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APPLICATION OF VARIOUS ELASTIC THIN SHELL THEORIES TO BLOOD FLOW PROBLEMS*

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Numerous theories for the analysis of thin-walled shells have been developed primarily for the solution of stress and stability problems that arise in the domain of solid mechanics. In this paper some of the existing theories are reviewed to ascertain their influence on the computation of phase velocities in fluid filled cylinders representing certain aspects of the behavior of arteries and veins in vivo. For physiologically meaningful parameters, including moderately large in plane prestrain that occurs in mammals, the results suggest that with one exception, the small differences in the formulations exercise little influence on the phase velocities. However, it is demonstrated that inclusion of the forces induced by the rotation of the hydrostatic pressure is essential or significantly erroneous torsional wave speeds result. The relevant term is often ignored in the literature since it is of no importance in many applications. Also the introduction of moderate implane prestrains that are present in living mammals is shown to lead to nonselfadjoint differential equations of motion, whose biorthogonal eigenvectors differ slightly from each other. While of theoretical interest this extension to the shell theories usually used in the solution of solid mechanics shell problems is not particularly important for the biomedical applications considered.

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INTRODUCTION

In many situations involving the dynamic behavior of blood vessels it is essential to account for the significant prestresses to which the vessels are subjected. Also, it is clearly established that the constitutive law of any tissue is very complex e.g., Fung [1]. In addition, King and Lawton [2], Anliker et al. [3], Rockwell et al. [4] have demonstrated that the elastic response of the vessel to the natural cardiac pulse can be significantly nonlinear. However, for a number of investigations such as those devoted to determination of the wall material properties, we are primarily concerned with the behavior of the vessel as it responds in a small neighborhood of a quasi-static prestressed state. In the experiments the wall material properties are deduced by measuring the amplitude and phase differences for mechanically induced high frequency waves propagating down the vessel, as described in [3], for example. For such cases the perturbation strains of interest are small and it is reasonable, at least as a first approximation, to treat the vessel wall material as perfectly elastic and consider the prestressed state as given. Once this is done the equations of elastic shell theory become applicable to blood vessels.

In reviewing the well-known literature on prestressed shells, e.g. [5] to [14], it is immediately obvious , as in many other fields of shell analysis, that the equations used by the various authors do not agree in all respects. It is appropriate to study whether such differences have any significant influence on the phase velocities in simplified models of veins and arteries. Another subject of considerable interest is the fact that unlike metal structures, the inplane strains induced by the prestress in the primary arteries are not small compared to unity. Furthermore Biot showed in [14] and reviewed in his book [15] the fact that the stress-strain law for the incremental deformations , relative to the prestress state is , in rectangular cartesian coordinates, non-symmetric, for prestress states which are not hydrostatic. The same conclusion is reached by Pflüger [16]. Therefore it is desirable to review the cylindrical elastic shell equations for linearized perturbations about a given prestressed state, in an attempt to illustrate some of the questions raised. Since many of the points to be made can be illustrated using cylinders of circular cross section we shall take advantage of the algebraic simplicity of having the second Lamé coefficient equal to the constant radius of the shell, in certain investigations while retaining general cross sections in others.

Bolotin's Dynamic Stability Equations for a Circular Cylinder

Since the dynamic stability of shells has been a research topic of considerable interest over the past decade and is likely to remain so for many years, it is tempting to utilize the equations presented in a well known text, namely [8] in which Bolotin considers the dynamic stability of circular cylinders and gives the differential equations describing small perturbations about a prestressed state:

$$[L_{ij}] \begin{vmatrix} u \\ v \\ w \end{vmatrix} + \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = 0 \qquad (1)$$

in which, after correcting some typographical errors we have:

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{(1-\nu)}{2R^2} \frac{\partial^2}{\partial \phi^2}$$

$$L_{12} = L_{21} = \frac{(1+\nu)}{2R} \frac{\partial^2}{\partial x \partial \phi} + \frac{(N_x^o - N_\phi^o)}{R} \frac{\partial^2}{\partial x \partial \phi}$$

$$L_{13} = -\frac{\nu}{R} \frac{\partial}{\partial x} - \frac{(N_x^o - N_\phi^o)}{R} \frac{\partial}{\partial x}$$

$$L_{22} = \frac{1}{R^2} \frac{\partial^2}{\partial \phi^2} + \frac{(1-\nu)}{2} \frac{\partial^2}{\partial x^2}$$

$$L_{23} = -L_{32} = -\frac{1}{R^2} \frac{\partial}{\partial \phi}$$

$$L_{31} = \frac{\nu}{R}\frac{\partial}{\partial x}$$

$$L_{33} = -d^2 \nabla^2 \nabla^2 + N_x^o \frac{\partial^2}{\partial x^2} + \frac{N_\phi^o}{R^2}\left(\frac{\partial^2}{\partial \phi^2} - 1\right)$$

and X, Y and Z are the applied midsurface loads per unit area. One of the most interesting aspects of these operators is that L_{13} is not the adjoint of L_{31} except when $N_x^o = N_\phi^o$ in which case the only prestress term appears in operator L_{33}.

Reviewing Bolotin's derivation we see that it is based on the determination of certain "reduced (membrane) loads". These are obtained by writing the linear membrane equations in terms of the Lamé coefficients and then replacing these by their values in the deformed state to obtain a set of geometrically non-linear (large displacement) membrane equations. This technique does not produce the same large displacement membrane equations as those described by most other authors. Also, Bolotin's equations exhibit non-self adjointness in one of the operator pairs even though his theory is based on the assumption of small strains. It is well known that all linear differential equations derivable from a variational principle must be self adjoint. Hence, his equations for the incremental reduced loads should be scrutinized. The basic idea of using the large deflection membrane equations to obtain the membrane prestress terms is valid, but obtaining these equations by replacing, in the linear equations, A by $A(1 + \ell_{11})$ and B by $B(1 + \ell_{22})$ is not wholly satisfactory as we shall now demonstrate.

If we restrict ourselves to $N_{\alpha\beta}^o = 0$ for this purpose, Washizu's [7] membrane equations for cylinders with $A = 1$ and $R_\alpha^{-1} = 0$, can be written as:

$$\frac{\partial}{\partial \alpha}\{N_\alpha(1 + \ell_{11})\} + \frac{1}{B}\frac{\partial}{\partial \beta}(N_\beta \ell_{12}) + X = 0$$

$$\frac{1}{B}\frac{\partial}{\partial \beta}\left[N_\beta(1 + \ell_{22})\right] + \frac{\partial}{\partial \alpha}(N_\alpha\, \ell_{21}) - \frac{N_\beta\, \ell_{32}}{R_\beta} + Y = 0 \qquad (2)$$

$$\frac{\partial}{\partial \alpha}(N_\alpha\, \ell_{31}) + \frac{1}{B}\frac{\partial}{\partial \beta}(N_\beta\, \ell_{32}) + \frac{N_\beta}{R_\beta}(1 + \ell_{22}) + Z = 0$$

