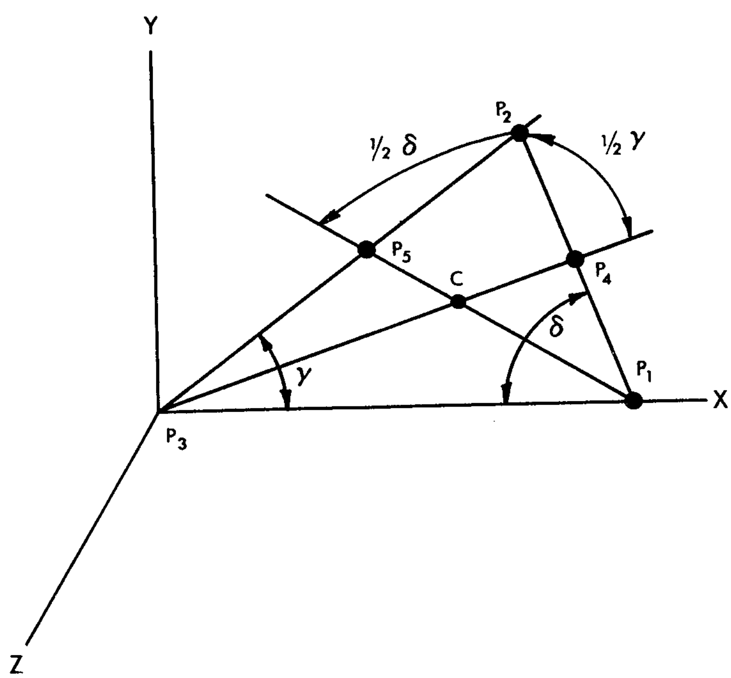


Figure 1.52

The angles γ and δ are found:

$$\text{Arctan} \quad \frac{y_2}{x_2} = \gamma \tag{1.17}$$

$$\text{Arctan} \quad \frac{y_2}{x_1 - x_2} = \delta$$

rotate P_2 about P_3 toward P_1, $^1/_2\ \gamma$ degrees

$$x_4 = x_2 \cos {}^1/_2\ \gamma + y_2 \sin {}^1/_2\ \gamma \tag{1.18}$$

$$y_4 = y_2 \cos {}^1/_2\ \gamma - x_2 \sin {}^1/_2\ \gamma$$

locate the origin at P_1, then rotate P_2 about P_1 toward P_3 $^1/_2\ \delta$ degrees.

$$x_5 = (x_2 - x_1) \cos {}^1/_2\ \delta - y_2 \sin {}^1/_2\ \delta + x_1 \tag{1.19}$$

$$y_5 = y_2 \cos {}^1/_2\ \delta + (x_2 - x_1) \sin {}^1/_2\ \delta$$

thus defining $\overline{P_1P_5}$ and $\overline{P_3P_4}$.

With P_3 at the origin formula 1.9 may be used to solve for the intersection of line $\overline{P_3P_4}$, $\overline{P_1P_5}$ finding center C. Rotate C back to three space using formula 1.11 . Then translate center C back to three space ("C" is located in the previously "exploded" window), figure 1.53.

$$C'x = Cx + P_3x \tag{1.20}$$

$$C'y = Cy + P_3y$$

$$C'z = Cz + P_3z$$

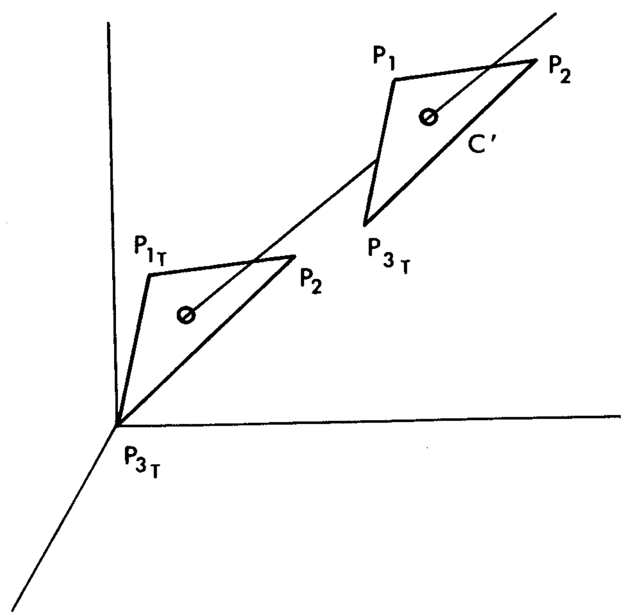


Figure 1.53

For Method I or Method II, the centers found are "exploded" to the surface of the sphere using formula 1.14 and formula 1.15 .

Using the coordinates, this program finds the lengths of the elements of the structure (ℓ), the angle between pairs of elements (face angle α), the angle between the elements and a radius from the origin to an endpoint of the element (axial angle Ω), and the angle between adjacent faces of the structure (dihedral angle β), Figure 1.54.

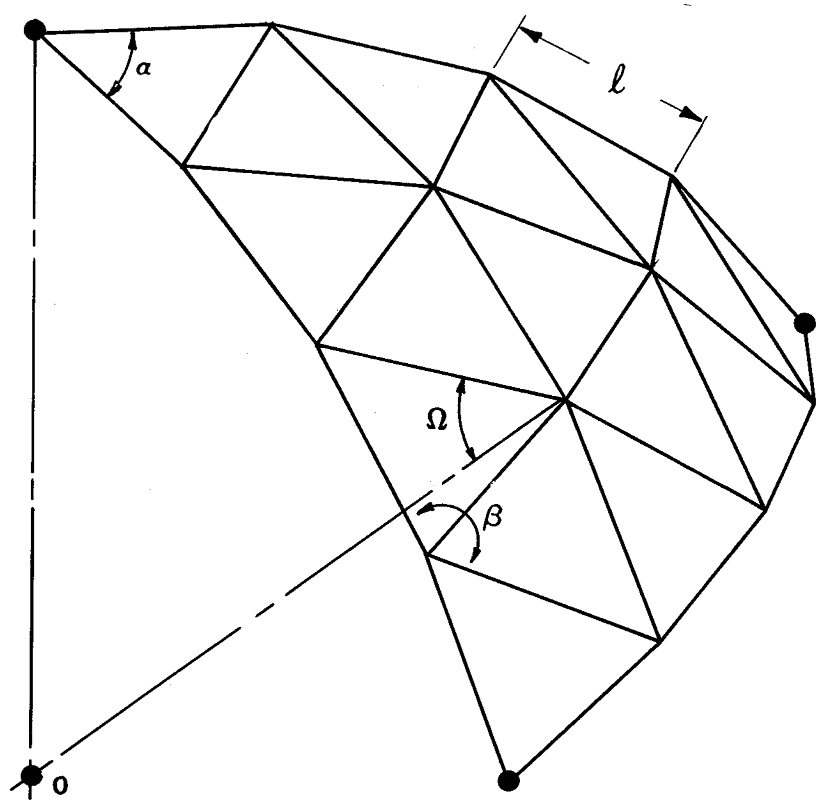


Figure 1.54

To find the angle between elements the face $\measuredangle \alpha$, we use the coordinates of their endpoints. The vertex of the angle is a common endpoint to each element and is translated to the origin. The other two endpoints P_1 and P_2 are translated in the same manner. Letting (x_1, y_1, z_1) and (x_2, y_2, z_2) be the points resulting from the translations of the endpoints P_1 and P_2,

$$\cos \alpha = \left|\frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{d_1 d_2}\right| \qquad 1.21$$

$$\text{where } d_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\text{and } d_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$$

α is the desired angle.

To find axial angles the above method is used except that the vertex is established at one end of an element and the origin is used with the other endpoint to define the angle. The desired angle is Ω.

The angle between two adjacent faces, the dihedral $\measuredangle\ \beta$, is found using

$$\cos \beta = \frac{\left|-A_1 A_2 + B_1 B_2 + C_1 C_2\right|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \qquad 1.22$$

where

β is the desired angle.

$A_1 X + B_1 Y + C_1 Z + D_1 = 0$ defines the plane containing one face and $A_2 X + B_2 Y + C_2 Z + D_2 = 0$ defines the plane containing the other face. The negative sign is used because the obtuse angle is desired.

The A,B, and C for each plane are computed as

$$A = \begin{vmatrix} Y_1 & Z_1 & 1 \\ Y_2 & Z_2 & 1 \\ Y_3 & Z_3 & 1 \end{vmatrix} \qquad 1.23$$

$$B = \begin{vmatrix} X_1 & Z_1 & 1 \\ X_2 & Z_2 & 1 \\ X_3 & Z_3 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix}$$

where (x_1, Y_1, Z_1), (X_2, Y_2, Z_2), and (X_3, Y_3, Z_3) lie in the plane. In particular the three vertices of each face are used.

The length of the elements ℓ are found by using the general equation:

$$\ell = \sqrt{(P_{x_1} - P_{x_2})^2 + (P_{y_1} - P_{y_2})^2 + (P_{z_1} - P_{z_2})^2} \qquad 1.24$$

ℓ is the desired length.

To reduce total output, this program takes into account certain symmetries and outputs only a part of the total angles and lengths. The rest of the values are the same as at least one outputed value and can easily be found using the following symmetries, Figure 1.55.

