AN ISOPARAMETRIC QUADRILATERAL MEMBRANE ELEMENT FOR NASTRAN

By

Howard M. Adelman, Joseph E. Walz, and James L. Rogers, Jr.
NASA Langley Research Center

SUMMARY

This paper describes the implementation of an improved quadrilateral membrane element in NASTRAN. Descriptions of the geometrical and kinematic properties of the element are included along with development of the matrices and vectors which characterize the element. The necessity for considering small deviations from planeness of the element is discussed and the approach taken to account for these deviations for the new element is described. The improved accuracy available from the element over the existing quadrilateral membrane is indicated by a sample calculation for which an analytical solution is available from beam theory. For the same finite element idealization, the errors in maximum displacement and stress were significantly reduced by the use of the new element.

INTRODUCTION

One of the more frequently used elements in the NASTRAN library is the quadrilateral membrane element (CQDMEM). This element is used to represent portions of structures for which membrane action constitutes the predominant contribution to the strain energy. An example of one such application is to the skin of aircraft wings. In addition, the quadrilateral membrane element is also used in conjunction with the quadrilateral bending element (CQDPLT) to form the membrane-bending elements CQUAD1 and CQUAD2 needed to represent more general deformation behavior.

The quadrilateral membrane element currently available in NASTRAN is composed of overlapping constant strain triangles as described in reference 1. It has been reported that the CQDMEM element does not accurately represent problems involving high stress gradients, suggesting that the current element needs improvement. The reason for this difficulty is generally attributed to the constant strain field provided by this element (ref. 2). Several improved quadrilateral membrane elements have been developed which have linear strain fields (e.g. refs. 3 and 4) and these elements should provide improved results in membrane element applications. The implementation of an improved quadrilateral membrane element in NASTRAN was undertaken as an in-house project to improve the element library by attempting to overcome the noted shortcomings.

The element chosen for implementation is the linear strain isoparametric quadrilateral membrane element described in references 3 and 4. The reasons for this choice are:

(1) The element is conforming, i.e. the displacements of adjoining elements are matched along their entire interface. As a result, the strain energy is an upper bound to the corresponding exact energy.

(2) The stresses and strains vary within the element thus providing improved accuracy over the existing element.

(3) The element is well-documented.

The purposes of the present report are to present a description of the element being implemented and to demonstrate the increased accuracy available through the new element by comparison with the existing quadrilateral element and an analytical solution.

SYMBOLS

[A]	matrix relating strains and displacements
[B]	transformation matrix relating displacements in mean plane to those at actual grid points
[C]	matrix which relates rotations of the mean plane to displacements of mean plane
[E]	matrix relating displacements in element coordinate system to those in basic coordinate system
e_x, e_y, e_{xy}	membrane strains
$\{f_a\}$	vector of forces at actual grid points in element coordinate system

$\{f_e\}$	vector of forces in mean plane
$[G_e]$	matrix relating stresses and strains
H	distance from actual grid point to the mean plane
h	thickness of membrane
J	Jacobian of transformation from x-y coordinates to ξ-η coordinates
$[K]$	stiffness matrix referred to global coordinate system
$[K^d]$	differential stiffness matrix referred to global coordinate system
$[K_{ee}]$	stiffness matrix in element coordinate system
$[K^d_{ee}]$	differential stiffness matrix in element coordinate system
$\{P\}$	thermal load vector referred to global coordinate system
$\{P_e\}$	thermal load vector in element coordinate system
$[S]$	matrix relating element stresses to global displacements
$[S_e]$	matrix relating element stresses to element displacements
$\{S_t\}$	vector relating element temperature to element stress
$[T]$	basic to global coordinate transformation matrix
t_o	reference or stress-free temperature of the element
$\bar{t}$	temperature of the element above the reference or stress-free temperature
u,v,w	displacements in x-, y-, and z-directions, respectively
$\bar{u},\bar{v},\bar{w}$	displacements in X-, Y-, and Z-directions, respectively, see table I
$\{u_a\}$	vector of displacements at actual grid point in element coordinate system
$\{u_e\}$	vector of displacements in mean plane
X,Y,Z	cartesian coordinates used in table I
x,y,z	element cartesian coordinate directions, see figure 1
η,ξ	element parametric coordinates, see figure 1
$\omega_x, \omega_y, \omega_z$	rotations of the mean plane element about x,y and z axes, respectively

Subscripts:

1, 2, 3, 4	refer to grid points 1, 2, 3 and 4 respectively, of the element

A subscript preceded by a comma indicates partial differentiation with respect to the subscript.