where the ℓ_{ij} are the linear strains and rotations, namely

$$\ell_{11} = \partial u/\partial\alpha; \quad \ell_{12} = (1/B)\ \partial u/\partial\beta; \quad \ell_{21} = \partial v/\partial\alpha$$

$$\ell_{22} = (1/B)\ \partial v/\partial\beta - w/R_\beta; \qquad \ell_{31} = \omega_\beta = \partial w/\partial\alpha$$

$$\ell_{32} = \omega_\alpha = v/R_\beta + (1/B)\ \partial w/\partial\alpha.$$

The technique for obtaining the equations for the incremental or "reduced" forces due to the prestress (labelled by some authors as variational equations) is to express each dependent variable as the sum of the value corresponding to the zeroth (basic membrane) state plus a small perturbation. Then we make use of the fact that the shell is in equilibrium in the zeroth state and retain only those terms that are of the first power in the perturbation quantities. To simplify the algebra without deleting anything significant for our demonstration, let us restrict ourselves to cylinders in which $B = R_\beta$, $N^o_{\alpha\beta} = 0$ and the membrane prestress resultants, N^o_α and N^o_β are constants. Then we have

$$N^o_\alpha \frac{\partial^2 u}{\partial \alpha^2} + \frac{N^o_\beta}{R_\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{R_\beta}\frac{\partial u}{\partial \beta}\right) + X' = 0$$

$$\frac{N^o_\beta}{R_\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\frac{\partial v}{\partial\beta}\right) + N^o_\alpha\frac{\partial^2 v}{\partial\alpha^2} - \frac{N^o_\beta v}{R^2_\beta} - \frac{N^o_\beta}{R_\beta}\left\{\frac{\partial}{\partial\beta}\left(\frac{w}{R_\beta}\right) + \frac{1}{R_\beta}\frac{\partial w}{\partial\beta}\right\} + Y' = 0 \qquad (3)$$

$$\frac{N^o_\beta}{R_\beta}\left\{\frac{\partial}{\partial\beta}\left(\frac{v}{R_\beta}\right) + \frac{1}{R_\beta}\frac{\partial v}{\partial\beta}\right\} + N^o_\alpha\frac{\partial^2 w}{\partial\alpha^2} + \frac{N^o_\beta}{R_\beta}\left\{\frac{\partial}{\partial\beta}\left(\frac{1}{R_\beta}\frac{\partial w}{\partial\beta} - \frac{w}{R_\beta}\right)\right\} + Z' = 0\ .$$

These equations are self adjoint in contrast to those given by Bolotin.

While this review of Bolotin's equations may be on interest, it does not help explain the differences in the equations presented by other authors, some of whom have been mentioned previously. Many have used the principles of virtual work or minimum potential energy, but the results depend on the strain displacement relations utilized. For theories applicable primarily to metal shells, the neglect of the nonlinear terms involving the inplane displacements in these relations, which are not valid in general, has been proven permissible in numerous applications. However, we shall retain these terms to study their effects in the case of blood vessels and present Washizu's theory [7] for a cylinder of general cross section.

Equations for Cylindrical Shells Under Initial Stress when the Strains are Small Compared to Unity

Let us now restrict our attention to those sets of equations in which the strains are considered small compared to unity, while retaining all terms involving the prestress resultants, but relax the assumption that the cross section is circular. Once this particular small strain assumption is invoked, Washizu's equations can be utilized directly and the matrix of differential operators for a membrane prestress state in which the axial and circumferential stress are constant and the inplane shear stress in zero, become

$$L_{11} = \frac{\partial^2}{\partial\alpha^2} + \frac{(1-\nu)}{2B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + N_\alpha^o\frac{\partial^2}{\partial\alpha^2} + \frac{N_\beta^o}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right)$$

$$L_{12} = L_{21} = \frac{(1+\nu)}{2B}\frac{\partial^2}{\partial\alpha\partial\beta}$$

$$L_{31} = -L_{13} = \frac{\nu}{R_\beta}\frac{\partial}{\partial\alpha} \qquad (4)$$

$$L_{22} = \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + \frac{(1-\nu)}{2}\frac{\partial^2}{\partial\alpha^2} - \frac{N^o_\beta}{R^2_\beta} + \frac{N^o_\beta}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + N^o_\alpha\frac{\partial^2}{\partial\alpha^2}$$

$$+ \frac{h^2}{12R^2_\beta}\left\{2(1-\nu)\frac{\partial^2}{\partial\alpha^2} + \frac{R_\beta}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{\cdot}{R_\beta}\right)\right)\right\}$$

$$L_{23} = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{\cdot}{R_\beta}\right) - \frac{2N^o_\beta}{R_\beta B}\frac{\partial}{\partial\beta} + \frac{h^2}{12R_\beta}\left\{\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right)\right) + \frac{(2-\nu)}{B}\frac{\partial^3}{\partial\alpha^2\partial\beta}\right\}$$

$$L_{32} = \frac{1}{R_\beta B}\frac{\partial}{\partial\beta} + \frac{2N^o_\beta}{B}\frac{\partial}{\partial\beta}\left(\frac{\cdot}{R_\beta}\right) - \frac{h^2}{12B}\left\{\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{\cdot}{R_\beta}\right)\right)\right)\right. \tag{4}$$

$$\left. + (2-\nu)\frac{\partial}{\partial\beta}\left(\frac{1}{R_\beta}\frac{\partial^2}{\partial\alpha^2}\right)\right\}$$

$$L_{33} = -\frac{\cdot}{R^2_\beta} + N^o_\alpha\frac{\partial^2}{\partial\alpha^2} + \frac{N^o_\beta}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) - \frac{N^o_\beta}{R^2_\beta} - \frac{h^2}{12}\nabla^2\nabla^2$$

As we would expect, the prestress terms are the same as those in (3). The operators L_{23} and L_{32} differ by more than the sign, but recalling the definition of a self adjoint differential operator it can be shown that they are the adjoints of each other. For two dimensional systems, the adjoint operator of $\partial\{a(\alpha,\beta)\, u(\alpha,\beta)\}/\partial\alpha$, where $a(\alpha,\beta)$ is a coefficient of the independent variables only, is by definition $(-1)\, a(\alpha,\beta)\, \partial u(\alpha,\beta)/\partial\alpha$ and similarly for higher order derivatives. We note that since the cylinders of concern here are developable surfaces it is always possible to choose coordinates such that both Lamé parameters are unity, i.e., we can replace $Bd\beta$ by ds wherever convenient. This is particularly advantageous when demonstrating that the operators are self adjoint. For a circular cross section, the operators of (4) can be easily simplified by putting $\alpha = x$, $\beta = \phi$ and $R_\beta = B = R$.

The prestress terms in these operators are then precisely those obtained by Aremenekas and Herrmann [17]. However, when the circumferential prestress is due to hydrostatic pressure, these authors show that certain prestress terms are cancelled by force components that arise from the rotation of the applied pressure. This fact is particularly important as we shall demonstrate later, but is not always mentioned in the literature.

