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4. An Input Adaptive, Pursuit Tracking Model of the Human Operator*

JOHN R. WARE

Naval Ship Research and Development Center

The purpose of this research was to develop and evaluate a simple model of the input adaptive behavior of the human operator (HO) in a pursuit tracking task in which the plant controlled consists of a pure gain. In such a task the HO must attempt to predict the future position of the input signal to overcome his inherent delay. Figure 1 is a block diagram of the pursuit tracking task.

If it is assumed that the HO is approximately an optimal predictor (in the mean square sense) using only position and velocity information, as suggested by J. I. Elkind, then there is a simple method of computing the values of the model parameters in terms of the autocorrelation function of the input signal. Experimental evidence indicates that the ability of the HO to use velocity information decreases with increasing signal velocity indicating that a biased estimator of the velocity weighting should be used. A suitable approximation is derived which has rapid convergence and low variance. The model thus derived is compared to actual subject transfer functions and is found to be in close agreement.

In addition to tracking random processes the model can adapt to and track deterministic signals, such as sine waves, up to approximately the frequency at which human operators begin to track precognitively.

The optimal position weighting is shown to be a good measure of effective bandwidth and it also possesses the attractive property of being easily calculable using the technique presented.

Possible application, in addition to modelling, are the use of the estimation procedure as an adaptive signal preprocessor to reduce operator workload and the use of the position weighting as an adaptive indicator of sampling rate in multiaxis tasks.

INTRODUCTION

Previously developed input adaptive models, such as those proposed by Angel and Bekey (ref. 1) and Fogel and Moore (ref. 2), are implemented using a finite-state machine approach. However models of this form have several drawbacks, not the least of which is that they are usually cumbersome to simulate.

Elkind (ref. 3) suggested a model for pursuit tracking as shown in figure 2. He then proposed that if the error signal is small, the error feedback loop could be neglected. If, in addition, it is assumed that the plant controlled is a pure gain the problem is reduced to a simple signal tracking task. This is the problem considered in this report, pursuit tracking in which the plant controlled consists of a pure gain.

MODEL FORM

The transfer function of the HO must contain a pure lag which is caused by neuromuscular delay and central processing in the brain. In order to minimize the error due to this delay the HO must make a prediction of the input signal based on previous knowledge of the signal and his present observations of the signal state. Elkind reasoned that the predictor would be such as to minimize the mean square error within the natural limitations of the HO. Minimizing mean square error is used as the criterion mainly because it is the only mathematically tractable function of error, and it is used as the major

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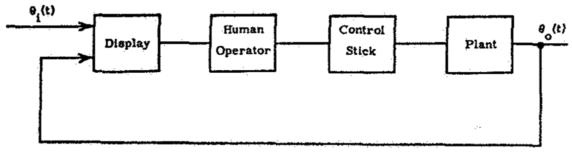

FIGURE 1.—**Block diagram of pursuit tracking task.**

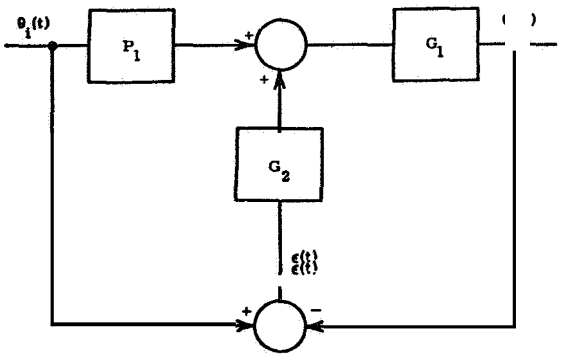

FIGURE 2.—**Elkind's (ref. 3) proposed pursuit tracking model.**

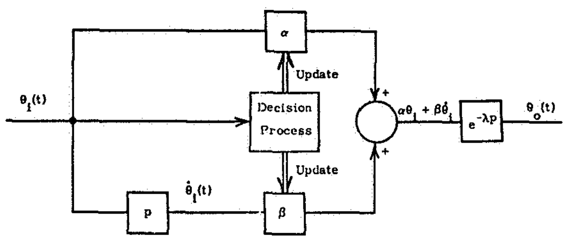

FIGURE 3.—**Schematic of proposed pursuit tracking model.**

criterion of performance in many studies. If it happens that the signal is a sample function of a Gaussian random process, then the predictor that minimizes mean square error will also minimize any error criterion that is symmetrical and a nondecreasing function for nonnegative argument. Since it is known that the HO can determine position quite accurately, is also able to perceive and use velocity information, but is much less able to use higher derivative information, the predictor model takes the form:

$$\theta_0(t) = (\alpha + \beta p) e^{-\lambda p} \theta_i(t)$$

in which λ is the prediction interval, equal to subject delay.

It is proposed, then, that if it is possible to construct a model in the form shown in figure 3 in which, by observation of the input signal α and β converge to the optimal predictor values, then such a model will be a satisfactory representation of human signal tracking behavior for a wide range of input signals.

It is easy to show (see appendix) that the optimal values of α and β are given by

$$\alpha = \frac{R(\lambda)}{R(0)}$$

$$\beta = \frac{R'(\lambda)}{R''(0)}$$

in which $' = d/d\tau$, if the following conditions are met:

(1) $E\{\theta_i(t)\} = 0$
(2) $R(\tau) = E\{\theta_i(t+\tau)\theta_i(t)\}$
(3) $\dot{\theta}_i(t)$ exists.

If $\hat{R}(0,t)$ is the output of the first order filter shown in figure 4, then it can be shown that $\hat{R}(0,t)$ is an unbiased estimator of $R(0)$. Similar circuits can be used to compute unbiased estimates of $R(\lambda)$, $R'(\lambda)$, $R''(0)$ (see appendix for details) and these may be used to form estimates of the optimal position and velocity weightings:

$$\hat{\alpha}(t) = \frac{\hat{R}(\lambda,t)}{\hat{R}(0,t)}$$

$$\hat{\beta}(t) = \frac{\hat{R}'(\lambda,t)}{\hat{R}''(0,t)}.$$

Experimental evidence indicates that the ability of the HO to use velocity information decreases with increasing signal velocity suggesting that a biased estimator of the velocity weighting should be used. Consider the expression for β

$$\beta = \frac{R'(\lambda)}{R''(0)}$$

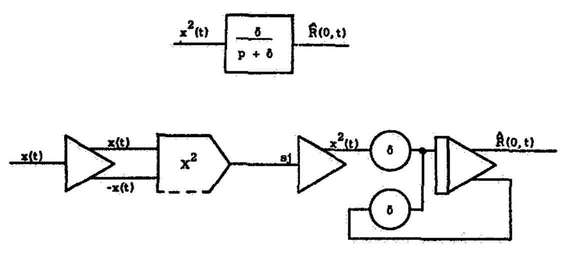

FIGURE 4.—**Circuit for generating $\hat{R}(0,t)$ (see appendix for circuit notation).**

but since $R'(0)=0$ (which follows from the fact that $R(\tau)$ is symmetrical and that $\dot{\theta}_i(t)$ exists), then

$$\beta/\lambda=\frac{[R'(\lambda)-R'(0)]/\lambda}{R''(0)}$$

and

$$\beta/\lambda\cong\frac{R''(\lambda)}{R''(0)}=\beta^*/\lambda.$$

This approximation is essentially the same as replacing

$$\frac{\theta_i(t+\lambda)-\theta_i(t)}{\lambda}$$

by $\dot{\theta}_i(\lambda)$.

This approximation to the optimal velocity weighting is exactly the same operator as used to generate the optimal position weighting but applied to the derivative of the input rather than the input itself.