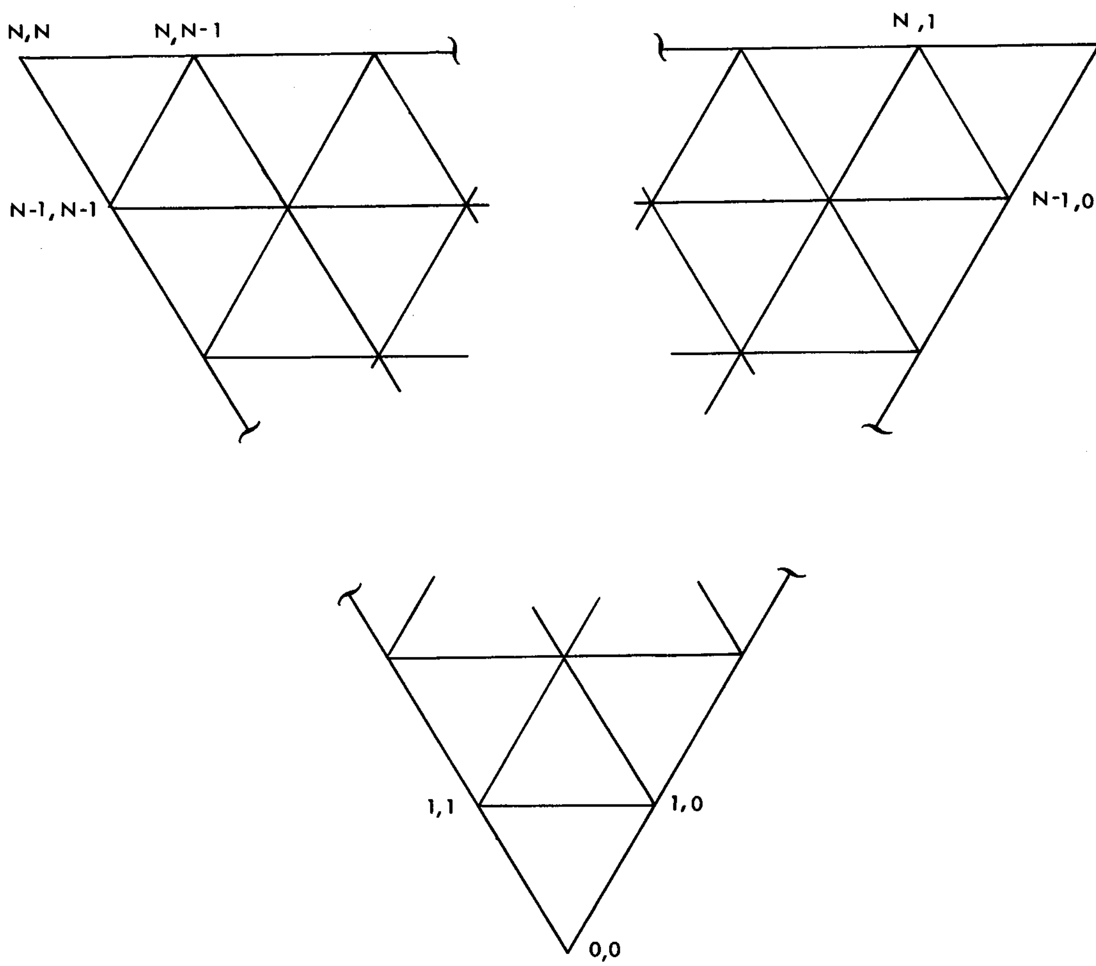


Figure 1.55

FACE ANGLES

For every face angle opening directly towards (or away from) the point (0,0), there are equal angles opening towards (or away from) the point (N,0) and (N,N). For example, the angle (1,1), (0,0), (1,0) with vertex at (0,0) is equal to the angle (N-1,0), (N,0), (N,1) and the angle (N,N-1), (N,N), (N-1, N-1). Thus, only the face angles facing directly towards or away from (0,0) are computed. If the vertex is to lie at (I,J), the angle will be either (I + 1, J + 1), (I,J), (I + 1, J) or (I-1, J-1), (I,J), (I-1,J). Also, only the face angles falling on the right of or on a line passing through (X_1, Y_1, Z_1,) and the mid-point of the opposite side are computed.

The elements of the structure can be put into one-to-one correspondence with the lengths and dihedral angles. The dihedral angle associated with an element is the angle between the two faces containing the element. For each element, there are two axial angles, one at each end, but since the element is a cord of the circle, the two angles are equal and may be considered one. In this case, we have a one-to-one correspondence between elements and axial angles. This program will only compute values around elements parallel to the side opposite (X_1, Y_1, Z_1) and on the right side of a line through (X_1, Y_1,Z_1) and the midpoint of the opposite side. All other lengths and angles are symmetric to one of the lengths and angles computed in this manner.

The computer program here contained was written for the IBM 7040/7044 computer, utilizing FORTRAN IV language. The program may be used for a Tetrahedron, Octahedron, or Icosahedron, depending upon the coordinates chosen as input data. The output is given in units based upon a radius of 1 for the spherical form and therefore, may be used as a basis for determining large structures.

The example of input data is given in Table 1.6. The example of output data is given for a six frequency Icosahedral sphere, and may be read as in Table 1.7. The output takes advantage of symmetries within the spherical Icosahedron as discussed in the text material. Figure 1.56

THE COMPUTER PROGRAM DESCRIBED ON

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IS AVAILABLE FROM COSMIC

1.8 METHODS 4 & 5

This mathematical and computer model was written for subdivision of a tetrahedron, octahedron, or icosahedron circumscribed by a unit sphere. The Icosahedron was chosen as an example to illustrate the geometry of the model. The polyhedron is oriented in a three dimensional rectangular coordinate system so that the vertices of one PPT are:

$$(x_1, y_1, z_1) = 0,\ \frac{\sqrt{\tau}}{\sqrt[4]{5}},\ \frac{1}{\sqrt[4]{5}\sqrt{\tau}}$$

$$\simeq (0,\ .850651,\ .525731)$$

$$(x_2, y_2, z_2) = \frac{1}{\sqrt[4]{5}\sqrt{\tau}},\ 0,\ \frac{\sqrt{\tau}}{\sqrt[4]{5}}$$

$$\simeq (.525731,\ 0,\ .850651)$$

$$(x_3, y_3, z_3) = \frac{\sqrt{\tau}}{\sqrt[4]{5}},\ \frac{1}{\sqrt[4]{5}\sqrt{\tau}},\ 0$$

$$\simeq (.850651,\ .525731,\ 0)$$

where: $\tau = \dfrac{1 + \sqrt{5}}{2}$ 1.25

with the intersection of the axis x, y, z located at the origin (0,0,0) of the polyhedron. Figure 1.58.

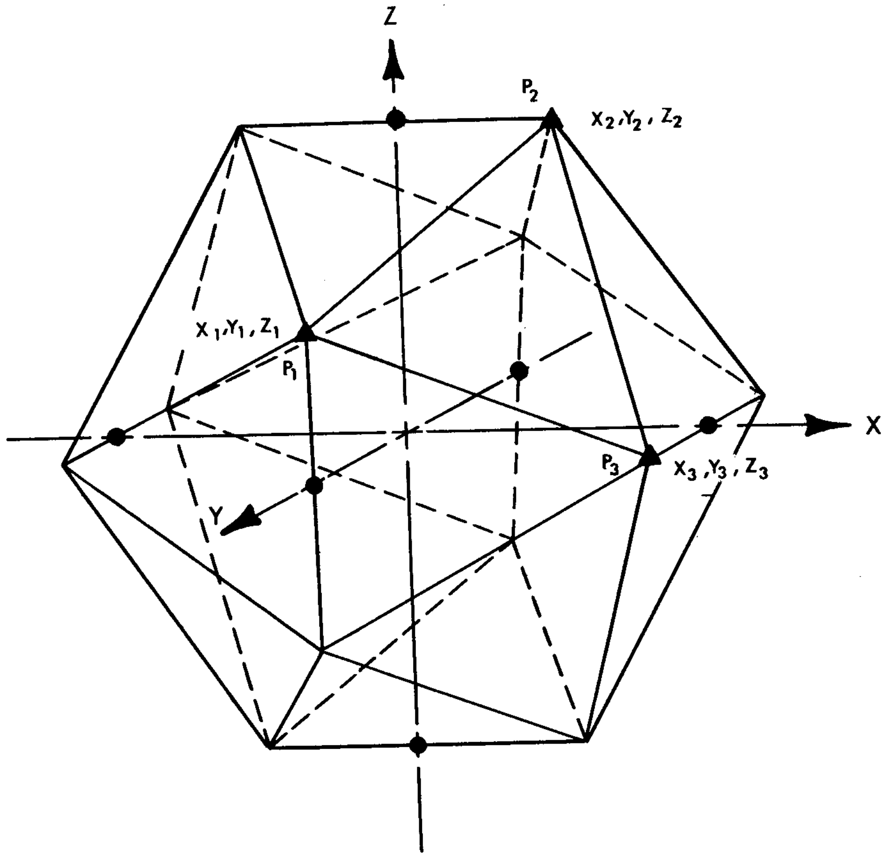


Figure 1.58

Due to the methods in which the three-way grids are generated only even frequency subdivisions may be used.

Subdivision and Gridding for Method 4

The PPT is subdivided into equal parts so that the vertices take the form:

$$\left(x_1 + I\,\frac{x_2 - x_1}{N},\; y_1 + I\,\frac{y_2 - y_1}{N},\; z_1 + I\,\frac{z_2 - z_1}{N} \right) \qquad 1.26$$

where N is the frequency of the structure and I is an interger such that $0 \leq I \leq N$. The values of I are unique for each vertex and are used to identify each vertex for each side as shown in Figure 1.59.