It is of interest to note that Flügge's equations, as used by Anliker and Maxwell [18] are identical with those of Herrmann and Armenakas [8] and the succeeding papers based on this work; if the same assumption is made concerning the application of the prestresses. In [18] the authors consider that after the axial prestress is applied, axial expansion of the cylinder is prevented while the cylinder is pressurized. This yields an axial prestress resultant of $N_x^o + \nu N_\phi^o$, rather than N_x^o if the shell is not restrained. We allow the vessel to be unrestrained. In our notation the corresponding self adjoint operators for the theories based on Flügge [5] and Herrmann-Armenakas [8] are

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{(1-\nu)}{2R^2}(1+d^2)\frac{\partial^2}{\partial\phi^2} + N_x^o\frac{\partial^2}{\partial x^2} + \frac{N_\phi^o}{R^2}\frac{\partial^2}{\partial\phi^2}$$

$$L_{12} = L_{21} = \frac{(1+\nu)}{2R}\frac{\partial^2}{\partial x\partial\phi}$$

$$L_{31} = -L_{13} = \frac{\nu}{R}\frac{\partial}{\partial x} - \frac{d^2}{R}\left\{\frac{\partial^3}{\partial x^3} - \frac{(1-\nu)\partial^3}{2R^2\,\partial x\partial\phi^2}\right\} - \frac{N_\phi^o}{R}\frac{\partial}{\partial x} \tag{5}$$

$$L_{22} = \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} + \frac{(1-\nu)}{2}(1+d^2)\frac{\partial^2}{\partial x^2} + N_x^o\frac{\partial^2}{\partial x^2} + \frac{N_\phi^o}{R^2}\frac{\partial^2}{\partial\phi^2}$$

$$L_{32} = -L_{23} = +\frac{1}{R}\frac{\partial}{\partial\phi} - \frac{d^2(3-\nu)}{2}\frac{\partial^3}{\partial x^2\,\partial\phi} - \frac{N_\phi^o}{R^2}\frac{\partial}{\partial\phi}$$

$$L_{33} = -\frac{1}{R^2} - d^2\left(R^2\nabla^2\nabla^2 + \frac{2}{R^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{R^2}\right) - N_x^o\frac{\partial^2}{\partial x^2} + \frac{N_\phi^o}{R^2}\frac{\partial^2}{\partial\phi^2} \tag{5}$$

The differences btween (4) and (5) arise from two different causes. Firstly in (5) the equilibrium equations in terms of the stress resultants, the last terms in the operators L_{13} and L_{31} do not occur in Washizu's derivation. However they have $-N^o_\phi(v + \partial w/\partial\phi)/R^2$ in the second equation and $-N^o_\phi(w + \partial v/\partial\phi)/R^2$ in the third that do not appear in (5). These discrepancies will be shown later to be caused by neglect of the changes of the hydrostatic pressure force component that occurs when the shell undergoes perturbation rotations and midsurface strains. Secondly the other differences that do not involve the prestress terms are due to Flügge [5] and Herrmann-Armenakas [8] using, for the definitions of the stress resultants expressions such as

$$N_x = \int_{-h/2}^{h/2} \sigma_x(1 + \frac{z}{R})\, dz$$

while in essentially all other works, z/R is neglected compared to unity in the expressions for the stress resultants. These terms are recognizable in (5) as those containing d^2 in all operators except L_{23}, L_{32}, and L_{33}.

In spite of the efforts of many investigators for decades, it is only recently that a generally accepted "correct" set of equations based on the Kirchoff-Love hypothesis which describe the behavior of prestressed thin shells. For many investigations, the assumption of inplane displacements that are small relative to that in the transverse direction has frequently been invoked to remove certain nonlinear terms in u and v from the strain displacement relations. While controversy over the linear formulation has been very pronounced until recently, there is as yet apparently no complete agreement on the equations that should be used in the general nonlinear problem. There is no intent here to devise a new approach to the overall problem but rather to review existing knowledge to explain certain facets of the behavior of prestressed cylindrical shells of general cross section as applied to certain problems in biomechanics while retaining the assumption of small strains. As an example let us consider the equations of Sanders [10], which allow for the rotation of the shell element about the normal to the middle surface. With this degree of complexity we have, for the strain-displacement relations

$$
\begin{aligned}
\epsilon_\alpha &= \frac{\partial u}{\partial \alpha} + \frac{1}{2}\left(\frac{\partial w}{\partial \alpha}\right)^2 + \underline{\frac{1}{8}\left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial u}{\partial \beta}\right)^2} \\
\epsilon_\beta &= \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{w}{R_\beta} + \frac{1}{2}\left(\frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_\beta}\right)^2 + \underline{\frac{1}{8}\left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial u}{\partial \beta}\right)^2} \\
\epsilon_{\alpha\beta} &= \frac{1}{2}\left\{\frac{\partial v}{\partial \alpha} + \frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{\partial w}{\partial \alpha}\left(\frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_\beta}\right)\right\} \qquad (6) \\
K_\alpha &= -\frac{\partial^2 w}{\partial \alpha^2} \qquad K_\beta = -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) + \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{v}{R_\beta}\right) \\
K_{\alpha\beta} &= \frac{1}{2}\left\{-\frac{2}{B}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{1}{R_\beta}\frac{\partial v}{\partial \alpha} + \underline{\frac{1}{2R_\beta}\left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial u}{\partial \beta}\right)}\right\}
\end{aligned}
$$

The terms underlined by dashes represent the influence of rotation about the normal to the middle surface. In many types of problems it is justifiable to state that the rotation normal to the middle surface is markedly smaller than the two rotations about axes in the plane of the middle surface. When this assumption is justified and the underlined terms are removed from (6) we have the strain displacement relations which are frequently used in investigations of shell behavior including stability. It is worth pointing out that nonlinear terms have been retained only in the direct strains, leaving the curvatures as linear in the displacements; a result of the "moderate rotations" assumption. Sanders [10] shows that retaining all terms in (6) yields the following differential operators in the displacement equations of equilibrium for a circular cross section when $N^o_{x\phi} = 0$ and the other two prestress resultants are constants:

$$
\begin{aligned}
L_{11} &= \frac{\partial^2}{\partial x^2} + \frac{(1-\nu)}{2R^2}\left(1 + \frac{d^2}{4}\right)\frac{\partial^2}{\partial \phi^2} + \underline{\frac{(N_x^o + N_\phi^o)}{2R^2}\frac{\partial^2}{\partial \phi^2}} \\
L_{12} &= L_{21} = \frac{(1+\nu)}{2R}\frac{\partial^2}{\partial x \partial \phi} - \frac{3(1-\nu)d^2}{8R}\frac{\partial^2}{\partial x \partial \phi} - \underline{\frac{(N_x^o + N_\phi^o)}{2R}\frac{\partial^2}{\partial x \partial \phi}} \\
L_{31} &= -L_{13} = \frac{\nu}{R}\frac{\partial}{\partial x} + \frac{(1-\nu)\,d^2}{2R^2}\frac{\partial^3}{\partial x \partial \phi^2} \\
L_{22} &= \frac{(1+d^2)}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{(1-\nu)}{2}\left(1 + \frac{9d^2}{4}\right)\frac{\partial^2}{\partial x^2} - \frac{N_\phi^o}{R^2} + \underline{\frac{(N_x^o + N_\phi^o)}{2}\frac{\partial^2}{\partial x^2}} \\
L_{32} &= -L_{23} = \frac{1}{R^2}\frac{\partial}{\partial \phi} - \frac{d^2}{R^2}\frac{\partial^3}{\partial \phi^3} - \frac{(3-\nu)d^2}{2}\frac{\partial^3}{\partial x^2 \partial \phi} + \frac{N_\phi^o}{R^2}\frac{\partial}{\partial \phi} \\
L_{33} &= -d^2R^2\nabla^2\nabla^2 - \frac{1}{R^2} + N_x^o\frac{\partial^2}{\partial x^2} + \frac{N_\phi^o}{R^2}\frac{\partial^2}{\partial \phi^2} \; .
\end{aligned}
\qquad (7)
$$

These operators are self adjoint and differ quite considerably from (4) as we would expect since certain large inplane displacements are neglected in (7).