PROPERTIES OF THE ESTIMATORS

In order to check the model and compare it with HO behavior tests were run with four different input signals. These consisted of essentially white noise passed through the filter,

$$G(j\omega)=\frac{\omega_0{}^N}{(\omega_0+j\omega)^N},$$

with $\omega_0=1.57, 3.14$ and $N=3,4$. In order to get a reasonable convergence time of the model without excessive variance in the parameters, comparisons were made of the model performance as a function of the time constant of the filter used to compute the autocorrelation functions. A time constant of *10* sec was selected **as** being a suitable compromise. Table *1* gives the properties of the estimators for the four test signals. Reference *4* contains a study of the linear correlation of the model indicating that the variation in the parameters is sufficiently small that, compared to the HO, the model can be considered as linear and time invariant.

PURSUIT TRACKING TESTS

In order to validate the steady state behavior of the model and compare it with actual **HO** performance a pursuit tracking experiment was conducted. Three right-handed male subjects practiced tracking each of the four signals with the data presented taken on the fourth day. This amounted to *15* minutes of tracking eachsignal per subject prior to the final data run. This was deemed sufficient in light of the simplicity of the task. A force stick was used throughout the experiment.

The output of the force stick was passed through a *40* rad/sec first order filter to reduce high frequency noise in the circuit. A plant with such a small time constant *(25* milliseconds) is not noticeable during tracking, the major effect being a small phase lag at high frequencies (8.5° at *6* rad/sec). All data are plotted including this filter and with the model outputs also through the same filter for uniformity.

The average subject data is compared with the model transfer function in figures *5, 6,* and *7* and also with the optimal predictor in figure 8.

A time delay of *0.200* sec ($\lambda=0.200$ sec.) is used throughout for the model delay. For a complete circuit diagram see appendix.

TABLE *1.—Estimator Properties for the Four Test Signals*

$\delta=.1$	Position weighting			Velocity weighting		
Signal	α	Mean of $\hat{\alpha}(t)$	Variance of $\hat{\alpha}(t)$	β/λ	Mean of $\hat{\beta}^*(t)/\lambda$	Variance of $\hat{\beta}^*(t)/\lambda$
1.57 4th	0.989	0.988	0.012	0.97	0.95	0.014
1.57 3rd	.983	.981	.013	.95	.89	.024
3.14 4th	.961	.958	.015	.94	.83	.027
3.14 3rd	.947	.934	.018	.88	.66	.033

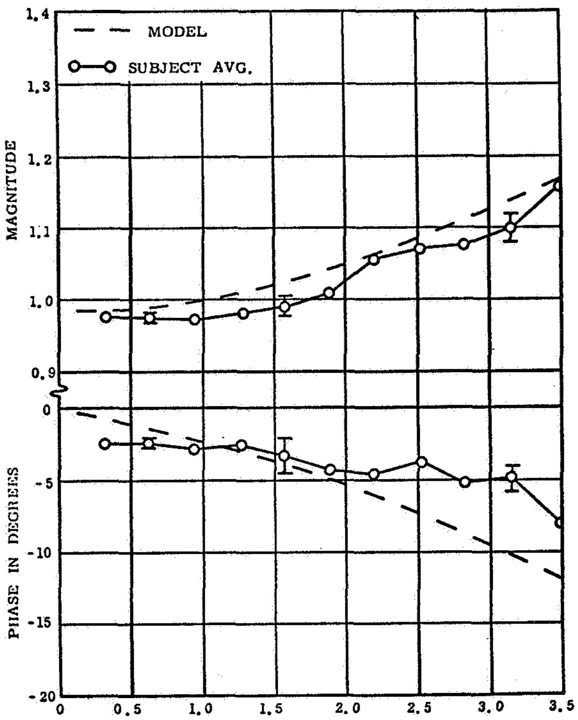

FIGURE 5.—Comparison of model and subject average transfer function: 1.57, 4th.