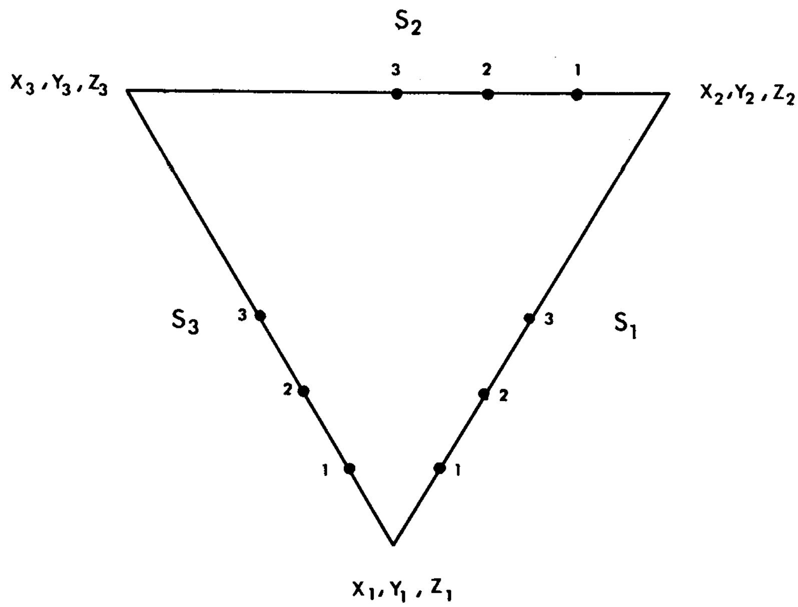


Figure 1.59

Having found the unit division along the principal side of PPT, the points of subdivision are connected so that a three-way grid is generated with the lines of the grid perpendicular to their respective sides. Figure 1.60

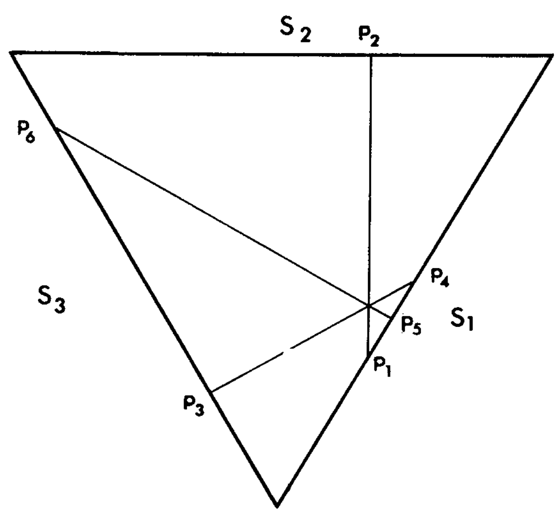


Figure 1.60

The gridding is formed by the following method: From the coordinates along the edges of S_1, S_2, & S_3 calculate the coordinates of the points of intersection of $\overline{P_1P_2}$ with $\overline{P_3P_4}$ by using the two point form of the equation of a line in three-space for the two lines and solving simultaneously.

$$\overline{P_1P_2} \text{ is: } \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad 1.27$$

$$\overline{P_3P_4} \text{ is: } \frac{x - x_3}{x_4 - x_3} = \frac{y - y_3}{y_4 - y_3} = \frac{z - z_3}{z_4 - z_3}$$

To find the intersection of $\overline{P_1P_2}$ with $\overline{P_3P_4}$ the equation take the following form:

(1) $\overline{P_1P_2}$ is: $x(y_2-y_1) + y(x_1-x_2) = y_1(x_1-x_2) + x_1(y_2-y_1)$

(2) $\overline{P_1P_2}$ is: $y(z_2-z_1) + z(y_2-y_1) = z_1(y_1-y_2) + y_1(z_2-z_1)$

(3) $\overline{P_3P_4}$ is: $x(y_4-y_3) + y(x_3-x_4) = y_1(x_3-x_4) + x_1(y_4-y_3)$

(4) $\overline{P_3P_4}$ is: $y(z_4-z_3) + z(y_3-y_4) = z_3(y_3-y_4) + y_3(z_4-z_3)$

for $\overline{P_1P_2}$ let: $(y_2-y_1) = a_1$

$(x_1-x_2) = b_1$

$y_1(x_1-x_2) + x_1(y_2-y_1) = c_1$

for $\overline{P_3P_4}$ let: $(y_4-y_3) = a_2$

$(x_3-x_4) = b_2$

$y_1(x_3-x_4) + x_1(y_4-y_3) = c_2$

Solve the equations for the x & y coordinates of the intersection of $\overline{P_1P_2}$ with $\overline{P_3P_4}$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} \qquad 1.28$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}$$

find the z coordinates

For P_1P_2 let: $(z_2-z_1) = a_1$

$(y_1 - y_2) = b_1$

$z_1(y_1-y_2) + y_1(z_2 - z_3) = c_1$ 1.29

For P_3P_4 let: $(z_4 - z_3) = a_2$

$(y_3 - y_4) = b_2$

$z_3(y_3 - y_4) + y_3(z_4 - z_3) = c_2$

The intersection of $\overline{P_5P_6}$ with $\overline{P_3P_4}$ & $\overline{P_1P_2}$ are coincident and need not be found.

All other points of intersection of the three-way grid are found in like manner and are stored for final translation to the surface of the circumscribed sphere.

Subdivision and Gridding for Method 5

With the following equations the planes consisting of the edge of the PPT and the origin (0,0,0) are rotated from 3-space, Figure 1.61.

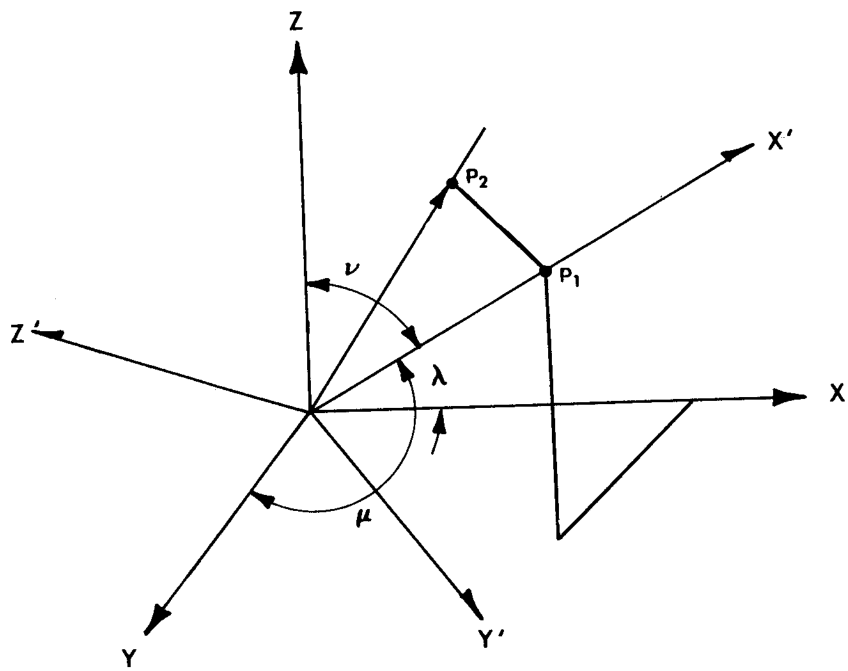


Figure 1.61

$$
\begin{aligned}
x' &= \lambda_1 x + \mu_1 y + \nu_1 z \\
y &= \lambda_2 x + \mu_2 y + \nu_2 z \\
z &= \lambda_3 x + \mu_3 y + \nu_3 z
\end{aligned}
\qquad 1.30
$$

Where λ, μ, ν are direction cosines of the X′-axis, Y′-axis, and Z′-axis respectively with respect to the old axis and are found by:

$$\lambda_1 = \sqrt{x_1 / \quad x_1^2 + y_1^2 + z_1^2}$$

$$\mu_1 = \sqrt{y_1 / \quad x_1^2 + y_1^2 + z_1^2}$$

$$\nu_1 = \sqrt{z_1 / \quad x_1^2 + y_1^2 + z_1^2}$$

λ_2, λ_3; μ_2, μ_3; and ν_2, ν_3 are found similarly.

Due to the method in which the three-way grid is generated only even frequency subdivisions may be used. The principal side of the PPT is subdivided into equal arc units by the following method: Figure 1.62 and 1.63.

FIND: the angle ϕ contained within the rotated triangle consisting of $\overline{P_1P_2}$, and the origin with the vertex located at the origin.

$$\phi = \text{Arctan}\left(\frac{P_{y2}}{P_{x2}}\right) \cdot r \qquad 1.31$$

Where r = 1 and is considered constant

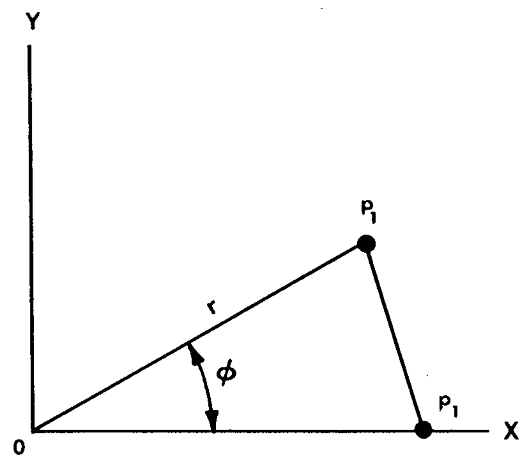


Figure 1.62

THEN: subdivide the angle ϕ into N angles

$$\theta = \frac{\phi}{N} \cdot T \qquad 1.32$$

where T = Increment 1 to N

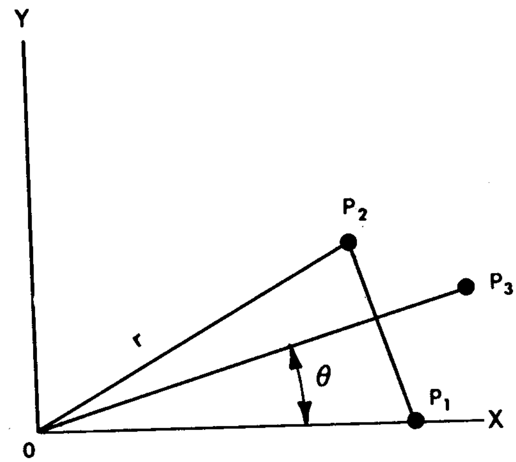


Figure 1.63

The points of intersection of $\overline{OP}_3$ and $\overline{P_1P_2}$ are found:

$$\overline{P_1P_2} \text{ is } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad 1.33$$

$$\overline{OP}_3 \text{ is } \frac{y - 0}{x - 0} = \frac{y_3 - 0}{x_3 - 0}$$

The equation takes the following form:

$$\overline{P_1P_2} \text{ is } x(y_2 - y_1) + y(x_1 - x_2) = y_1(x_1 - x_2) + x\ (y_2 - y_1)$$

let $(y_2 - y_1) = a_1$

$(x_2 - x_1) = b_1$

$y_1(x_1 - x_2) + x_1(y_2 - y_1) = c_1$

$\overline{OP}$ is $xy_3 - yx_3 = 0$

let $y_3 = a_2$

$-x_3 = b_2$

$0 = c_2$

Solve the equations for the point of intersection:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad X = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} \qquad 1.34$$

$$Y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}$$

Rotate the points of intersection along the PPT edge from 2-spaces back to 3-spaces.