Thin Circular Cylindrical Shell Analyses Allowing For Membrane Strains that are Nonlinear in the Inplane Displacements

In studying most of the contributions mentioned that are based on the small strain assumption, it is soon demonstrated that the displacement equilibrium equations are self adjoint and thus, for the thin circular cylinder can be written in symmetric form. Since self adjointness is evidenced by symmetry of the differential operators in cartesian coordinates we have reason to be curious about this apparent conflict with Biot's [5] and Pflüger's [16] demonstrations that the equations of elasticity for incremental deformations are non-symmetric in cartesian coordinates. The clue to this paradox comes from the fact that the majority of authors concerned primarily with the applications of shell theory, neglect, in the strain displacement relations, all non-linear terms involving the tangential displacements u and v . This enables the stress function to be introduced into the equations in a simple manner and means that they assume small strains and rotations, from the start. Self adjointness of differential operators, at least as usually defined [see e.g. Morse and Feshbach [19] has meaning only for linear equations. Hence when the potential energy contains displacement terms to powers higher than the second, whether the final equations from which the eigenmatrix is formed are self adjoint or not, depends on the assumptions made to linearize the Euler equations of the variational method. To illustrate the situation let us consider two contributions directly usable for numerical studies and which retain the nonlinear terms in u and v , namely Washizu [7] and Herrmann-Armenakas [8] whose strain displacement relations for a circular cylinder are

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\
\epsilon_\phi &= \frac{1}{R}\left(\frac{\partial v}{\partial \phi} - w\right) + \frac{1}{2R^2}\left\{\left(\frac{\partial u}{\partial \phi}\right)^2 + \left(\frac{\partial v}{\partial \phi} - w\right)^2 + \left(v + \frac{\partial w}{\partial \phi}\right)^2\right\} \\
\epsilon_{x\phi} &= \frac{1}{R}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x} + \frac{1}{R}\left\{\frac{\partial u}{\partial x}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial \phi} - w\right) + \frac{\partial w}{\partial x}\left(v + \frac{\partial w}{\partial \phi}\right)\right\} \\
K_x &= \frac{\partial^2 w}{\partial x^2}; \quad K_\phi = \frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial v}{\partial \phi}\right); \quad K_{x\phi} = \frac{1}{R}\left(\frac{\partial^2 w}{\partial x \partial \phi} + \frac{\partial v}{\partial x}\right) .
\end{aligned} \tag{8}$$

Inserting (8) into the expression for the strain energy, neglecting z/R relative to unity and utilizing the calculus of variations in the time honored manner, we obtain the large displacement equilibrium equations which were derived by Washizu using the theorem of virtual displacements. They are

$$\frac{\partial}{\partial x}\left\{N_x(1+\ell_x)+\frac{N_{x\phi}}{R}\frac{\partial u}{\partial \phi}\right\}+\frac{1}{R}\frac{\partial}{\partial \phi}\left\{\frac{N_\phi}{R}\frac{\partial u}{\partial \phi}+N_{x\phi}(1+\ell_x)\right\}+X=0$$

$$\frac{1}{R}\frac{\partial}{\partial \phi}\left\{N_\phi(1+\ell_\phi)+N_{x\phi}\frac{\partial v}{\partial x}\right\}+\frac{\partial}{\partial x}\left\{N_x\frac{\partial v}{\partial x}+N_{x\phi}(1+\ell_\phi)\right\}$$

$$-\frac{1}{R}\left\{N_\phi\,\omega_x+N_{x\phi}\,\omega_\phi+\frac{1}{R}\frac{\partial M_\phi}{\partial \phi}+\frac{\partial M_{x\phi}}{\partial x}\right\}+Y=0 \quad (9)$$

$$\frac{\partial}{\partial x}\left\{N_x\,\omega_\phi+N_{x\phi}\,\omega_x\right\}+\frac{1}{R}\frac{\partial}{\partial \phi}\left\{N_\phi\,\omega_x+N_{x\phi}\,\omega_\phi\right\}+N_\phi(1+\ell_\phi)+\frac{1}{R}N_{x\phi}\frac{\partial v}{\partial x}$$

$$+\frac{\partial^2 M_x}{\partial x^2}+\frac{2}{R}\frac{\partial^2 M_{x\phi}}{\partial x\partial \phi}+\frac{1}{R^2}\frac{\partial^2 M_\phi}{\partial \phi^2}+Z=0$$

in which we have used the linearized strain displacement relations

$$\ell_x=\frac{\partial u}{\partial x};\quad \ell_\phi=\frac{1}{R}\left(\frac{\partial v}{\partial \phi}-w\right);\quad \omega_x=\frac{1}{R}\left(\frac{\partial w}{\partial \phi}+v\right);\quad \omega_\phi=\frac{\partial w}{\partial x}\ .$$

These equilibrium equations consist of terms of two types, namely stress resultants and their derivatives plus stress resultants times midsurface rotations, which constitute the equations usually found in shell analysis which invoke the "small strain" assumption. The second type of terms are those consisting of stress resultants times midsurface strains. These latter terms result from a variational formulation only when the nonlinear terms in the inplane displacements are retained in the strain-displacement relations. When equation (4) is particularized for a circular cross section and we introduce into (9) the assumption that the strains are small compared to unity, the resulting equations are identical.

The complexity of the equations describing the motions relative to the prestressed state, depend, to a large measure on the prestressed state. A simple situation, of frequent interest, particularly in the field of biomechanics is when uniform axial stretch and internal pressure constitute the prestressed state. When solving the linear membrane equations for this loading, we obtain for a circular cross section N_x^o and N_ϕ^o as constants and $N_{x\phi}^o$ vanishes. For non-circular shells the prestress solution is much more involved. If we now assume an isotropic shell in which the Young's modulus is independent of the prestressed state the perturbation equations derived from (9) assume a relatively elegant form. To obtain them, we use the standard technique of inserting $N_x = N_x^o + N_x'$, $\partial u/\partial x = \partial u^o/\partial x + \partial u'/\partial x$ etc., extracting all terms involving only the zeroth state and assuming that the primed quantities are all small enough for their products to be negligible.