DESCRIPTION OF THE ELEMENT

In this section of the paper, descriptions of the geometry and kinematic behavior of the isoparametric element (figure 1) will be given. The element, since it is defined by four points, need not be planar; however the development of the necessary matrices is carried out for a flat element. The treatment of the case for which the four points are not coplanar will be discussed in a later section of the paper. The element parametric coodinates ξ and η shown in figure 1 vary linearly between zero and one with the extreme values occurring on the sides of the quadrilateral. Further, lines of constant ξ and η are straight as indicated in the figure. A set of element cartesian coordinates x, y, z is defined as follows: the x-axis is along the line connecting the first two grid points; the y-axis is perpendicular to the x-axis and lies in the plane of the element; and the z-axis is normal to the plane and forms a right handed system with the x- and y-axes. Displacement components in x, y, and z directions are denoted by u, v, and w, respectively. As given in reference 3, the displacement field is assumed to have the following form:

$$\left.\begin{aligned} u(\xi,\eta) &= (1 - \xi)(1 - \eta)\ u_1 + \xi(1 - \eta)\ u_2 + \xi\eta u_3 + (1 - \xi)\ \eta u_4 \\ v(\xi,\eta) &= (1 - \xi)(1 - \eta)\ v_1 + \xi(1 - \eta)\ v_2 + \xi\eta v_3 + (1 - \xi)\ \eta v_4 \end{aligned}\right\} \qquad (1)$$

where the subscript on a displacement component denotes the grid point value of the component.

It may be observed that on lines of constant ξ, u and v vary linearly with η and on lines of constant η, u and v vary linearly with ξ. In particular u and v vary linearly on each edge between grid points and as a result the displacements of adjacent elements are matched all along their common edges. The element is therefore of the "conforming" type which guarantees that the element will converge as an upper bound on the strain energy.

The required membrane strains are related to the displacements u and v by the familiar relations

$$e_x = u_{,x} \qquad e_y = v_{,y} \qquad e_{xy} = u_{,y} + v_{,x} \tag{2}$$

where the comma indicates partial differentiation. Since the displacements are expressed in terms of ξ and η, the operations in equations (2) cannot be carried out without knowing the relationships between the (x,y) coordinates and the (ξ,η) coordinates. These relations are given in reference 3 as

$$\left.\begin{aligned} x &= (1 - \xi)(1 - \eta)\, x_1 + \xi(1 - \eta)\, x_2 + \xi\eta x_3 + (1 - \xi)\, \eta x_4 \\ y &= (1 - \xi)(1 - \eta)\, y_1 + \xi(1 - \eta)\, y_2 + \xi\eta y_3 + (1 - \xi)\, \eta y_4 \end{aligned}\right\} \tag{3}$$

By use of familiar relations involving partial derivatives, the operations indicated in equations (2) may be performed. Thus, for example,

$$u_{,x} = u_{,\xi}\, \xi_{,x} + u_{,\eta}\, \eta_{,x} \tag{4}$$

where

$$\xi_{,x} = \frac{1}{J}\, y_{,\eta} \qquad \eta_{,x} = -\frac{1}{J}\, y_{,\xi} \tag{5}$$

and

$$J = \begin{vmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{vmatrix} \tag{6}$$

It is noted in passing that equations (1) which relate displacements within the element to its grid point values or "parameters" are identical in form to equations (3) for coordinates x and y. Thus the term "isoparametric" is applied to characterize the element.

For the special case of a rectangle it can be shown that the x and ξ directions are identical as are the y and η directions. In this case e_x is linear with respect to y and constant with respect to x, whereas e_y is linear with respect to x and constant with respect to y. The shear strain varies linearly with respect to both x and y. In contrast to the strains, all three stress components vary linearly in both the x and y directions. This is a direct consequence of the constitutive equations. For nonrectangular shapes the behavior of the stress and strain components is more complicated and is not easily characterized.

IMPLEMENTATION OF THE ELEMENT

The addition of an element to NASTRAN requires the derivation of a set of characteristic matrices and vectors as described in reference (1) and those necessary for the new element are:

- stiffness matrix, $[K_{ee}]$
- lumped mass matrix, $[M_{ee}]$
- thermal load vector, $\{P_e\}$
- stress recovery matrices, $[S_e]$ and $\{S_t\}$
- differential stiffness matrix, $[K^d_{ee}]$

The development of these matrices is presented in this section along with a description of the procedure used when the four grid points defining the element are not coplanar. Finally, this section contains an outline of the matrix transformations required so that the new element will be compatible with others in NASTRAN.