$$x = \lambda_1 x' + \mu_1 y' + \nu_1 z \qquad 1.35$$

$$y = \lambda_2 x' + \mu_2 y' + \nu_2 z$$

$$z = \lambda_3 x' + \mu_3 y' + \nu_3 z$$

where λ, μ, ν, are direction cosines of the X'-axis, Y'-axis, and Z'-axis with respect to the old axis and are found:

$$\lambda_1 = x_1 / \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\mu_1 = y_1 / \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\nu_1 = z_1 / \sqrt{x_1^2 + y_1^2 + z_1^2}$$

λ_2, λ_3; μ_2, μ_3; and ν_2, ν_3 are found similarly.

Retain the co-ordinates along the edges S_1, S_2 and S_3 as shown in Figure 1.64.

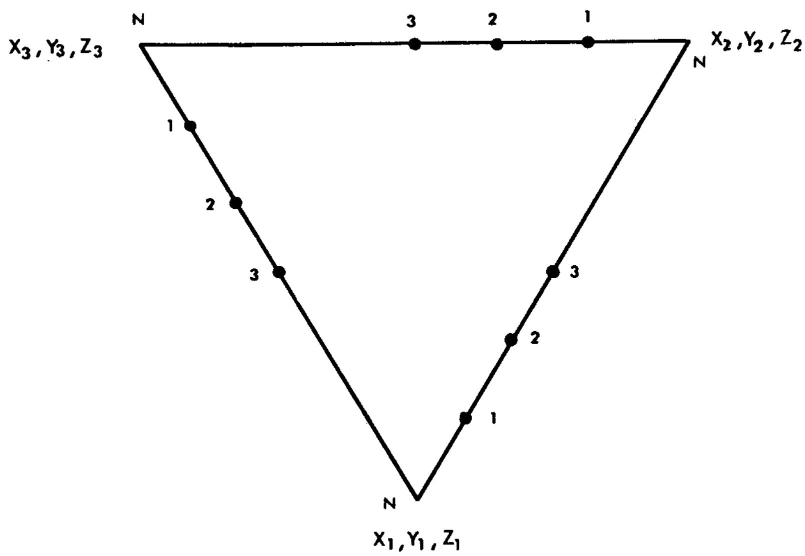


Figure 1.64

After finding the unit divisions along the principal sides of the PPT, the points of subdivison are connected thereby creating a grid network. Since the units along the principal sides are not of equal length, the gridding will create "windows". The centers of these "windows" must be found to establish the three-way grid on the PPT. Figure 1.65.

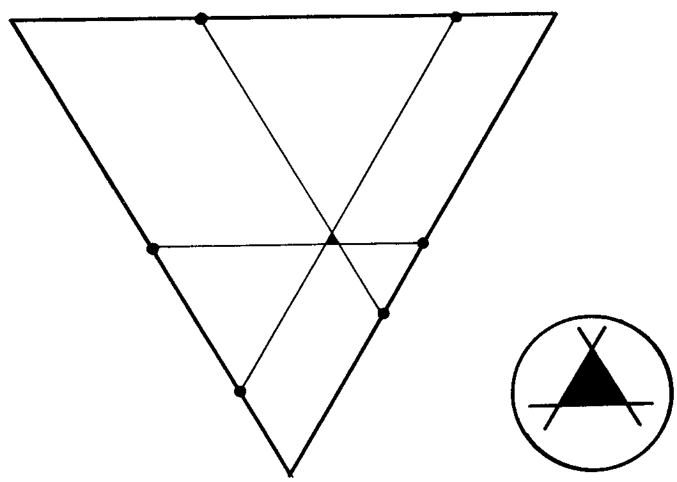


Figure 1.65

The gridding and windows are found by the following method: From the coordinates along the edges of S_1, S_2 and S_3, calculate coordinates of the window by finding the intersection of $\overline{P_1P_2}$ with $\overline{P_3P_4}$ and $\overline{P_1P_2}$ with $\overline{P_5P_6}$ and $\overline{P_3P_4}$ with $\overline{P_5P_6}$ by using the two point form of the equation of a line in three-space for the three lines and solve simultaneously for the points of intersection.

$\overline{P_1P_2}$ is: $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad 1.36$$

$\overline{P_3P_4}$ is: $$\frac{x - x_3}{x_4 - x_3} = \frac{y - y_3}{y_4 - y_3} = \frac{z - z_3}{z_4 - z_3}$$

$\overline{P_5P_6}$ is: $$\frac{x - x_5}{x_6 - x_5} \quad \frac{y - y_5}{y_6 - y_5} \quad \frac{z - z_5}{z_6 - z_5}$$

To find the intersection of $\overline{P_1P_2}$ with $\overline{P_3P_4}$ the equation takes the following form:

(1) $\overline{P_1P_2}$ is: $x(y_2-y_1) + y(x_1-x_2) = y_1(x_1-x_2) + x_1(y_2-y_1)$

(2) $\overline{P_1P_2}$ is: $y(z_2-z_1) + z(y_2-y_1) = z_1(y_1-y_2) + y_1(z_2-z_1)$

(3) $\overline{P_3P_4}$ is: $x(y_4-y_3) + y(x_3-x_4) = y_1(x_3-x_4) + x_1(y_4-y_3)$

(4) $\overline{P_3P_4}$ is: $y(z_4-z_3) + z(y_3-y_4) = z_3(y_3-y_4) + y_3(z_4-z_3)$

For $\overline{P_1P_2}$ let:

$$(y_2-y_1) = a_1$$
$$(x_1-x_2) = b_1$$
$$y_1(x_1-x_2) + x_1(y_2-y_1) = c_1$$

For $\overline{P_3P_4}$ let:

$$(y_4-y_3) = a_2$$
$$(x_3-x_4) = b_2$$
$$y_1(x_3-x_4) + x_1(y_4-y_3) = c_2$$

solve for x and y coordinates of the intersections of $\overline{P_1P_2}$ with $\overline{P_3P_4}$.

Find the z coordinate:

For $\overline{P_1P_2}$ let:

$$(z_2 - z_1) = a_1$$
$$(y_1 - y_2) = b_1$$
$$z_1(y_1 - y_2) + y_1(z_2 - z_3) = c_1$$

For $\overline{P_3P_4}$ let:

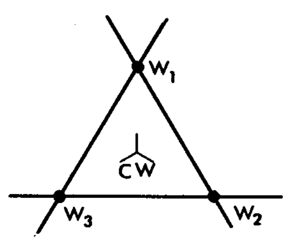
$$(z_4 - z_3) = a_2$$
$$(y_3 - y_4) = b_2$$
$$z_3(y_3 - y_4) + y_3(z_4 - z_3) = c_2$$

The other two vertices of the window are found in a similar manner. Once the coordinates for the vertices of the window are determined, its center is found by the following method:

$$cw = (w_1 + w_2 + w_3)/3$$

where: cw = center of the windows

w = the x, y, or z coordinate of the vertices of the window.



Translation of the Grid for Method 4 & 5

The vertices of the three-way grid are then translated to the surface of the sphere along a line passing through the respective vertex and the origin (0,0,0) of the polyhedron by:

$$x_1' = \frac{rx_1}{d}$$
$$y_1' = \frac{ry_1}{d} \qquad 1.37$$
$$z_1' = \frac{rz_1}{d}$$

Where: $d = \sqrt{x_1^2 + y_1^2 + z_1^2}$

and d = distance from origin to P_1

and r = the radius of the unit sphere

where: $r = 1$

Using the translated coordinates, this program finds the lengths of the elements of the structure (ℓ), the angle between pairs of elements (face angle α), the angle between the elements and a radius from the origin to an endpoint of the element (axial angle Ω), and the angle between adjacent faces of the structure (dihedral angle β), Figure 1.66

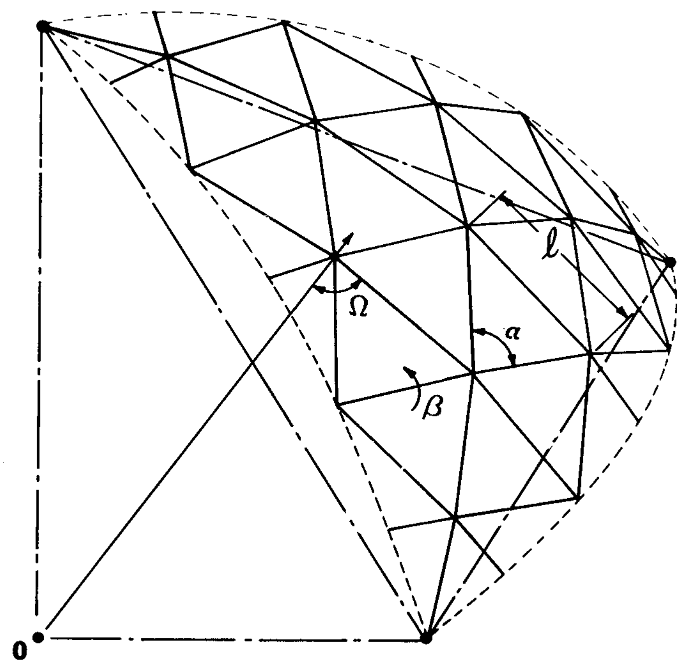


Figure 1.66

To find the angle between elements the face $\sphericalangle\ \alpha$, we use the coordinates of their endpoints. The vertex of the angle is a common endpoint to each element and is translated to the origin. The other two endpoints P_1 and P_2 are translated in the same manner. Letting (x_1, y_1, z_1) and (x_1, y_1, z_1) be the points resulting from the translations of the endpoints P_1 and P_2,

$$\cos \alpha = \left| \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{d_1 d_2} \right| \qquad 1.38$$

$$\text{where } d_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\text{and } d_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$$

α is the desired angle.