Now we may write the equations for perturbations about a prestressed state defined by the constant strains in differential operator form as

$$[L_{ij}] \begin{vmatrix} u \\ v \\ w \end{vmatrix} + \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = 0 \tag{10}$$

where

$$L_{11} = (1 + 2\ell_x^o + \nu\ell_\phi^o)\frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\left\{\frac{(1-\nu)}{2}(1 + \ell_x^o) + \ell_\phi^o + \nu\,\ell_x^o\right\}\frac{\partial^2}{\partial\phi^2}$$

$$L_{12} = \frac{(1+\nu)}{2R}(1 + \ell_x^o)\frac{\partial^2}{\partial x\partial\phi}$$

$$L_{13} = -(1 + \ell_x^o)\frac{\nu}{R}\frac{\partial}{\partial x}$$

$$L_{21} = \frac{(1+\nu)}{2R}(1 + \ell_\phi^o)\frac{\partial^2}{\partial x\partial\phi}$$

$$L_{22} = \frac{(1 + 2\ell_\phi^o + \nu\ell_x^o)}{R^2}\frac{\partial^2}{\partial\phi^2} + \left\{\frac{(1-\nu)}{2}(1 + \ell_\phi^o) + \ell_x^o + \nu\ell_\phi^o\right\}\frac{\partial^2}{\partial x^2}$$
$$- \frac{(\ell_\phi^o + \nu\ell_x^o)}{R^3} + d^2\left\{\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} + \frac{2(1-\nu)}{R}\frac{\partial^2}{\partial x^2}\right\}$$

$$L_{23} = -\frac{1}{R^2}(1 + 3\ell_\phi^o + 2\nu\,\ell_x^o)\frac{\partial}{\partial\phi} + d^2\left\{\frac{1}{R^2}\frac{\partial^3}{\partial\phi^3} + (2-\nu)\frac{\partial^3}{\partial x^2\partial\phi}\right\}$$

$$L_{31} = (1 + \ell_\phi^o)\frac{\nu}{R}\frac{\partial}{\partial x}$$

$$L_{32} = \frac{1}{R^2}(1 + 3\ell_\phi^o + 2\nu\,\ell_x^o)\frac{\partial}{\partial\phi} - d^2\left\{\frac{1}{R^2}\frac{\partial^3}{\partial\phi^3} + (2-\nu)\frac{\partial^3}{\partial x^2\partial\phi}\right\}$$

$$L_{33} = (\ell_x^o + \nu\ell_\phi^o)\frac{\partial^2}{\partial x^2} + \frac{(\ell_\phi^o + \nu\ell_x^o)}{R^2}\frac{\partial^2}{\partial\phi^2} - \frac{(1 + 2\ell_\phi^o + \nu\ell_x^o)}{R^2} - d^2\left(R^2\frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2\partial\phi^2} + \frac{1}{R^2}\frac{\partial^4}{\partial\phi^4}\right).$$

It is clear that these operators are self adjoint for large strains only when $\ell_\phi^o = \ell_x^o$, i.e., when the prestress state is hydrostatic, substantiating completely Biot's and Pflüger's conclusion. The usual manner in which the influence of the prestressed state is illustrated [see for example Biot [15] and Pflüger [16]] is to show that the non-symmetry in Cartesian coordinates occurs in the stress strain relations and the degree of asymmetry is of the order of the initial stress as divided by the elastic modulus. In our derivation the classic symmetric stress strain law was assumed and the degree of non-self adjointness was also shown to be of the order of the initial strain. In virtually all metallic structures the initial membrane strains are indeed small compared to unity and the operators in (10) can be simplified to those resulting from (4) and (9). However, in blood flow problems they can be as high or higher than 0.6 and neglecting them is less justified. Throughout our discussion it must be borne to mind that while an elastic stress strain law is reasonable for the perturbation stresses in the blood vessel, but it is not valid for the prestressed state. The determination of the actual prestressed state in blood vessels is extremely difficult compared with virtually all metal structures experiments. Another word of warning is relevant at this time. The shell equations we are utilizing are not valid for arbitrarily large strains and rotations. A completely general analysis is beyond our scope.

The equations we have discussed are considered typical of the literature and to see how significant some of the differences are, let us turn to the numerical evaluation of the phase velocities.

To study the importance of the assumptions made in the shell theories that have been discussed in relation to certain biomechanics problems, it is of interest to compute the phase velocity as a function of frequency and prestress levels. The displacement relations for axially propagating waves in a cylinder of circular cross section were substituted into the equations of motion and the resulting eigenvalue problem solved for specified wave numbers. The form of the displacement relations was

$$
\begin{aligned}
u(x, \phi, t) &= A \operatorname{Cos} kx \operatorname{Cos} s\phi \operatorname{Sin} \sigma t \\
v(x, \phi, t) &= B \operatorname{Sin} kx \operatorname{Sin} s\phi \operatorname{Sin} \sigma t \\
w(x, \phi, t) &= C \operatorname{Sin} kx \operatorname{Cos} s\phi \operatorname{Sin} \sigma t
\end{aligned} \tag{11}
$$

These were inserted into (7) for Sanders" [10] theory and (10) for Washizu's [7] theory and into (5) for the Anliker-Maxwell [18] utilization of Flügge's [5] theory and Herrmann-Armenakas' [8] theory. The computations were also accomplished using Budiansky's theory [13]. The substitution of (11) into (5) and (7) leads to a symmetric matrix because of the self adjointness of the operators, but that derived from (10) naturally remains nonsymmetric.

Numerical Results and Discussion

A digital program was written using a readily available non-self-adjoint eigenvalue routine to extract the modes and frequencies for prescribed shell-fluid parameters and wave number. After using the results presented by Anliker and Maxwell [18] to check the accuracy, the nondimensional wave speeds were found as functions of the nondimensional frequency for the theories mentioned previously. The results are shown in Figs. 1 and 2 and correspond to axisymmetric waves. The former shows the influence of axial stretch, as represented by N_x^o, while the latter illustrates how the wave speeds vary as a function of transmural pressure as represented by N_ϕ^o. Data for the $s = 2$ waves did not show any differences from the results of [17] that could be considered significant for physiological applications and therefore have not been illustrated.

For axisymmetric waves and $N_\phi^o = 0$, the predictions based on Sanders [10] Budiansky [13] and Herrmann-Shaw [20] and our equation (5) agree very closely with each other for the pressure and axial waves. The apparent lack of agreement with [18] is solely due to the previously discussed different assumption concerning the prestress application. However, for nonzero circumferential prestress there are some significant variations among the results. They are most easily discussed by considering the uncoupled torsion waves separately from the pressure and axial waves, whose equations are coupled.

