A DESIGN STUDY FOR THE ADDITION OF HIGHER-ORDER PARAMETRIC DISCRETE ELEMENTS TO NASTRAN*

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SUMMARY

Higher-order parametric discrete elements are a significant modeling advance over similar elements with straight-sided triangular or quadrilateral planforms. However, the addition of such discrete elements to NASTRAN poses significant interface problems with the Level 15.1 assembly modules and geometry modules. The present paper systematically reviews potential problems in designing new modules for higher-order parametric discrete elements in both areas. An assembly procedure is suggested that separates grid point degrees of freedom on the basis of admissibility. New geometric input data are described that facilitate the definition of surfaces in parametric space.

SYMBOLS

c^k	Denotes continuity through k derivatives
$f,_{\alpha\beta}$	The partial derivative of f with respect to α and β
df	Total differential of f
R	Cylindrical radius
S	Arc length
u^α	Elastic displacement in the curvilinear coordinate direction α
u^3	Elastic displacement normal to the midsurface
UA	Grid point displacement parameters required for admissibility

UH   Grid point displacement parameters not required for admissibility

x^i Grid point coordinates in a reference coordinate system

θ^i Rotations about the x^i coordinate directions

(ξ, η) Patch parameters analogous to curvilinear coordinates

Subscripts:

FP   Flat plate

CP   Cylindrical panel

INTRODUCTION

In the main, joining problems arise with higher-order discrete elements because the additional grid point degrees of freedom contain terms directly proportional to element strains. A complete one-to-one joining of displacement parameters between geometrically similar elements implies a strain continuity that is erroneous if the elements are of different materials. A complete one-to-one joining of displacement parameters between geometrically dissimilar elements implies a strain discontinuity that is erroneous if the elements are of similar materials. An example of this latter behavior will be given in which the in-plane displacement gradients are erroneously linked between cylindrical panel elements and flat plate elements. To obtain the correct solution these parameters must either be allowed to vary independently or be joined by a constraint equation for in-plane strain continuity. In general, any variational problem can be solved by using only the minimum constraints required to produce an admissible displacement field for the Ritz procedure. As a practical matter this approach is of little help if used uncritically since it can add many unnecessary degrees of freedom. At the intersection of four MDAC parametric discrete elements, for example, there are 48 degrees of freedom that are reduced by admissibility constraints to 22 independent degrees of freedom. When strain continuity constraints are applied these are reduced to 12 independent degrees of freedom. In this case there is nearly a 50 percent reduction when strain continuity is valid. Any modification of the NASTRAN assembly modules to process higher-order discrete elements must be flexible enough to take advantage of this situation if it is to be efficient.

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The basic geometric entity used by NASTRAN is the grid point. The basic geometric entity need for parametric discrete elements is a mapping of two surface coordinates into points on the surface in three dimensions. This mapping, called a patch, is approximated locally by interpolation functions. These functions must be input to NASTRAN in order to generate element matrices (stiffness, etc.). It is, of course, feasible to input the patch for each element directly as part of the property data for the element. This has the obvious disadvantage of requiring a great deal of input data; up to 48 items for a bicubic patch. To reduce input data requirements, the MDAC parametric plate element program uses the boundary curve for the entire plate to generate patches for each discrete element once a topological mesh has been specified (Reference 1). The present paper considers new geometric input data for NASTRAN to facilitate the introduction of parametric discrete elements.

ADMISSIBLE DISPLACEMENT FIELDS

Admissibility conditions for discrete element displacement functions are an especially important topic for higher-order discrete elements. In classical variational mechanics the material properties are usually assumed either constant or continuously differentiable functions of the spatial coordinates. This leads to simple smoothness requirements based on the order of the differential operator in the equilibrium equations (Reference 2). The displacement u^3 in a homogeneous plate bending problem, for example, must be C^4 in the interior and C^3 on a free edge. The latter condition is a consequence of the natural boundary condition for shear. These are of course conditions on the continuum displacement solution that the discrete element model must converge to in the limit. Convergence is measured by an energy norm and it is the existence of this norm that sets the admissibility conditions for the piecewise polynominal displacement function formed by assembling individual discrete elements. Returning to the plate example, the energy norm is derived from the strain energy density which involves at most second derivatives of u^3. As long as the discrete element displacement field is at least C^1 between elements the energy norm is well defined. Physically this condition corresponds to the absence of a hinge between plate elements and it is imposed at the grid points to assemble or build a discrete element model of a plate structure. How closely the assumed displacement functions approach