To find axial angles the above method is used except that the vertex is established at one end of an element and the origin is used with the other endpoint to define the angle. The desired angle is Ω.

The angle between two adjacent faces, the dihedral $\angle$, β, is found using

$$\cos \beta = \frac{-\left| A_1 A_2 + B_1 B_2 + C_1 C_2 \right|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \qquad 1.39$$

where

β is the desired angle.

$A_1 X + B_1 Y + C_1 Z + D_1 = 0$ defines the plane containing one face and $A_2 X + B_2 Y + C_2 Z + D_2 = 0$ defines the plane containing the other face. The negative sign is used because the obtuse angle is desired.

The A, B, and C for each plane are computed as

$$A = \begin{vmatrix} Y_1 & Z_1 & 1 \\ Y_2 & Z_2 & 1 \\ Y_3 & Z_3 & 1 \end{vmatrix} \qquad 1.40$$

$$B = \begin{vmatrix} X_1 & Z_1 & 1 \\ X_2 & Z_2 & 1 \\ X_3 & Z_3 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix}$$

where (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), and (X_3, Y_3, Z_3) lie in the plane. In particular the three vertices of each face are used.

The length of the elements are found by using the general equation:

$$\ell = \sqrt{(P_{x_1} - P_{x_2})^2 + (P_{y_1} - P_{y_2})^2 + (P_{z_1} - P_{z_2})^2} \qquad 1.41$$

ℓ is the desired length.

To reduce total output, this program takes into account certain symmetries and outputs only a part of the total angles and lengths. The rest of the values are the same as at least one outputed value and can easily be found using the following symmetries, Figure 1.67.

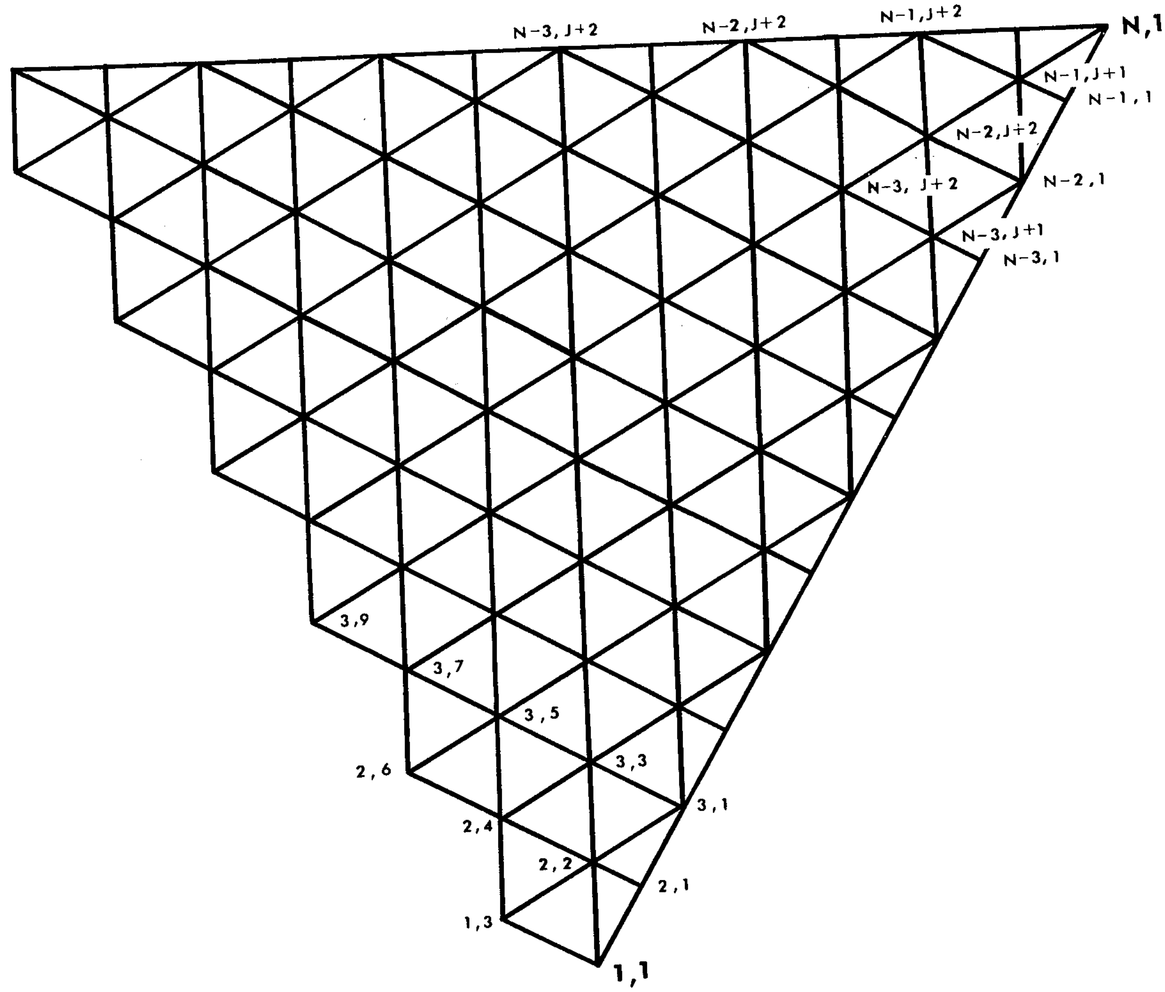

N,1
N-3, J+2
N-2,J+2
N-1,J+2
N-1,J+1
N-1,1
N-2,J+2
N-3, J+2
N-2,1
N-3,J+1
N-3,1
3,9
3,7
3,5
3,3
3,1
2,6
2,4
2,2
2,1
1,3
1,1

Figure 1.67

THE COMPUTER PROGRAM DESCRIBED ON

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1.9 Methods 6 & 7

This mathematical and computer model was written for subdivision of a tetrahedron, octahedron, or icosahedron circumscribed by a unit sphere. The icosahedron was chosen as an example to illustrate the geometry of the model. The polyhedron is oriented in a three dimensional rectangular coordinate system so that the vertices of one PPT are:

$$(x_1, y_1, z_1) = \left(0, \frac{\sqrt{\tau}}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}\sqrt{\tau}}\right)$$

$$\simeq (0, .850651, .525731)$$

$$(x_2, y_2, z_2) = \left(\frac{\sqrt{\tau}}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}\sqrt{\tau}}, 0\right)$$

$$\simeq (.850651, .525731, 0)$$

$$(x_3, y_3, z_3) = \left(\frac{1}{\sqrt[4]{5}\sqrt{\tau}}, 0, \frac{\sqrt{\tau}}{\sqrt[4]{5}}\right)$$

$$\simeq (.525731, 0, .850651)$$

where: $\tau = \dfrac{1 + \sqrt{5}}{2}$

The intersections of the axis X, Y, Z is located at the origin (0,0,0) of the polyhedron. Figure 1.68.

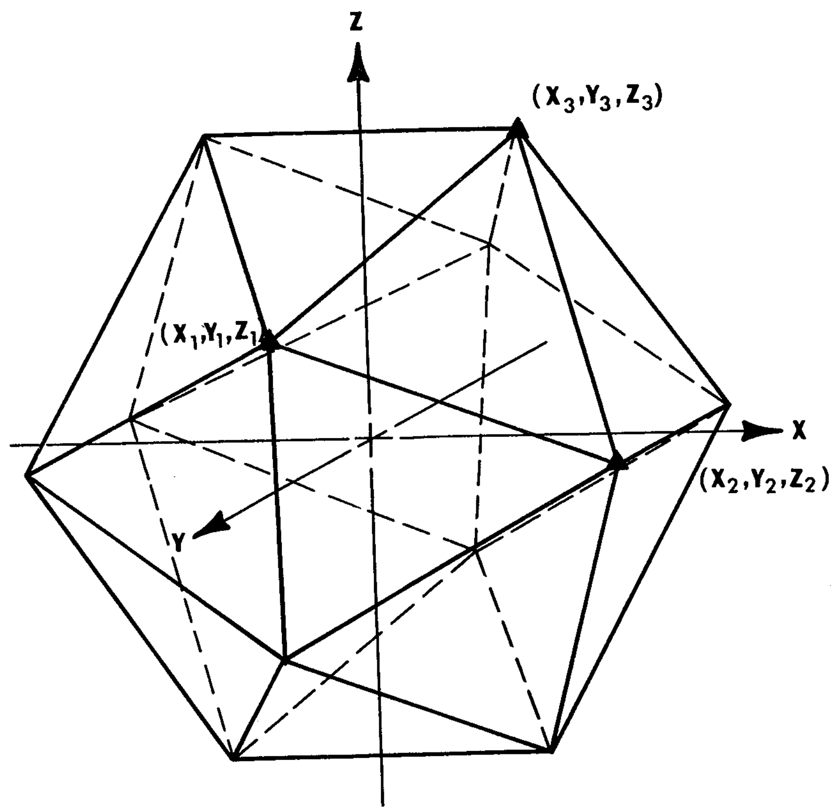


Figure 1.68

The PPT is first translated by the following equations such that the origin lies at the first point

$$x' = x - T_x$$

$$y' = y - T_y \qquad 1.42$$

$$z' = z - T_z$$

where: (T_x, T_y, T_z) is the point in the original coordinate system which will become the new origin.