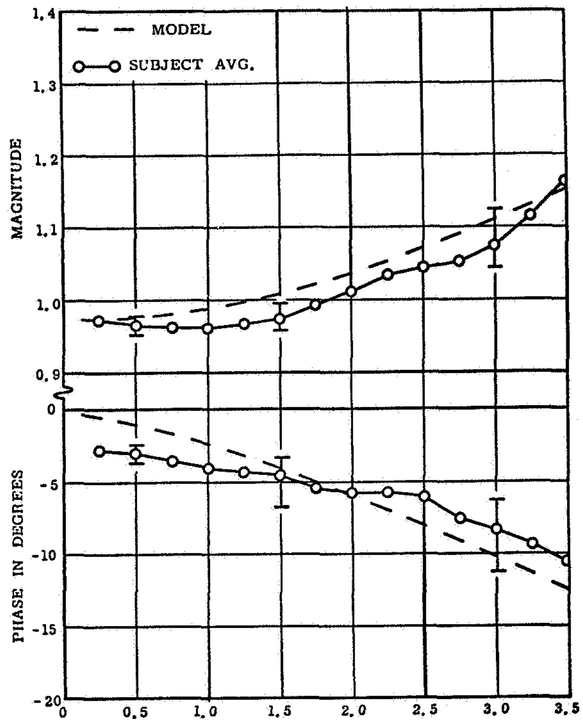

FIGURE 6.—Comparison of model and subject average transfer function: 1.57, 3rd.

As can be seen the model provides a very good fit for the subject data. The major discrepancy appears in the magnitude plots in the low frequency region and the discrepancy is greater for high frequency signals. It can also be seen that in no case does the subject phase lag approach 0°. These effects are no doubt related.

As a check on the quality of the model fit to the subject data, the average absolute difference in magnitude ratio and phase as well as the linear correlation between the two was computed. The results are presented in table 2.

PROBABLE CAUSES OF SUBOPTIMAL BEHAVIOR

Possible explanations for the suboptimal behavior and the low frequency phase droop are:

(1) The subjects are tracking in a partially compensatory manner. Kreifledt (ref. 5) was able to duplicate the suboptimal magnitude ratio at low frequencies obtained by Elkind (ref. 3) by including a compensatory loop in his model.

(2) Subjects are reluctant to correct a small error in a low frequency portion of the signal fearing that a sudden reversal will cause a high error.

(3) For higher frequency signals it becomes increasingly difficult for the subjects to estimate velocity and they therefore appear to decrease their dependence on velocity for prediction. The corresponding increase in the uncertainty of the position weighting is not as pronounced.

POWER MATCH (PM)

A commonly used index of a model performance is called the "power match" (ref. 6). If the difference between model output and subject output is denoted as $e(t)$ then the power match is defined to be

$$PM = 1 - \frac{\overline{e^2(t)}}{\overline{\theta_i^2(t)}}.$$

If $PM = 1$ then the model is a perfect match of subject behavior. Table 3 presents the power match and the mean square error averaged over the three subjects for the last day of trials.

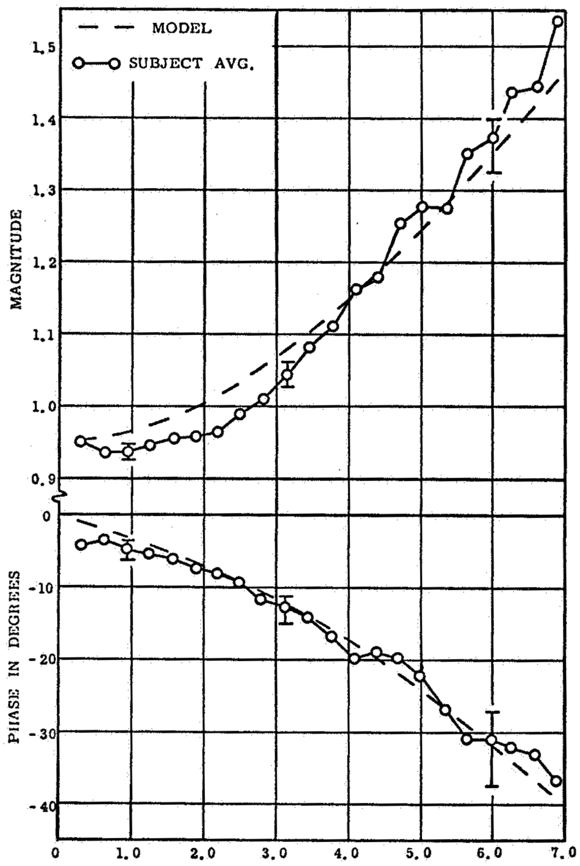

FIGURE 7.—Comparison of model and subject average transfer function: 3.14, 4th.