Stiffness Matrix

Using equations (1) through (5) results in the following relation between strains and grid point displacements

$$\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [A] \{u_e\} \tag{7}$$

where the elements of the 3x8 matrix [A] are functions of ξ and η. The stress-strain relation is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [G_e] \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} - [G_e] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} \bar{t} \tag{8}$$

where the 3x3 matrix $[G_e]$ represents most generally a completely anisotropic material. The terms α_x, α_y and α_{xy} are thermal expansion coefficients and $\bar{t}$ is the average temperature of the element above the stress-free temperature, given by

$$\bar{t} = \frac{1}{4} (t_1 + t_2 + t_3 + t_4) - t_0 \tag{9}$$

The strain energy, V, (apart from thermal effects) is

$$V = \frac{h}{2} \int_0^1 \int_0^1 \{u_e\}^T [A]^T [G_e][A]\{u_e\} J \, d\xi \, d\eta \tag{10}$$

and from this expression, the stiffness matrix can be identified as

$$[K_{ee}] = h \int_0^1 \int_0^1 [A]^T [G_e][A] J \, d\xi \, d\eta \tag{11}$$

The required integration is performed numerically by the use of Gaussian quadrature using a 4x4 grid. For a discussion of Gaussian quadrature as used for isoparametric elements, see references 3 and 4.

Lumped Mass Matrix

The mass matrix developed for the isoparametric element is a lumped mass matrix since coupled mass matrices for membrane elements generally result in overly stiff representations in dynamic problems (see ref. 1 p. 5.5-5). One method of lumping is to assign one quarter of the mass of the element to each of the four grid points. However, this method usually does not preserve the location of the center of mass of the element. Accordingly, the method used to generate the mass matrix for the new element is that presently used in NASTRAN for the existing quadrilateral membrane element. This later method is based on an averaging procedure which always preserves the center of mass.

Thermal Load Vector

For the purpose of developing the thermal load vector, the contribution to the potential energy, U, of the element temperature above some stress-free value is written as

$$U = h \int_0^1 \int_0^1 \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} \bar{t} \; J \; d\xi \; d\eta \qquad (12)$$

Using equations (7) and (8) in equation (12) and discarding an irrelevant constant term not involved in the solution, results in

$$U = h \int_0^1 \int_0^1 \{u_e\}^T [A]^T [G_e] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} \bar{t} \; J \; d\xi \; d\eta \qquad (13)$$

and the thermal load vector is then recognized as

$$\{P_e\} = h \int_0^1 \int_0^1 [A]^T J \, d\xi \, d\eta \, [G_e] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} \bar{t} \qquad (14)$$

Stress Recovery Matrices

Expressions for the element stress components σ_x, σ_y, and σ_{xy} at any point written in terms of displacements measured in the element cartesian coordinate system are obtained by combining equations (7) and (8) to give

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [S_e]\{u_e\} - \{S_t\} \bar{t} \qquad (15)$$

where

$$[S_e] = [G_e][A] \text{ and } \{S_t\} = [G_e] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} \qquad (16)$$

Although the stress components may vary within the element, they are computed only at the intersection of the element diagonals for the purpose of stress output. Once the stress components are obtained, they are used to compute the principal stresses and directions by appropriate formulas given in reference 5.

Differential Stiffness Matrix

The differential stiffness matrix, which is used in NASTRAN primarily for linear buckling analyses, is developed by a consideration of the work done by stress components σ_x, σ_y, and σ_{xy} during small rotations ω_x, ω_y, and ω_z about the three element cartesian axes. The expression for the work done is taken from reference 1 (section 7.1, equation (16)) given as

$$W = -\frac{h}{2} \int_0^1 \int_0^1 \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}^T [K^d_{\omega\omega}] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} J \, d\xi \, d\eta \tag{17}$$

where

$$[K^d_{\omega\omega}] = \begin{bmatrix} \sigma_y & -\sigma_{xy} & 0 \\ -\sigma_{xy} & \sigma_x & 0 \\ 0 & 0 & \sigma_x+\sigma_y \end{bmatrix} \tag{18}$$

and the rotation components are given by

$$\begin{aligned} \omega_x &= w,_y \quad \omega_y = -w,_x \\ \omega_z &= \frac{1}{2}(v,_x - u,_y) \end{aligned} \tag{19}$$

In order to evaluate ω_x and ω_y, the behavior of w in the element is required, and for this purpose w is assumed to have the same parametric variation as u and v, thus

$$w = (1-\xi)(1-\eta)\, w_1 + \xi(1-\eta)\, w_2 + \xi\eta w_3 + (1-\xi)\, \eta w_4 \tag{20}$$

Combining equations (1), (19), and (20) results in

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [C]\{u_e\} \tag{21}$$

where the elements of the 3x8 matrix [C] are functions of ξ and η. Substituting equation (21) into equation (17) gives

$$W = -\frac{h}{2}\int_0^1 \int_0^1 \{u_e\}^T [C]^T [K^d_{\omega\omega}][C]\{u_e\} J \, d\xi \, d\eta \qquad (22)$$

and the differential stiffness matrix is recognized as

$$[K^d_{ee}] = h \int_0^1 \int_0^1 [C]^T [K^d_{\omega\omega}][C] J \, d\xi \, d\eta \qquad (23)$$

The integration is performed by use of Gaussian quadrature using a 4x4 grid. The values of σ_x, σ_y, and σ_{xy} at each of the sixteen quadrature points are obtained from equation (15) for appropriate values of ξ and η.