this condition between nodes is a problem in approximation theory that is intimately related to the question of completeness. This is another issue entirely and for the present discussion completeness will be assumed. Again returning to the plate example if there are no line moments between elements and the material is continuous, then the strains are continuous. Additional constraints can then be used to impose inter-element strain continuity but these conditions are not required for admissibility. The solutions obtained with and without these additional constraints will often have the same mean error (Reference 3) but the solution with strain continuity will require solving fewer equations. When dealing with higher-order elements the constrained stiffness matrix may be less than one-half the dimension of the original.

When higher-order discrete elements with distinctly different strain-displacement equations must be assembled, a clear understanding of the admissibility conditions is essential. Perhaps the best way to describe the possible pitfalls is with an illustrative example. Consider adjacent flat plate and cylindrical panel elements from the discrete element model of the pear-shaped cylinder shown in Figure 1. Using the Bogner, Fox, Schmit (BFS) plate and cylindrical panel elements (Reference 4), there are 12 degrees of freedom per grid point per element, the three displacement components relative to a local curvilinear coordinate frame u^α, u^3 and the nine gradients of these displacement components $u^\alpha,_1$, $u^\alpha,_2$, $u^\alpha,_{12}$, $u^3,_1$, $u^3,_2$, $u^3,_{12}$. There is a tendency to erroneously assume the plate displacement gradient components are equal the panel displacement gradient components at a common node, in particular $(u^2,_2)_{FP} = (u^2,_2)_{CP}$. A substantial error in the normal displacement component u^3 is caused by this assumption as shown in Figure 2 for the pear-shaped cylinder under a uniform axial load. The admissibility conditions merely require $(u^\alpha)_{FP} = (u^\alpha)_{CP}$ and $(u^3,_\alpha)_{FP} = (u^3,_\alpha)_{CP}$ where the curvilinear coordinates have been parameterized such that $d\alpha = dS$ in each element. This allows discontinuities in $u^2,_2$ which of course must exist if we are to have continuous midsurface strains,

$$\left(u^2,_2\right)_{FP} = \left(u^2,_2\right)_{CP} + \frac{u^3}{R} \tag{1}$$

It should be noted that it is <u>not</u> necessary to use Equation (1) as a constraint. The minimum potential energy theorem ensures that the Ritz procedure for <u>admissible</u> displacement fields will find $(u^2,_2)_{FP}$ and $(u^2,_2)_{CP}$ such that equilibrium is satisfied in the limit. In this problem equilibrium implies strain continuity and Equation (1) can be used as in Reference 5 as an additional constraint which reduces the number of equations to be solved. If we modify the pear-shaped cylinder such that the flat panel material is different from the cylindrical panel material then Equation (1) <u>cannot</u> be used since equilibrium now requires a strain discontinuity.

There are at least two other situations in the assembly of higher-order discrete elements that deserve attention. Consider the stiffened cylindrical panel shown in Figure 3 again modeled using the BFS discrete elements. In this case, even though adjacent cylindrical panel elements have the same strain displacement equations and are made of the same material, there can be discontinuities in $u^1,_2$ caused by load transfer between the stiffener and the cylindrical panel. Consider next an error that can occur when constraining the three rotations to be equal between adjacent higher-order plate elements. Let x^1 = constant be the common edge between two BFS plate elements and recall that the elastic rotations are one-half the curl of the displacement vector. Using a Cartesian coordinate system, the rotations are

$$\theta^1 = \frac{1}{2} u^3,_2$$

$$\theta^2 = -\frac{1}{2} u^3,_1$$

$$\theta^3 = \frac{1}{2}\left(u^2,_1 - u^1,_2\right) \tag{2}$$

To ensure the same displacement along the common edge requires $u^\alpha,_2$ and $u^3,_2$ to be continuous. If, in addition, we now require θ^3 to be continuous this will imply $u^2,_1$ is continuous. This in turn implies the shear strains are continuous which is erroneous if the plates are of different materials. Admissibility requires only that θ^1 and θ^2 be continuous.