Following the translation, rotation angles are found which will bring the PPT from three-space into two-space such that one of the PPT sides will be on the x-axis. Rotation angles for the points (0,0,0) (x_1,y_1,z_1) (x_2,y_2,z_2) around a particular axis take the form:

X axis

$$T_x = \text{Arctan}\left\{\frac{[(-x_2\text{Cos}(T_z)-y_2\text{Sin}(T_z)][\text{Sin}(T_y)+z_2\text{Cos}(T_y)]}{-x_2\text{Sin}(T_z)+y_2\text{Cos}(T_z)}\right\}$$

Y axis

$$T_y = \text{Arctan}\left\{\frac{z_1}{x\ \text{Cos}(T_z)+y_1\text{Sin}(T_z)}\right\} \qquad 1.43$$

Z axis

$$T_z = \text{Arctan}\left\{\frac{y_1}{x_1}\right\}$$

The rotation of the PPT may be found by the following equations:

$$x' = x\,\mathrm{Cos}(T_z)\mathrm{Cos}(T_y) + y\,\mathrm{Sin}(T_z)\mathrm{Cos}(T_y) + z\,\mathrm{Sin}(T_y)$$

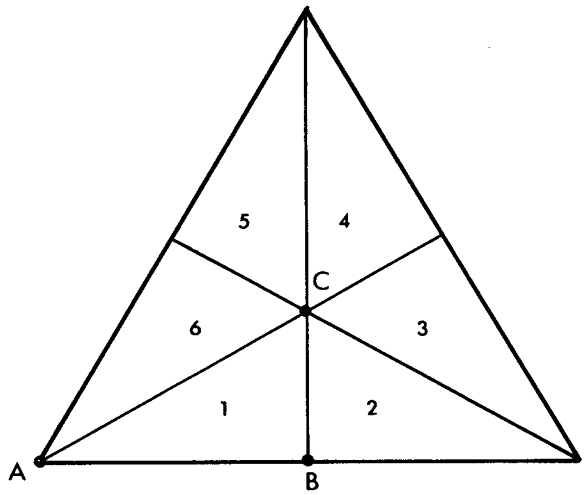
$$\begin{aligned} y' &= x[-\mathrm{Sin}(T_z)\mathrm{Cos}(T_x) - \mathrm{Cos}(T_z)\mathrm{Sin}(T_y)\mathrm{Sin}(T_x)] + y[\mathrm{Cos}(T_z) \\ &\quad \mathrm{Cos}(T_x) - \mathrm{Sin}(T_z)\mathrm{Sin}(T_y)\mathrm{Sin}(T_x)] + z\,\mathrm{Cos}(T_y)\mathrm{Sin}(T_x) \end{aligned} \qquad 1.44$$

$$\begin{aligned} z' &= x[\mathrm{Sin}(T_z)\mathrm{Sin}(T_x) - \mathrm{Cos}(T_z)\mathrm{Sin}(T_y)\mathrm{Cos}(T_x)] + y[-\mathrm{Cos}(T_z) \\ &\quad \mathrm{Sin}(T_x) - \mathrm{Sin}(T_z)\mathrm{Sin}(T_y)\mathrm{Cos}(T_x)] + Z\,\mathrm{Cos}(T_y)\mathrm{Cos}(T_x) \end{aligned}$$

Through matrix multiplication, the equation takes the following form:

$$(x', y', z') = (x,y,z) \begin{vmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}(T_x) & -\mathrm{Sin}(T_x) \\ 0 & \mathrm{Sin}(T_x) & \mathrm{Cos}(T_x) \end{vmatrix} \bullet \begin{vmatrix} \mathrm{Cos}(T_y) & 0 & -\mathrm{Sin}(T_y) \\ 0 & 1 & 0 \\ \mathrm{Sin}(T_y) & 0 & \mathrm{Cos}(T_y) \end{vmatrix} \bullet \begin{vmatrix} \mathrm{Cos}(T_z) & -\mathrm{Sin}(T_z) & 0 \\ \mathrm{Sin}(T_z) & \mathrm{Cos}(T_z) & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad 1.45$$

The PPT may be described as six right triangles, each being a reflection of the other. In these methods of subdivision, only triangle ABC is subdivided. The remaining sections of the PPT are found through rotations and reflections of the basic unit.



Note: ABC is a right triangle

Figure 1.69

Subdivision for Method 6

The Line AB is subdivied into parts chosen as equal are divisions of the central angle of the polyhedron by the following equation. Figure 1.70.

$$\phi = 2\left[\text{Arcsin}\ \left(\sqrt{(x_2-x_1)^2 + (y_2-y_1)^{2}/2}\right)\right]$$

$$x = \frac{2_2 - x_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}\ \text{Sin}\left(\frac{I\Theta}{N}\right) \qquad 1.46$$

$$y = \frac{y_2-y_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}\ \text{Sin}\left(\frac{I\Theta}{N}\right)$$

where: $\Theta = {}^1/_2\ \delta$

δ = the central angle of the polyhedron

$0 \le I \le N$

(x_1, y_1) and (x_2, y_2) represent any two points on the PPT edge

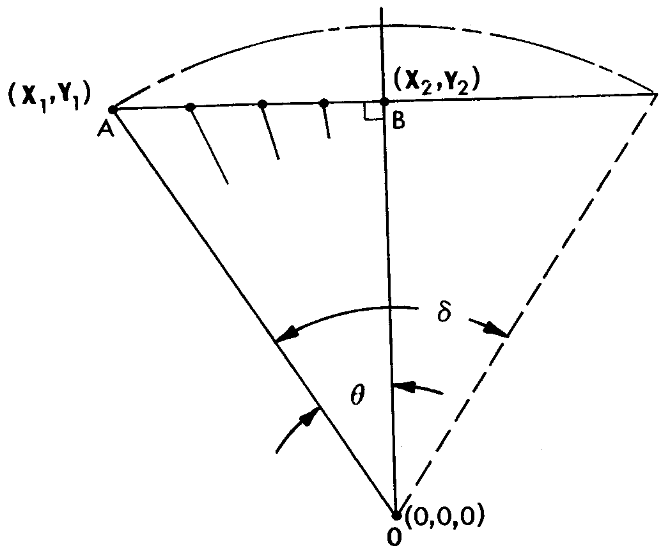


Figure 1.70

Subdivision for Method 7

The Line AC is subdivided into parts chose as equal arc divisions of an angle made up of the triangle AC and the origin of the polyhedron with the origin (0, 0, 0) being the center of the triangle of subdivision. The following equations are used for this subdivision. Figure 1.71

$$\Delta = \text{Arcsin}\left[\,{}^{2}/_{3}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\,\right]$$

$$x = \frac{2\ (x_2-x_1)}{3\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\ \text{Sin}\left(\frac{I\,\Delta}{N}\right) \qquad 1.47$$

$$y = \frac{2\ (y_2-y_1)}{3\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\ \text{Sin}\left(\frac{I\,\Delta}{N}\right)$$

where: Δ = the angle between $\overline{AO}$ and $\overline{OC}$

$0 \leq I \leq N$

(x_1, y_1) and (x_2, y_2) represent any two points on the PPT face bisector.

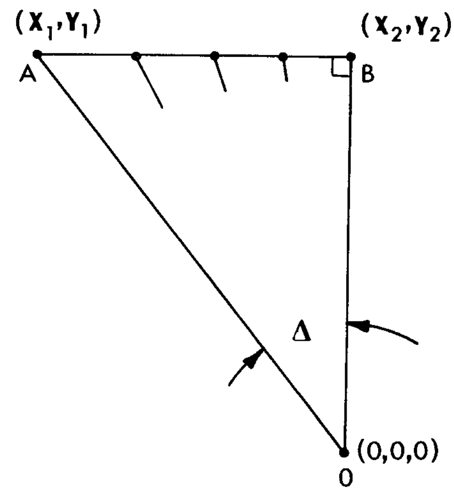


Figure 1.71

Gridding & Projection for Methods 6 & 7

The points of subdivision for methods 6 and 7 are stored in a matrix and are used in the gridding process for each respective method. Figure 1.72

Matrix PT(A,B,C)

sphere: A = the type of point

1-on the edge of triangle

2-on bisector of angle

3-on internal points of triangle

4-external points

B = line value and x, y, z values

1-3 represents x, y, z values of line 1

4-6 represents x, y, z values of line 2

7-9 represents x, y, z values of line 3

C = the number of points on the line.