Use of the Mean Plane

As mentioned previously, the matrices and vectors associated with the new quadrilateral membrane element were derived for a planar element. If the four grid points are not coplanar they are projected onto a so-called mean plane which is defined in the following manner (see figure 2). Two skewed lines whose end points are grid points 1 and 3 and grid points 2 and 4, respectively, are defined to be the diagonals of the element. The mean plane is defined so that it passes through the midpoint of the perpendicular connector of the two diagonals and parallel to them (see ref. 5 p. 4.87-105.). As shown in figure 2, if the length of the perpendicular connector is 2H, then the user defined grid points are alternatively H units above and below the mean plane as one progressively moves around the element. Once the actual grid points are projected onto a mean plane, a planar element is defined and the previously derived matrices are applicable.

The authors would like to emphasize that the use of the mean plane concept does not finally resolve the question of how to deal with a quadrilateral element whose grid points are not coplanar and that further research on this subject is warranted. One alternative to the mean plane is to assume the element to be planar with the plane defined by three of the grid points. This alternative was found to be undesirable for the new element because numerical results were sensitive to the grid point numbering sequence in a single element. This numerical experiment to determine this sensitivity is described in the Appendix.

Transformation of Matrices and Vectors to Global Coordinates

Up to this point the characteristic matrices and vectors for the element have been derived in terms of displacements in the element cartesian coordinate system. In order that the new element be compatible with other elements in NASTRAN, it is necessary to transform the matrices and vectors so that they are expressed in the global coordinate system. This procedure involves three transformations:

(1) transformation from displacements in the mean plane in the element cartesian system to the displacements at the user-defined grid points in the element cartesian system

(2) transformation from displacements in the element coordinate system to displacements in the NASTRAN basic coordinate system

(3) transformation from the NASTRAN basic coordinate system to the NASTRAN global coordinate system.

The first transformation is based on replacing the set of forces in the mean plane $\{f_e\}$ by a statically equivalent set at the user defined grid points $\{f_a\}$. The first set consists of forces in the x and y directions only, whereas the second set consists of forces in the x, y, and z directions. This statement of static equivalence can be written as

$$\{f_a\} = [B]\{f_e\} \tag{24}$$

where

$$\{f_a\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \\ f_{x2} \\ \cdot \\ \cdot \\ \cdot \\ f_{z4} \end{Bmatrix} \text{ and } \{f_e\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{Bmatrix} \tag{25}$$

The matrix [B] depends on the geometry of the element and H which is a measure of the nonplaneness of the element. This same matrix is used to transform displacements, thus

$$\{u_e\} = [B]^T \{u_a\} \tag{26}$$

The second and third transformations are required of all elements in NASTRAN and are discussed fully in reference 1. In terms of the notation of reference 1 the transformation matrix from element coordinates to NASTRAN basic coordinates is denoted by a 12x12 matrix [E], and the transformation matrix from NASTRAN basic coordinates to NASTRAN global coordinates is denoted by a 12x12 matrix [T]. The relation between element and global displacements may finally be written as

$$\{u_e\} = [B]^T [E]^T [T] \{u_g\} \tag{27}$$

All that is now required to express the matrices and vectors in the global coordinate system is to substitute equation (27) into equations (10), (13), (15), and (22). The results are summarized below

Stiffness matrix:

$$[K] = [T]^T [E][B][K_{ee}][B]^T [E]^T [T] \tag{28}$$

Thermal load vector:

$$\{P\} = [T]^T [E][B]\{P_e\} \tag{29}$$

Stress recovery matrix:

$$[S] = [S_e][B]^T [E]^T [T] \tag{30}$$

Differential stiffness matrix:

$$[K^d] = [T]^T [E][B][K^d_{ee}][B]^T [E]^T [T] \tag{31}$$

PRELIMINARY RESULTS

At the time of writing, the element is in the early stage of being verified. A stand-alone program has been written for the purpose of testing the element in a NASTRAN-like environment. The first available test case for this element is a cantilever beam modeled by 16 equal elements as shown in figure 3. Displacements and stresses were computed using the existing CQDMEM element and the new isoparametric element based on the same finite element idealization. The finite element results are then compared to an analytical solution based on elementary beam theory. In figure 3 the analytical solution for displacements is indicated by the solid line and the finite element results by dashed lines. The new element is seen to be a significant improvement over the CQDMEM element. When comparison is made with the analytical solution, the largest error exhibited by CQDMEM is 50% whereas the corresponding error for the new element is 9%. The comparison of stresses is presented in figure 4 where the results have been normalized to the maximum stress predicted by elementary beam theory. Although the finite element stresses were computed at a single point in each element, these stresses are presented as continuous curves. Again the exact solution is shown by the solid line and the finite element results by dashed lines. The results indicate a significant improvement in accuracy by using the new element. The maximum error in stress using the CQDMEM element is about 49% whereas that for the improved element is 14%.