Torsion Waves

A somewhat surprising result is that unlike other shell equations, those of Sanders [10] and Washizu [7] plus a number of other authors predict a cut-off frequency for the type II or torsion waves whenever the shell is subjected to transmural pressures (see Fig. 2). The cause for this can be seen by noting that the equation for axisymmetric torsion waves is

$$L_{22}\, v - \frac{\partial^2 v}{\partial t^2} = 0$$

since for $s = 0$, $L_{12} = L_{21} = L_{32} = L_{23} = 0$ and the equilibrium equation for the circumferential direction becomes uncoupled from the other two. For the Anliker-Maxwell utilization of Flügge's theory we have the characteristic equation

$$c^2 = \left(\frac{\sigma}{k}\right)^2 = \left[\frac{(1-\nu)\frac{(1+3d^2)}{2} + N_x^o + \nu N_\phi^o}{\rho}\right]$$

in which the term νN_ϕ^o vanishes when we use (5). According to this equation the torsion waves are nondispersive and have no cut-off frequency below which they do not propagate. This is in contradiction with the other two theories mentioned. For example, from (10) we have according to Washizu's theory

$$c^2 = \left(\frac{\sigma}{k}\right)^2 = \left[\frac{(1-\nu)\frac{(1+\epsilon_\phi^o)}{2} + 2d^2(1-\nu) + N_x^o}{\rho}\right] + \frac{N_\phi^o}{\rho k^2}$$

while Sanders' theory yields

$$c^2 = \left(\frac{\sigma}{k}\right)^2 = \left[\frac{(1-\nu)\frac{(1+9d^2/4)}{2} + \frac{(N_x^o + N_\phi^o)}{2}}{\rho}\right] + \frac{N_\phi^o}{\rho k^2} .$$

In both of these cases it is the last term $(N_\phi^o/\rho k^2)$ that introduces both the dispersion and the cut-off frequency for non-zero transmural pressure. As the wave number approaches zero, so does the frequency and the wave speed becomes very large as the frequency approaches zero, and we obtain the cut-off frequency. The existence of the cut-off frequencies is also demonstrated by noting that for finite values of σ, k can be zero. In fact, the above equations show that $\sigma^2 = N_\phi^o/\rho$ for zero wave number. Therefore $\sigma = (N_\phi^o/\rho)^{1/2}$ is the cut-off frequency below which waves do not propagate.

This behavior for the rotationally symmetric torsion waves is so completely contrary to all experimental evidence that it cannot be glossed over. An explanation for the case of shells subjected to hydrostatic pressures is given by

Herrmann and Shaw [20] which uses the equations derived in [8]. It is shown that the erroneous cut-off frequency arises when certain terms are omitted in the derivation of the equations by neglecting the change in the direction of the hydrostatic pressure force induced by the perturbation rotations. If we retain these terms and use equation (16) of [20], the equilibrium condition for the rotationally symmetric torsion mode and for pressures acting on the shell midsurface is

$$\left\{(1-\nu)\frac{(1+3d^2)}{2} + N_x^o\right\}\frac{\partial^2 v}{\partial x^2} - \frac{N_\phi^o v}{R^2} - \frac{\partial^2 v}{\partial t^2} + \Delta F_\phi = 0 \qquad (12)$$

where $\Delta F_\phi = N_\phi^o v/R^2$, is the change of the circumferential force component arising from the displacement. By taking the pressure to act on the shell midsurface the moment components Δm_ϕ and Δm_z, discussed in [17], [18] and [20], are identically zero. If we substitute the above expression for ΔF_ϕ in (12) it becomes

$$\left\{\frac{(1-\nu)}{2}(1+3d^2) + N_x^o\right\}\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} = 0 \quad . \qquad (13)$$

The questionable terms (which arise from the treatment of the applied loading, not the description of shell behavior) in Washizu's and Sanders' plus certain other theories have been cancelled out by ΔF_ϕ and the torsion waves are now also nondispersive. This demonstrates the difficulties that can arise when the displacements and rotations of a prestressed system are being neglected while the stress resultants are retained. For many metal shell analyses except those involving torsional wave propagation this approximation may well be entirely satisfactory. In the case of blood vessels however, the displacements and rotations are generally much larger than in metal structures and must be properly accounted for. For a circular cylindrical shell, this is accomplished by using either the equations of Budiansky [13] or those of Herrmann-Armenakas [8] and

Herrmann-Shaw [20], which reduce to those of Flügge [5]. Hence the results given by Anliker and Maxwell [18] are indeed correct.

However, if we accept the reasoning of Armenakas and Herrmann [17] which is also employed in the subsequent papers, e.g. [20] torsion waves retain their cut-off frequency in the case of constant directional pressure since ΔF_{ϕ} is zero in this situation. This produces a result which we also find to be at variance with our intuition.

Consider an infinitely long shell whose internal pressure does not change direction and which is undergoing rotationally symmetric torsional oscillations. The rotation about the tangential base vector, namely $\partial w/\partial x$ does indeed vanish. Therefore, ΔF_x is zero. However, the other component of rotation of the normal to midsurface is v/R, and therefore not zero and ΔF_{ϕ} is again equal to $N^o_{\phi} v/R$. Thus the extraneous term causing the cut-off frequency is always cancelled out. This small modification is of little physical significance since the fluid pressure is hydrostatic in virtually all cases of practical interest.

Another item of interest in connection with the torsion modes is that the results of Anliker and Maxwell [18] do not exhibit the same dependence on axial prestress as do similar theories illustrated in Fig. 2. This discrepancy has been explained by noting the different methods in which the prestress is applied. In our applications, we did not restrict the axial motion and hence applying the pressure does not contribute to the axial prestress. It should be mentioned that this latter assumption affects the torsional phase velocities. However, it is not necessarily representative of blood vessels under "in vivo" conditions since they are anchored by branches and connective tissue and therefore do not change their length with pressure.

Pressure and Axial Waves

The contributions to the equilibrium equations of the terms involving the applied pressure times the perturbation rotations and strains are accounted for by Budiansky [13]. The two inplane equilibrium equations contain the rotations times the hydrostatic pressure while in the normal equation the pressure is multiplied by the sum of the inplane strains. A situation entirely analogous to the one involving the behavior of the torsion waves, occurs in the third equilibrium equation. The variation of the strain energy yields a term which is cancelled by the pw/R element in the expression accounting for the hydrostatic pressure. The neglect of this pressure term is the cause of the behavior of Washizu's equations and all others which make the same approximation. Thus Sanders' [10] theory predicts phase velocities that are independent of internal pressurization for types I (pressure) and III (axial) waves by properly accounting for the change in direction fo the pressure. This is also true in the theories of Budiansky [13] and Herrmann-Shaw [20] in addition to our form of Flügge's theory in (5). The only reason for the dependence on internal pressure of phase velocities for the rotationally symmetric modes in Anliker and Maxwell is due to the method in which the prestressed are applied.

Also illustrated in Fig. 1 is the consistent nondispersive character of the type III (axial) waves. Only Sanders' theory predicts axial phase velocities independent of axial stretch while the other theories applied, in particular Washizu's, show a strong influence of axial stretch. The reason for this is to be found in the absence of the $(\partial u/\partial x)^2$ term in the expression for the axial strain and its influence on the operator L_{11}. For rotationally symmetric waves, the L_{11} coefficient in the frequency determinant for Sanders' theory is simply k^2 while in the other two theories it is k^2 times a function of the axial stretch which accounts for the noted dependence.