These examples illustrate the pitfalls that can occur in the assembly of high-order discrete elements when constraints are used that exceed those

necessary for admissibility. Unfortunately it is not practical to use only admissibility constraints when strain continuity or other grid point constraints are valid. These constraints not only reduce the number of equations, they usually do not change the structure of the stiffness matrix (if it was banded it will remain banded) and in most cases they do not increase the mean error. The design of a new structural matrix assembly module for use with higher-order elements in NASTRAN must take these factors into account.

STRUCTURAL MATRIX ASSEMBLY MODULE

The structural matrix assembly module in Level 15.1 of NASTRAN cannot process discrete elements with more than six degrees of freedom per grid point. A new module is required for higher-order elements that accounts for accurate and efficient design requirements. Assembly based on simply the admissibility conditions must be available as a default and grid point constraints must be available for efficiency. The module should be able to assemble the existing general elements in NASTRAN with higher-order elements of different types. This suggests two categories of grid point degrees of freedom for each element; those directly involved in admissibility conditions, UA, and all others, UH. All grid point degrees of freedom (in element coordinates) for all general elements now in NASTRAN fall in the first category. The new grid point degrees of freedom, UH, are somewhat like scalar point variables except they are elastically coupled to all the other grid point degrees of freedom for an element. As a default value, the number of UH at a grid point is equal the sum of the UH associated with that grid point from each element connected to that grid point. This corresponds to assembly based simply on admissibility. Next, it is necessary to provide for grid point constraints that are linear equations, usually identities, among the UH and UA at a grid point. This is analogous to multipoint constraint equations with all the degrees of freedom occurring at the same grid point. As a practical matter a unique identification scheme for the UH will be needed. The UA of course already are identified uniquely by component numbers 1 to 6. Also, as a practical matter, an automated grid point constraint generator is needed; one that could set all UH components equal for elements of the same type at a grid point. To fix some of these ideas consider a BFS plate element (Reference 4),

a CKLO plate element (Reference 5) and a CQDPLT plate element (Reference 6) all modeling the behavior of a plate having one common grid point. At the element level the CQDPLT element has five degrees of freedom per grid point, the BFS element has twelve and the CKLO element has twelve. These are listed in Table 1, divided into UA and UH degrees of freedom.

Table 1. Grid Point Degrees of Freedom

Element	UA_1	UA_2	UA_3	UA_4	UA_5	UA_6	UH_1	UH_2	UH_3	UH_4	UH_5	UH_6	UH_7
BFS	u^1	u^2	u^3	$u^3_{,2}$	$-u^3_{,1}$		$u^1_{,1}$	$u^1_{,2}$	$u^1_{,12}$	$u^2_{,1}$	$u^2_{,2}$	$u^2_{,12}$	$u^3_{,12}$
CKLO	u^1	u^2	u^3	$u^3_{,2}$	$-u^3_{,1}$		$u^1_{,1}$	$u^1_{,2}$	$u^2_{,1}$	$u^2_{,2}$	$u^3_{,11}$	$u^3_{,22}$	$u^3_{,12}$
CQDPLT	u^1	u^2	u^3	$u^3_{,2}$	$-u^3_{,1}$								

The two higher-order elements have elastic rotations about the x^3 axes (c.f. Equation 2) but these are not UA degrees of freedom as described earlier. If the UA_6 degree of freedom is removed with an SPC1 card there are 19 degrees of freedom at the common grid point. If the elements are all of the same material then grid point constraint equations can reduce this to 14. Suppose there is a rod normal to the plate that transmits torsion. This does not add a degree of freedom since UA_6 in this case is related to UH_2 and UH_4 of the BFS element by Equation (2) or equivalently UH_2 and UH_3 of the CKLO element. As is obvious from this example, admissibility cannot be determined for all combinations of general elements without the analyst making judgments about load paths in his structure. These decisions will be input via grid point constraint equations. An analogous situation now exists with frame structures when the analyst uses pin flags (cuts and releases) on the CBAR card to input his decisions about joints.