CONCLUDING REMARKS

An isoparametric quadrilateral membrane element which is being implemented in NASTRAN by the NASTRAN Systems Management Office is described. Included are descriptions of the basic geometry and deformation characteristics of the element, as well as an outline of the derivation of each of the required matrices and vectors for the element. Derivations of the stiffness matrix, thermal load vector, stress recovery matrices, and differential stiffness matrices are given. A discussion of the need for some method of accounting for deviations from planeness is included along with a description of the method used for this purpose in the new element. Finally, a sample calculation is carried out in which the displacements and stresses in a cantilever beam are computed by the new element and by the existing quadrilateral membrane (CQDMEM) and compared with an analytical solution from beam theory. The significant increase in accuracy obtainable with the new element is demonstrated by the calculation.

APPENDIX

SENSITIVITY OF MEMBRANE RESULTS TO

ELEMENT GRID POINT NUMBERING SEQUENCE

A numerical experiment suggested by Dr. Raphael Haftka, currently a National Research Council postdoctoral Fellow at the Langley Research Center, is performed on the structure sketched at the top of table I. The top member is non-planar and the vertical member is planar. Both members were represented by the new quadrilateral membrane element assumed to be planar with the plane of each element being defined by three of its grid points. Displacements under two loading conditions were computed by means of a stand-alone program. In the calculations, the grid point sequence for the top member was varied as shown in table I, and the effects of these variations are shown in table II. Examination of table II (a) shows that for a load applied in the X-direction, the displacements both in the X-Y plane and normal to it are sensitive to the grid point numbering. For example, there is a 12% difference among the displacements in the Y-direction as well as among the displacements in the Z-direction at grid point 3. For the case of loads in the Z-direction there is a difference among the displacements in the Y-direction of 46%. The consequences of these percentage errors are tempered by the observation that the large errors noted always occurred in displacement components which were not in the direction of the applied load. By contrast the maximum difference was 1.5% for a displacement in the X-direction at grid point 2 in the first loading condition. Although the difference for displacements in the direction of the load is small, it is felt that the existence of a situation where different answers can be obtained with the same finite element model by merely changing the grid point numbering sequence for individual elements should be avoided.

REFERENCES

1. MacNeal, Richard H., editor: The NASTRAN Theoretical Manual, NASA SP-221, Sept. 1970.

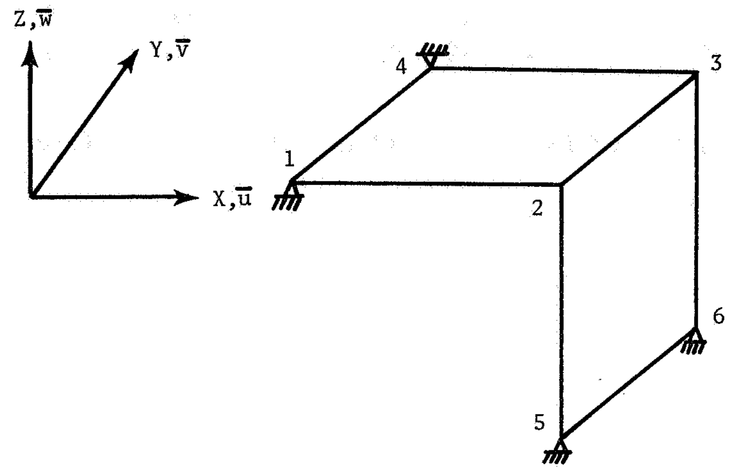
2. Bergmann, H. W.; Robinson, J. C.; and Adelman, H. M.: Analysis of a Hot Elevon Structure, NASTRAN Users Experiences, NASA TM X-2378, 1971, pp. 163-180.

3. Przemieniecki, J. S.: Theory of Matrix Structural Analysis, McGraw-Hill, 1968, pp. 102-107.

4. Zienkiewicz, O. C., and Cheung, Y. K.: The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill, 1968, pp. 67-70.

5. Douglas, F. J., editor: The NASTRAN Programmer's Manual, NASA SP-223, Sept. 1970.

TABLE I.- STRUCTURE USED TO DETERMINE EFFECT OF NONPLANENESS



Young's Modulus = 125 GN/m^2

Poisson's ratio = .3

thickness = .0374 cm

(a) Gridpoint Locations
(all coordinates given in centimeters)

Gridpoint	X	Y	Z
1	1280.45	480.44	9.38
2	1367.30	480.44	7.36
3	1391.87	536.68	5.52
4	1305.01	536.68	5.78
5	1367.30	480.44	-87.00
6	1391.87	536.68	-87.00

(b) Loading

Loading condition 1	$F_{X2} = F_{X3} = 1N$
Loading condition 2	$F_{Z2} = F_{Z3} = 1N$

(c) Grid Numbering Sequence for Top Element

Case	1st Point	2nd Point	3rd Point	4th Point
I	1	2	3	4
II	2	3	4	1
III	3	4	1	2
IV	4	1	2	3

Table II.- Effect of Grid Point Numbering on Displacement of a Warped Quadrilateral. (All Displacements are in cm.)