Another interesting result of the method of application of the prestress comes to light when we note that in (5), all the phase velocities are virtually independent of internal pressurization. This disagrees with Anliker and Maxwell only because of the previously mentioned application of the prestress, which results in terms involving Poisson's ratio times the circumferential prestress resultants. In our application of Flügge's theory, all three types of waves are independent of pressurization within drawing accuracy of the curves shown in Fig. 2.

Consideration of Non-Self-Adjointness of Washizu's Equations

Attention has been drawn to the fact that the "moderate strain" from of Washizu's accounting for nonlinear inplane displacements, as discussed, yields displacement equilibrium equations which are non-self-adjoint and thus we know from well established theory [e.g. [18]] that the eigenvalues of the basic system and its adjoint are identical, but that the eigenvectors of the two systems are different and form a biorthogonal set. The adjoints of the operators in (12) are easily found by replacing ℓ_x^o by ℓ_ϕ^o and vice versa in the off-diagonal terms. The modal amplitude coefficients were indeed slightly different and we provide the following table as typical results.

TABLE I

TYPICAL MODAL COEFFICIENTS FOR THE ANLIKER-MAXWELL WASHIZU THEORY FOR $N_x^o = 0.4$, $N_\phi^o = 0$ AND $k = 5.0$

Wave Type	Shell Theory Modal Displacement								
	Anliker-Maxwell [18]			Washizu [7] Basic System			Washizu [7] Adjoint System		
	Axial	Torsion	Radial	Axial	Torsion	Radial	Axial	Torsion	Radial
I	-0.044	0	1.0	-0.0469	0	1.0	-0.0335	0	1.0
II	0	1.0	0	0	1.0	0	0	1.0	0
III	1.0	0	0.044	1.0	0	0.0335	1.0	0	0.0469

For this demonstration all coefficients less than 10^{-5} in absolute value have been replaced by zeros in Table I.

These results illustrate the theory by showing that for the self adjoint operators of [18], the modes are orthogonal, while those of Washizu [7] for both the basic and adjoint equations are not orthogonal, but they are biorthogonal. However, the fact that the modes predicted by [7] are not quite orthogonal is of negligible importance in the study of high frequency waves induced mechanically in the cardiovascular system.

CONCLUSIONS

The phase velocities predicted by the small strain theories of Herrmann and Shaw [20], Budiansky [13], and our application of Flügge's [5] theory are considered to be correct. Our results are nothing more than those of Anliker and Maxwell [18] with a different assumption concerning the application of the prestresses. All three wave speeds are essentially independent of internal hydrostatic pressure if the cylinder walls are not axially constrained.

It is demonstrated that the neglect of the force component induced by the perturbation rotation of the applied pressure leads to the prediction of a cut-off frequency below which no torsion waves are propagated for non-zero transmural pressure. The importance of including these contributions to the equilibrium, which are frequently neglected, accounts for the correctness of the theories. For future studies of this type on shells of general geometry, those of Budiansky [13] are recommended.

The nonself adjointness of the moderate membrane strain theory of Washizu [7] isshown to occur due to the inclusion of nonlinear inplane displacements in the strain displacement relations. The modes obtained from the resulting eigenmatrix are very similar to those of the adjoint eigenmatrix. Thus, the nonselfadjointness character of the equations is of negligible importance for the physiological problems of wave propagation in the arterial and venous systems.

NOTATION

A, B	Lamé parameters
$c = \bar{c}/\bar{c}_0$	Nondimensional phase velocity
$c_0^{-2} = \bar{E}/\bar{\rho}(1-\nu^2)$	Normalizing phase velocity
$d^2 =$	$h^2/12$
$\bar{E}$	Normalizing Young's modulus
e_α, e_β, $e_{\alpha\beta}$	Linear midsurface strains
$h = \bar{h}/\bar{R}_0$	Nondimensional shell wall thickness
$k = 2\pi \bar{R}_0/\bar{\lambda}$	Nondimensional wave number
L_{ij}	Differential operators
ℓ_{ij}	Midsurface strains and rotations
M_α, M_β, $M_{\alpha\beta} = \bar{M}_\alpha(1-\nu^2)/\bar{E}$, etc.	Nondimensional moment resultants
N_α, N_β, $N_{\alpha\beta} = \bar{N}_\alpha(1-\nu^2)\ \bar{E}\bar{h}$, etc.	Nondimensional force resultants
R_α, $R_\beta = \bar{R}_\alpha/\bar{R}_0$, etc.	Nondimensional radii of curvature

$\bar{R}_0$	Normalizing geometric dimension
$u, v, w = u/R_0$, etc.	Nondimensional midsurface displacements in the axial, tangential and normal directions
x, ϕ	Axial and circumferential coordinates of a general cylinder
$X, Y, Z = \bar{X}(1-\nu^2)/\bar{E}\bar{h}$, etc.	Nondimensional applied force resultants
α, β	Axial and circumferential coordinates of a general cylinder
$\epsilon_\alpha, \epsilon_\beta, \epsilon_{\alpha\beta}$	Nonlinear midsurface strains
$K_\alpha, K_\beta, K_{\alpha\beta}$	Curvatures
$\lambda = \bar{\lambda}/\bar{R}_0$	Nondimensional wave length
$\rho = \bar{\rho}\,\bar{R}_0(1-\nu^2)\bar{\sigma}_0^{-2}/\bar{E}$	Nondimensional density
$\sigma = \bar{\sigma}/\sigma_0$	
$\bar{\sigma}_0^{-2} = \bar{E}/\bar{\rho}\,\bar{R}_0(1-\nu^2)$	Normalizing frequency
$\omega_\alpha, \omega_\beta$	Rotations
$\nabla^2\nabla^2(\cdot) =$	$\partial^4(\cdot)/\partial\alpha^4 + 2\partial^4(\cdot)/B^2\partial\alpha^2\partial\beta^2 + \partial^4(\cdot)/B^4\partial\beta^4$