$A \in \{1,2,3\}$

$B \in \{4,5,6\}$

$C \in \{7,8,9\}$

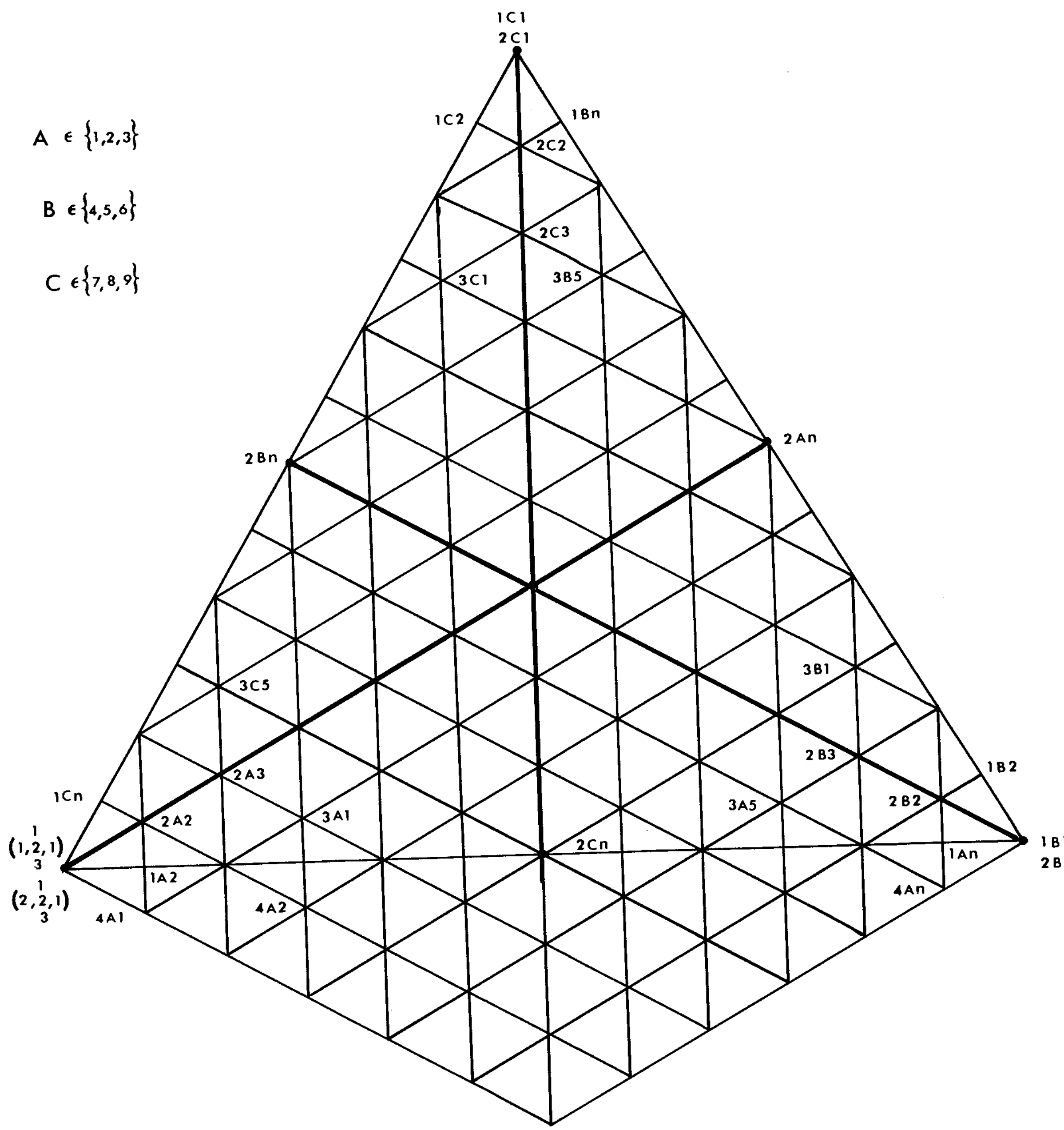


Figure 1.72

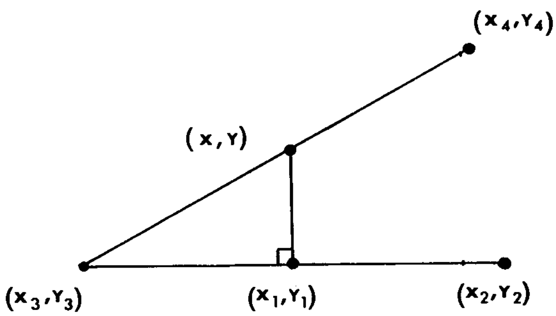
Once the subdivisions are found, they are used to find points of divisions on the other two sides of the right triangle by the following equations. Figures 1.73 & 1.74.

$$x = \frac{(Y_1 + x_1\ [M_{21}]) \ - \ ({}^{1}/_{3} - x_3\ [M_{34}])}{M_{34} - M_{21}}$$

$$y = x(M_{21}) + y_1 - x_1(M_{21})$$

where: $M_{21} = \frac{-x_2 - x_1}{y_2 - y_1}$ (neg. reciprocal of slope) 1.48

$M_{34} = \frac{y_3 - y_4}{x_3 - x_2}$ (slope)



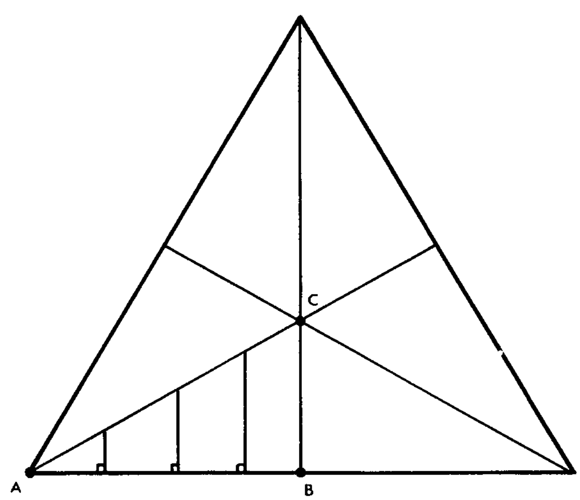


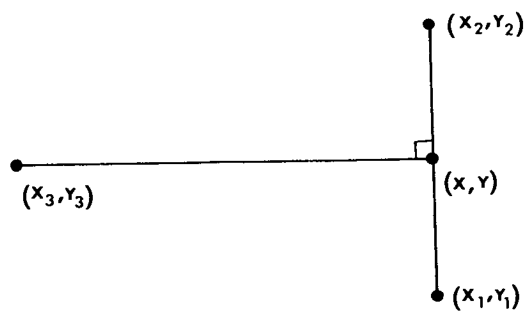
Figure 1.73

and: $$x = \frac{(y_3 + M_1 x_3) - (y_1 + M_2 x_1)}{(M_2 - M_1)} \qquad 1.49$$

$$y = xM_1 + y_3 + x_3 M_1$$

where: $M_1 = -\left(\frac{1}{M_2}\right)$ (neg. reciprocal of slope)

$M_2 = \frac{y_2 - y_1}{x_2 - x_1}$ (slope)



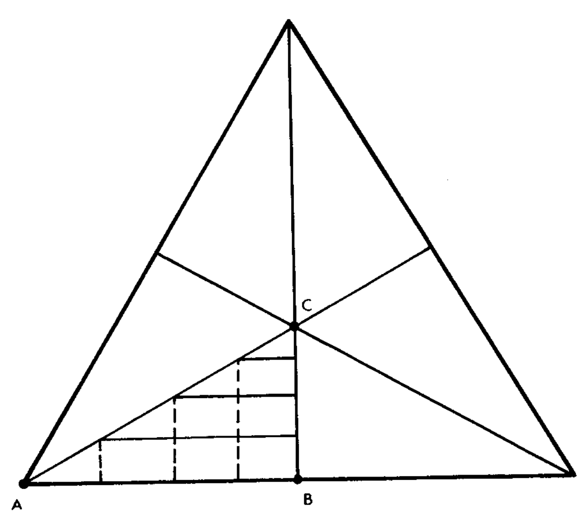


Figure 1.74

Reflection of points in two-space are found by:
Figure 1.75.

$$x_4 = \frac{\left[(y_3 + M_1 x_3) - (y_1 + M_2 x_1)\right]}{(M_2 - M_1)}$$

$$y_4 = x_4 M_1 + {}^1/_3 + x_3 M_1$$

where $M_1 = - {}^1/M_2$ 1.50

$$M_2 = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

and the reflected points are:

$$x = x_1 + 2\,(x_4 - x_1)$$

$$y = y_1 + 2\,(y_4 - y_1)$$

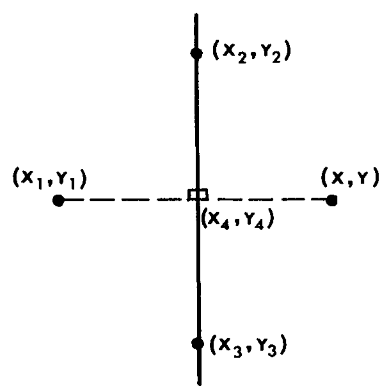


Figure 1.75

Rotations of points in two-space are found by:
Figure 1.76.

$$x = (x_2 - x_1)\,\mathrm{Cos}\,\Phi - (y_2 - y_1)\,\mathrm{Sin}\,\Phi + x_1$$

$$y = (x_2 - x_1)\,\mathrm{Sin}\,\Phi + (y_2 - y_1)\,\mathrm{Cos}\,\Phi + y_1$$ 1.51

in matrix form

$$(x,y) \begin{vmatrix} \text{Cos } \Phi & -\text{ Sin } \Phi \\ \text{Sin } \Phi & \text{Cos } \Phi \end{vmatrix}$$

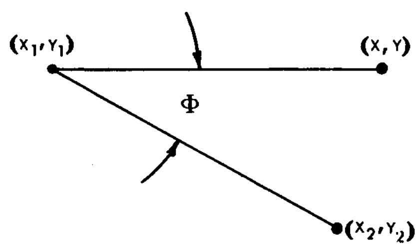


Figure 1.76

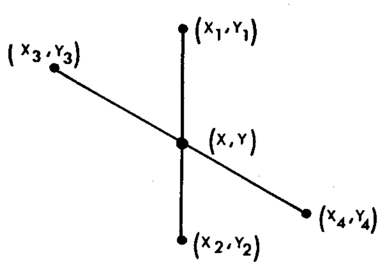
The points for the internal gridding are found by the equations: Figure 1.77.

$$x = \frac{(B_2C_1 - B_1C_2)}{(A_2B_1 - A_1B_2)} \qquad 1.52$$

$$y = \frac{(A_2 C_1 - A_1 C_2)}{(A_1 B_2 - B_1 A_2)}$$

where:

$$A_1 = y_2 - y_1$$
$$B_1 = x_1 - x_2$$
$$C_1 = x_1 (y_1 - y_2) + y_1 (x_2 - x_1)$$
$$A_2 = y_4 - y_3$$
$$B_2 = x_3 - x_4$$
$$C_2 = x_3 (y_3 - y_4) + y_3 (x_4 - x_3)$$