PARAMETRIC DISCRETE ELEMENT GEOMETRY

The initial geometric representation of a complicated structure is a formidable design problem but it is one that has been solved by the time a discrete element analysis is required. Some form of a geometric model (loft lines, offsets, etc.) has been prepared and serves as a data base for

the analyst. Increasingly these models are computer generated and in several industries piecewise polynomial surface representation is now used (Reference 7). This form of surface representation is the same as that used for parametric discrete element models and consists of patches that map two parameters (ξ, η) into spatial coordinates ($x^1(\xi, \eta)$, $x^2(\xi, \eta)$, $x^3(\xi, \eta)$, on the midsurface of the discrete element. The patches are constructed such that the edges of the element coincide with constant values of the patch parameters (ξ, η) as Figure 4 illustrates. The data required to define a patch with curved edges is obviously more than the grid point coordinates of the corners. If bicubic Hermite polynomials (Coons' surface patches) are used then x^i, $x^i_{,\xi}$, $x^i_{,\eta}$ and $x^i_{,\xi\eta}$ are required at each corner. To a large extent the increased data per grid point is offset by a reduction in the number of grid points required to model the geometry but this is a separate issue. The immediate problem is how to efficiently introduce into NASTRAN the geometric data required by parametric discrete elements. This data can be input as property data for each element or as a separate entity like grid point coordinates that can then be referenced on a broader basis by all elements. The first approach would require a minimum change to NASTRAN but could be very inefficient in that the same boundary data might be input over and over again, once for each element sharing a common edge. Primarily for this reason only the second approach will be considered further.

The patches used for parametric discrete elements are almost always bivariate polynomials although other interpolatory functions are possible. These polynomials can be uniquely determined in several different ways, each related to the other by a linear transformation. A bicubic Hermite polynomial interpolate determined from corner coordinates and derivatives also can be uniquely determined by the coordinates of sixteen points (Reference 1) where four of these must be interior points for the $x^i_{,\xi\eta}$. Boundary curves can also be used to define patches. Mallet provides an excellent example (Reference 8) using piecewise cubic interpolation of a grid line to obtain C^1 continuity. This approach uses grid line data in much the same way grid point data is now used in NASTRAN. Another useful representation is the super patch that defines several patches within its boundaries. The super patch is constructed using spline constraints (Reference 9) and has been used by Timmer (Reference 10) to form bicubic patches for aerodynamic surface modeling. Both the grid line modeling and super patch modeling offer the additional

benefit of C^1 continuity along the entire common boundary between adjacent patches. Patches derived with the spline constraints of Reference 9 also have C^2 continuity and require far less input data. This suggests a simple way of constructing grid line data from a near minimal data base. The parametric slopes at the two end points, $x^i,_{\xi 1}$ and $x^i,_{\xi N}$, and the grid point identification numbers G1, G2, ..., GN of points on the line are all that's required to define the line. If cross derivative data is desired for Coons surface patches then $x^i,_{\xi \eta 1}$ and $x^i,_{\xi \eta N}$ are also required but only at the four corners of a super patch. A prototype data card for generating a grid line from spline constraints is shown in Figure 5. There are of course situations where spline constraints produce a wavy line that does not model the initial geometry well. In this instance the parametric slopes should be input for every grid point on the line. Although no mention of grid line parameterization has been made, it will be necessary to adopt some standard such as $0 \le \xi \le 1$ between adjacent grid points.

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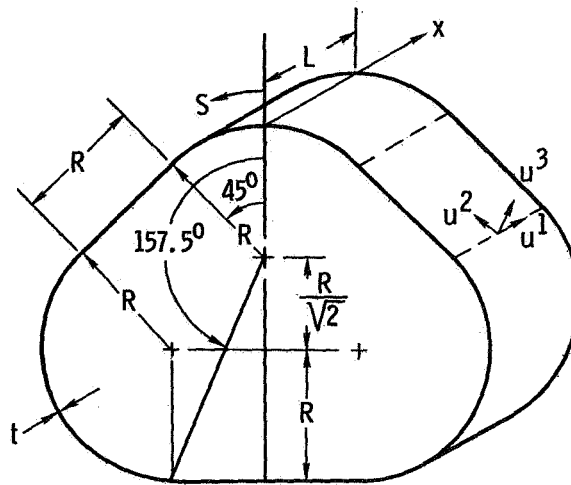MATERIAL PROPERTIES:

$E = 68.95 \times 10^9$ NEWTONS/METER^2

$\nu = 0.3$

$\rho = 689.5$ NEWTONS/METER^2

$R = 2.54$ CM

$L = 2.032$ CM

$t = 0.254$ CM

UNIFORM AXIAL LOAD = 1.751 NEWTONS

Figure 1   Cylindrical Shell with Pear-Shape Cross-Section
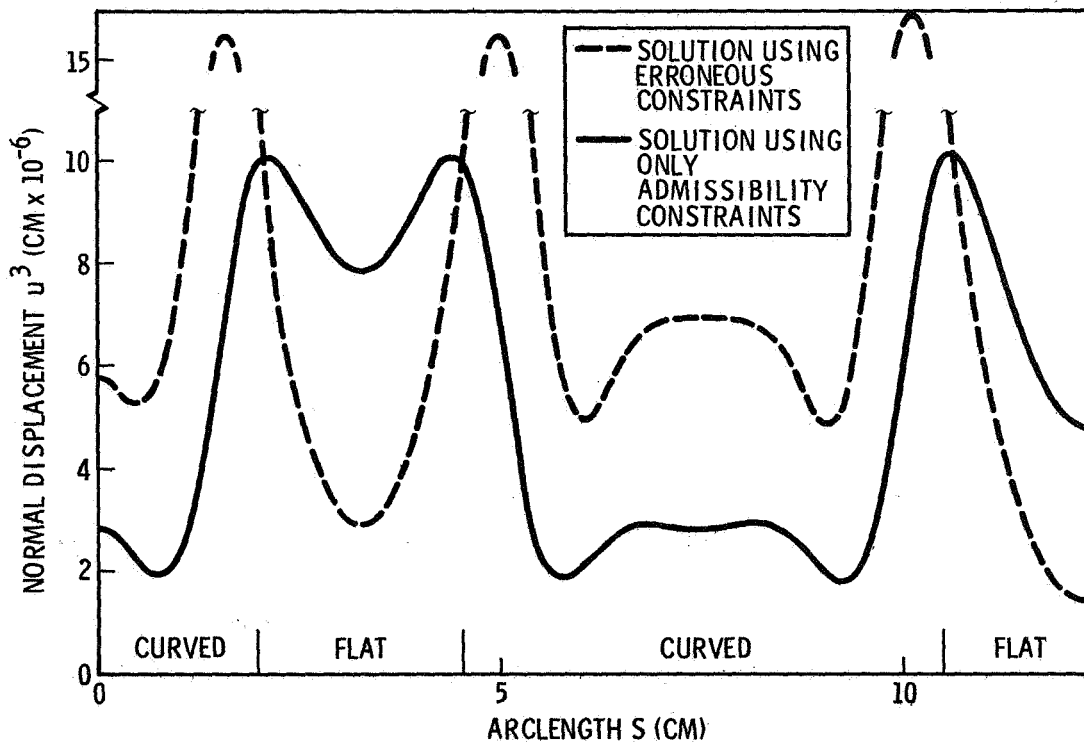
Figure 2   Effect of Erroneous Constraints Between Higher-
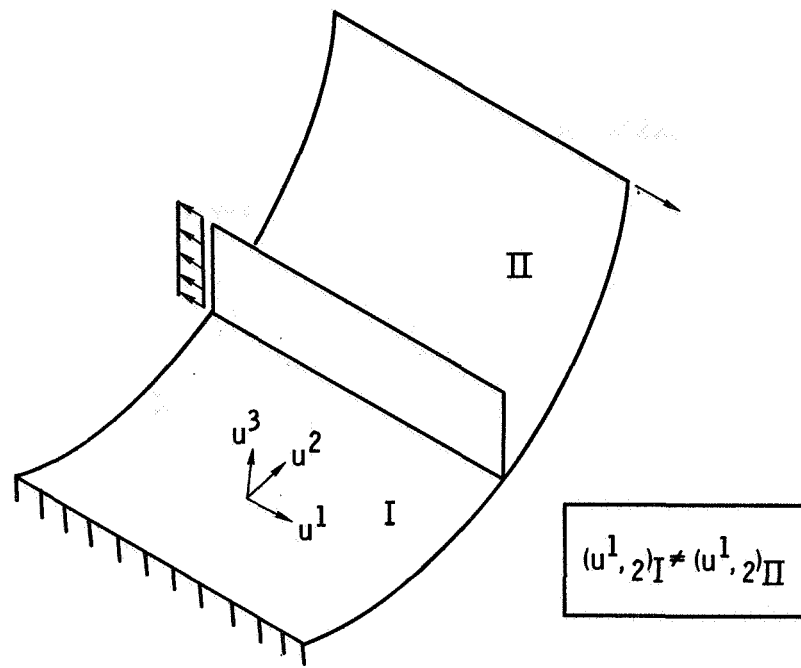Order Curved and Flat Discrete Elements