superscripts

$\bar{\ }$	Denotes a dimensional quantity
^{o}	Denotes a prestress quantity

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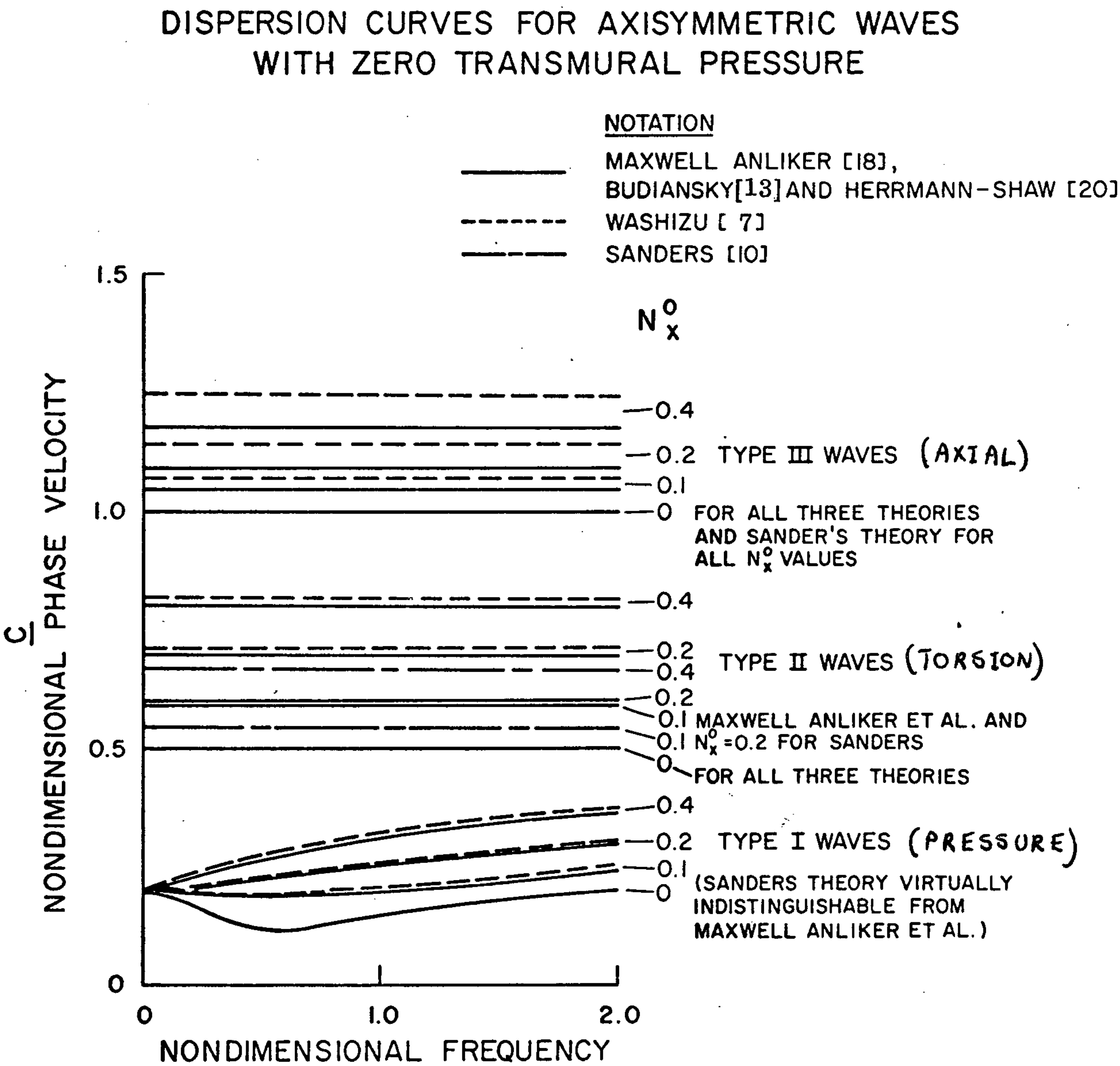

Fig. 1. Dispersion Curves of Axisymmetric Waves for Zero Transmural Pressure Predicted by Different Shell Theories

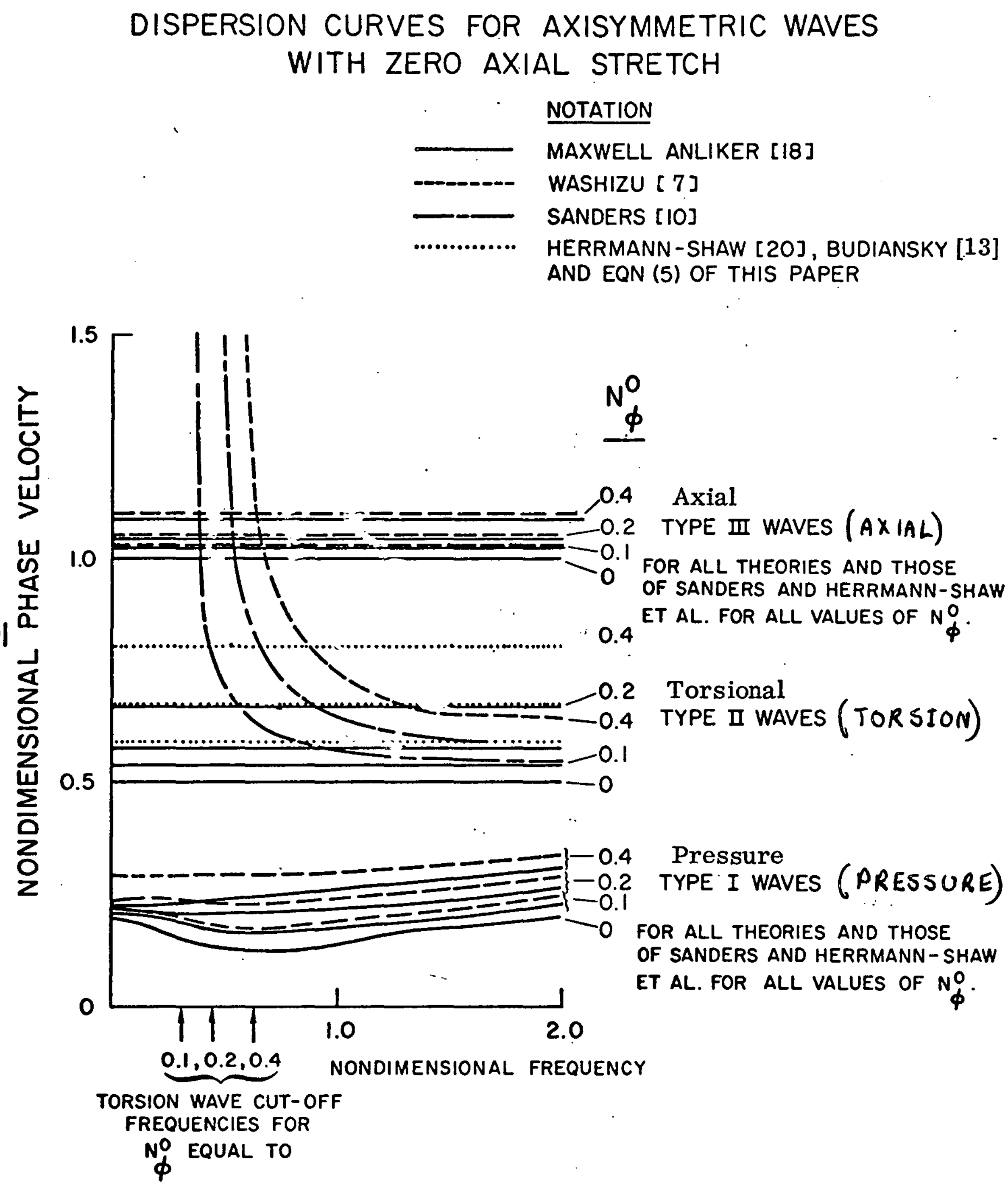

Fig. 2. Dispersion Curves of Axisymmetric Waves for Zero Axial Sketch Predicted by Different Shell Theories

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