(a) Displacement due to loads in X-direction						
Grid point Sequence (table I)	$\bar{u}_2 \times 10^6$	$\bar{v}_2 \times 10^6$	$\bar{w}_2 \times 10^6$	$\bar{u}_3 \times 10^6$	$\bar{v}_3 \times 10^6$	$\bar{w}_3 \times 10^6$
I	5.46	1.59	1.46	6.85	.498	-1.07
II	5.49	1.57	1.48	6.84	.526	-1.08
III	5.44	1.56	1.35	6.86	.511	-1.20
IV	5.41	1.57	1.33	6.88	.470	-1.19

(b) Displacements due to loads in Z-direction						
Grid point Sequence	$\bar{u}_2 \times 10^6$	$\bar{v}_2 \times 10^6$	$\bar{w}_2 \times 10^6$	$\bar{u}_3 \times 10^6$	$\bar{v}_3 \times 10^6$	$\bar{w}_3 \times 10^6$
I	.485	.535	6.41	-.095	-.298	6.05
II	.506	.365	6.33	-.099	-.447	6.12
III	.377	.421	6.33	-.223	-.392	6.12
IV	.354	.591	6.41	-.217	-.246	6.05

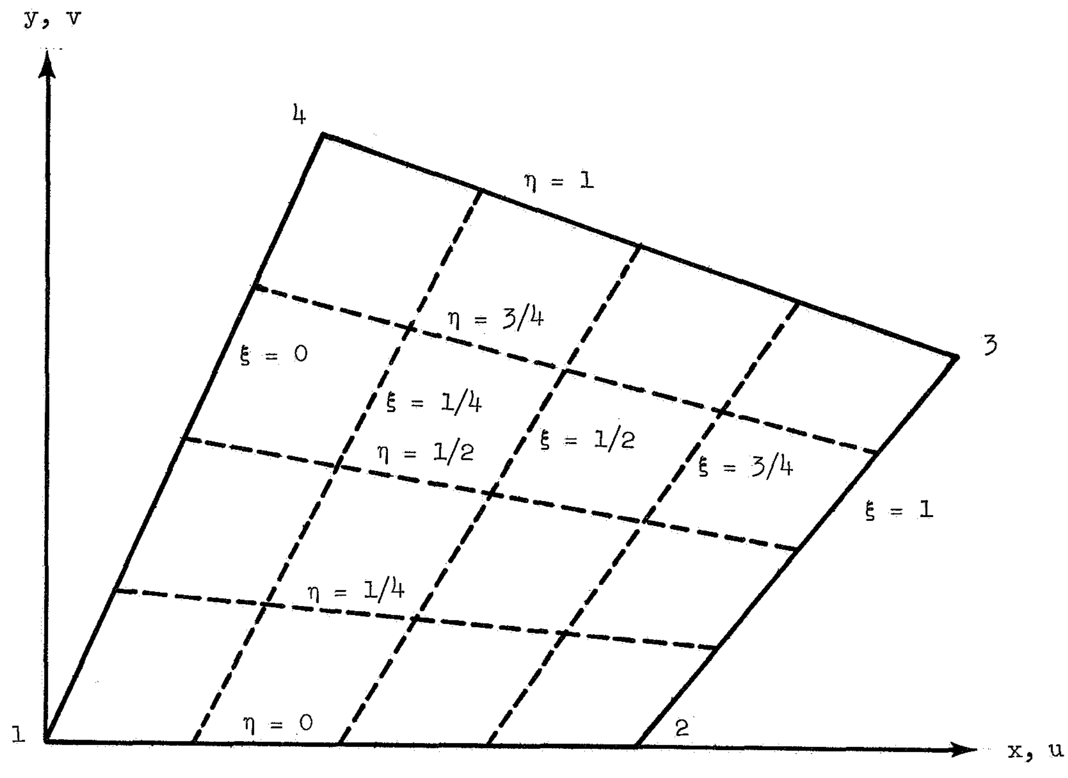


Figure 1. — Coordinate systems for quadrilateral membrane element.