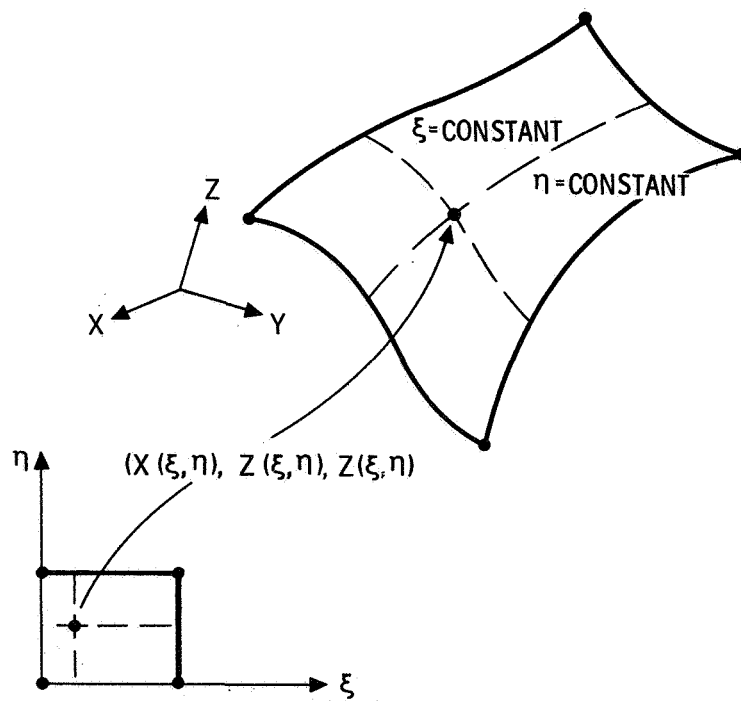Figure 3   Stiffened Panel Joining Example

Figure 4   Discrete Element Geometry Represented by a Patch

BULK DATA DECK

INPUT DATA CARD <u>SPLIN1</u>   GRID LINE SPLINE CONSTRAINTS

<u>DESCRIPTION</u>: DEFINE END SLOPES AND INTERMEDIATE GRID POINTS FOR
A TYPE I SPLINE CONSTRAINT EQUATION

SPLIN1	GLID	CD	X1C1	X2C1	X3C1	X1CN	X2CN	X3CN	abc
+bc	G1	G2	.	.	.	GN			

FIELD   CONTENTS

GLID   GRID LINE IDENTIFICATION NUMBER

CD   IDENTIFICATION NUMBER OF COORDINATE SYSTEM
                        IN WHICH THE PARAMETRIC SLOPES ARE INPUT

X1C1   PARAMETRIC SLOPE $X^1,_\xi$ AT GRID POINT 1

X3CN   PARAMETRIC SLOPE $X^3,_\xi$ AT GRID POINT N

G1, G2, .. GN   GRID POINT IDENTIFICATION NUMBERS OF POINTS
                        ON GRID LINE GLID IN SEQUENCE

Figure 5   Prototype Bulk Data Card for Spline Constraints