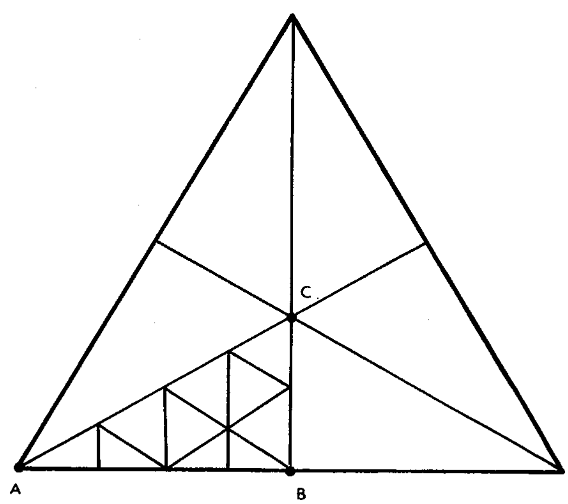


Figure 1.77

Through rotations and reflections (equations 1.50, 1.51) of the basic unit, the entire PPT three-way grid is found. Figure 1.78

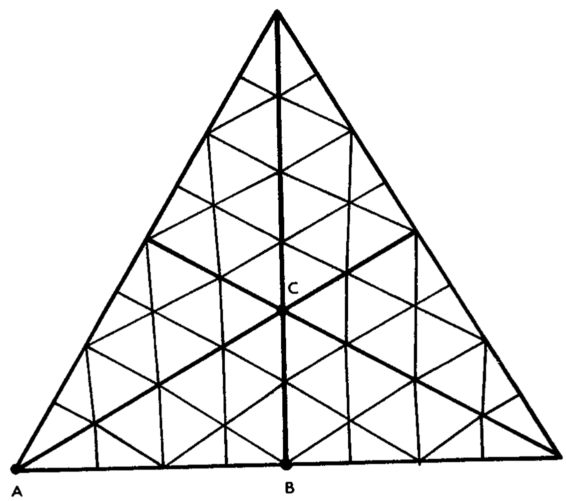


Figure 1.78

The external points of the PPT are found by equation 1.52 and are rotated into their respective plane by equation 1.44 where: the angles of rotation are found by equation 1.43. The external points are rotated by equation 1.51 to their respective positions.

All points of the three-way grid are then rotated into three-space using equation 1.45

where:
$$T_z = 2\Pi - T_z$$
$$T_y = 2\Pi - T_y \qquad 1.53$$
$$T_x = 2\Pi - T_x$$

are the angles of rotation.

The origin is then retranslated to its original position by:

$$x' = x + T_x$$
$$y' = y + T_y \qquad 1.54$$
$$z' = z + T_z$$

where: (T_x, T_y, T_z)

the coordinates to which the origin was originally translated to.

All points of the three-way grid are then projected to the surface of the sphere by: Figure 1.79

$$x' = \frac{x}{Dis}$$
$$y' = \frac{y}{Dis} \qquad 1.54$$
$$z' = \frac{z}{Dis}$$

where: $Dis = \sqrt{(x)^2 + (y)^2 + (z)^2}$

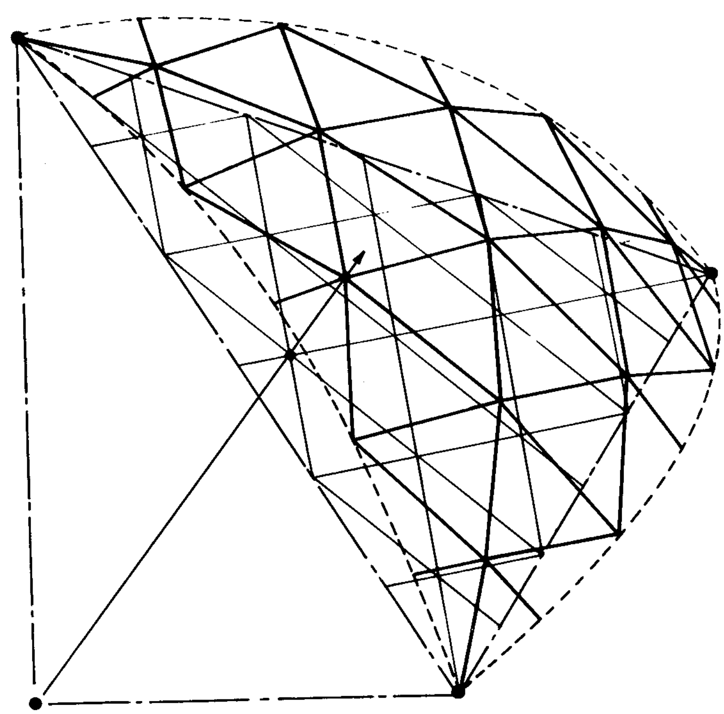


Figure 1.79

For the PPT, the number of:

$$\text{edges} = \frac{3\nu(3\nu - 2)}{8}$$

$$\text{half edges} = \frac{3\nu}{2}$$

$$\text{faces} = \frac{3\nu(\nu - 2)}{4} \qquad 1.55$$

$$\text{half faces} = 3\nu$$

$$\text{vertices} = \frac{3\nu(\nu + 2)}{8} + 1$$

where: ν = frequency and must be even

For the total spherical form, the number of:

$$\text{Edges} = 3\,\frac{E\nu^2}{4}$$

$$\text{Faces} = \frac{E\nu^2}{2}$$

$$\text{Vertices} = \frac{E\nu^2}{4} + 2$$

where: E = no. of edges in polyhedral unit.

Using the coordinates, the lengths of the elements of the structure (ℓ), the angle between pairs of elements (face angle α), the angle between the elements and a radius from the origin to an endpoint of the element (axial angle Ω), and the angle between adjacent faces of the structure (dihedral angle β), are calculated. Figure 1.80

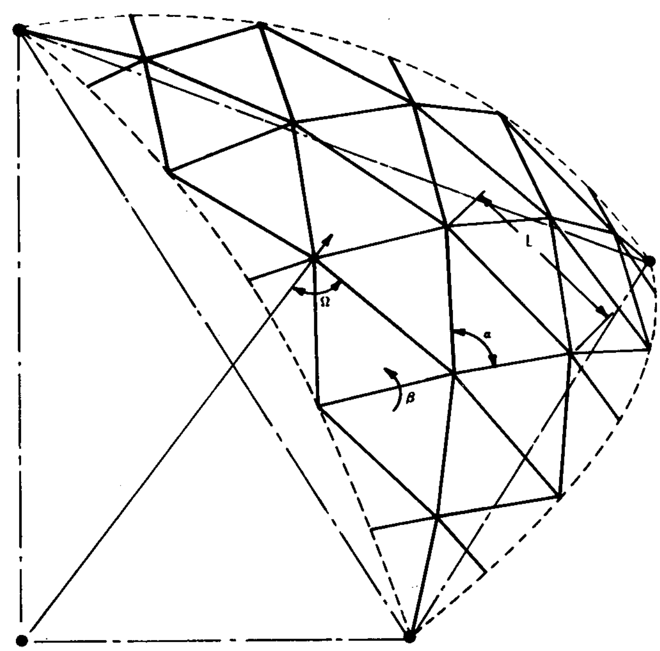


Figure 1.80

To find the angle between elements, the face $\measuredangle\ \alpha$, we use the coordinates of their endpoints. The vertex of the angle is a common endpoint to each element and is translated to the origin. The other two endpoints, P_1 and P_2 are translated in the same manner. Letting (x_1, y_1, z_1) and (x_2, y_2, z_2) be the points resulting from the translations of the end-points P_1 and P_2,

$$\cos \alpha = \left| \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{d_1 d_2} \right|$$

$$\text{where} \quad d_1 = \sqrt{x_1^2 + y_1^2 + z_1^2} \qquad 1.57$$

and $\quad d_2 = \sqrt{x_2{}^2 + y_2{}^2 + z_2{}^2}$

α is the desired angle.

To find axial angles, the above method is used except that the vertex is established at one end of an element and the origin is used with the other endpoint to define the angle. The desired angle is Ω.

The angle between two adjacent faces, the dihedral $\measuredangle\beta$, is found using

$$\cos \beta = \frac{-\left|A_1A_2 + B_1B_2 + C_1C_2\right|}{\sqrt{A_1{}^2 + B_1{}^2 + C_1{}^2}\ \sqrt{A_2{}^2 + B_2{}^2 + C_2{}^2}} \qquad 1.58$$

where

β is the desired angle.

$A_1X + B_1Y + C_1Z + D_1 = 0$ defines the plane containing one face and

$A_2X + B_2Y + C_2Z + D_2 = 0$ defines the plane containing the other face.

The negative sign is used because the obtuse angle is desired.

The A, B, and C for each plane are computed as

$$A = \begin{vmatrix} Y_1 & Z_1 & 1 \\ Y_2 & Z_2 & 1 \\ Y_3 & Z_3 & 1 \end{vmatrix}$$

$$B = \begin{vmatrix} X_1 & Z_1 & 1 \\ X_2 & Z_2 & 1 \\ X_3 & Z_3 & 1 \end{vmatrix} \qquad 1.59$$

$$C = \begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix}$$

where (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), and (X_3, Y_3, Z_3) lie in the plane. In particular, the three vertices of each face are used.

The length of the elements ℓ are found by using the general equation:

$$\ell = \sqrt{(P_{x_1} - P_{x_2})^2 + (P_{y_2} - P_{y_2})^2 + (P_{z_3} - P_{z_3})^2} \qquad 1.60$$

ℓ is the desired length

At the time of publication, the computer programs for these methods had not been completed and therefore, have not been included in this report.