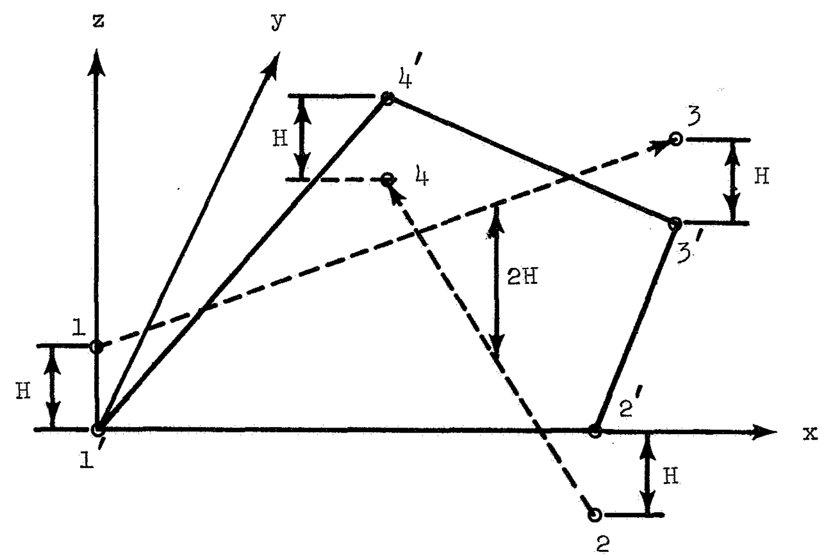


Figure 2. — Mean plane for quadrilateral membrane element. (Actual grid points are indicated by unprimed numbers and projection of grid points onto mean plane are indicated by primed numbers.)

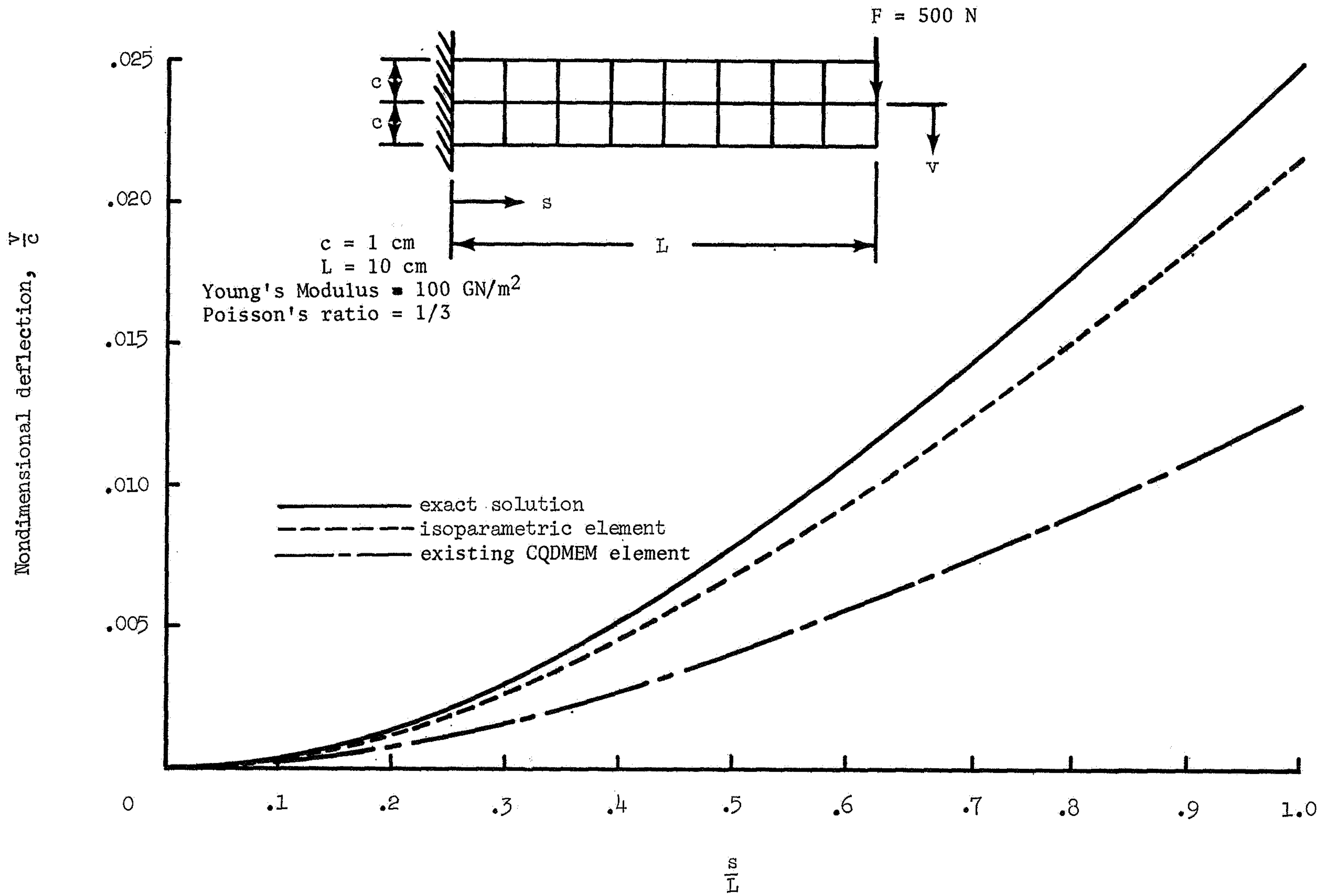


Figure 3.- Deflection of cantilever beam idealized by quadrilateral membrane elements.

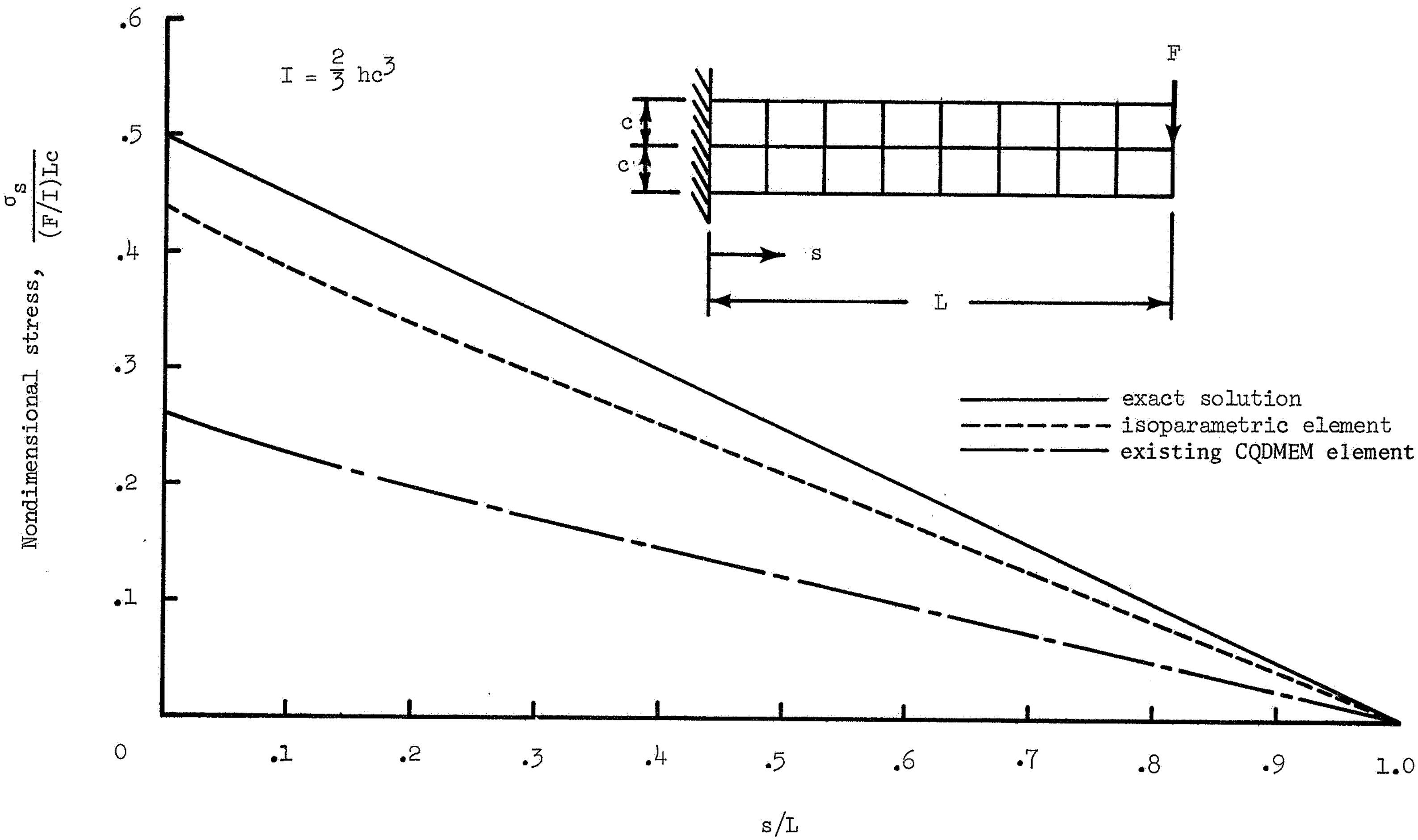


Figure 4. - Longitudinal stress at distance c/2 above neutral axis in cantilever beam idealized by quadrilateral membrane elements.