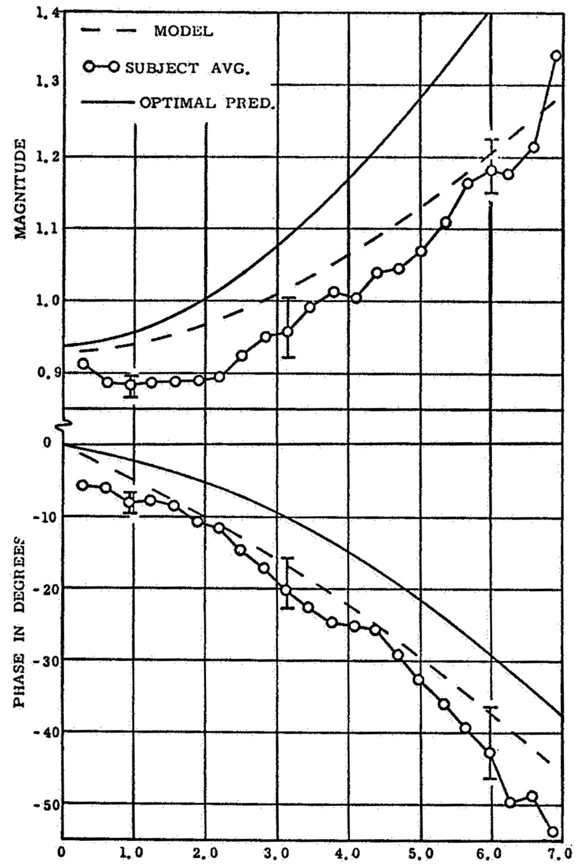

FIGURE 8.—Comparison of model and subject average transfer function: 3.14, 3rd.

TABLE 2.—*Results of Computation of Magnitude Ratio, Phase, and Linear Correlation*

Signal	|ϵ| Magnitude ratio	|ϵ| Phase lag, degree	Linear correlation magnitude	Linear correlation phase
1.57 4th	0.023	2.2	0.982	0.926
1.57 3rd	.024	1.4	.978	.985
3.14 4th	.033	1.7	.995	.993
3.14 3rd	.043	3.9	.982	.992

TABLE 3.—Power Match and Mean Square *Error* Averages

Signal	PM	ϵ^{-2}
1.57 4th	0.998	0.00654
1.57 3rd	.996	.0151
3.14 4th	.995	.0275
3.14 3rd	.989	.0720

Because of the low error scores, it is no surprise that the power match is very nearly unity for all signals. A visual comparison of model and subject error, although resulting in no quantitative measure, nevertheless reveals significant qualitative similarities. Figures 9 and 10 are typical time histories.

SINE WAVE TRACKING

Although the model was constructed to follow signals ccnsisting of low-pass filtered white noise it is also useful as a model for tracking low frequency predictable signals such as sine waves. It is not unreasonable to assume that in sine wave tracking subjects change their mode of behavior at a frequency of *2.5* to 3.5 rad/sec (0.4 to **0.5 Hz)** (ref. 7). Below this frequency subjects follow the signal apparently using short term prediction but above this frequency subjects use the regularity of the signal to establish a rhythmic pattern which they attempt to match with the input signal. One would expect then that the model, if valid for short term prediction **as** presented, should be useful for tracking such signals up to about 3.5 rad/sec.

Figure 11 shows the magnitude ratio of the model versus input sinusoid frequency. The magnitude ratio is very nearly unity to 2.0 rad/sec, drops to **0.95** at 3.0 rad/sec, and falls off rapidly at frequencies above that. This is due primarily to the fact that $\hat{\beta}^*$ is a biased estimator. In fact, for sine wave tracking $\beta^*/\lambda=\alpha$.

Figure 12 compares model performance with data obtained by Magdaleno et al. (ref. **7)** for pursuit tracking of sine waves.

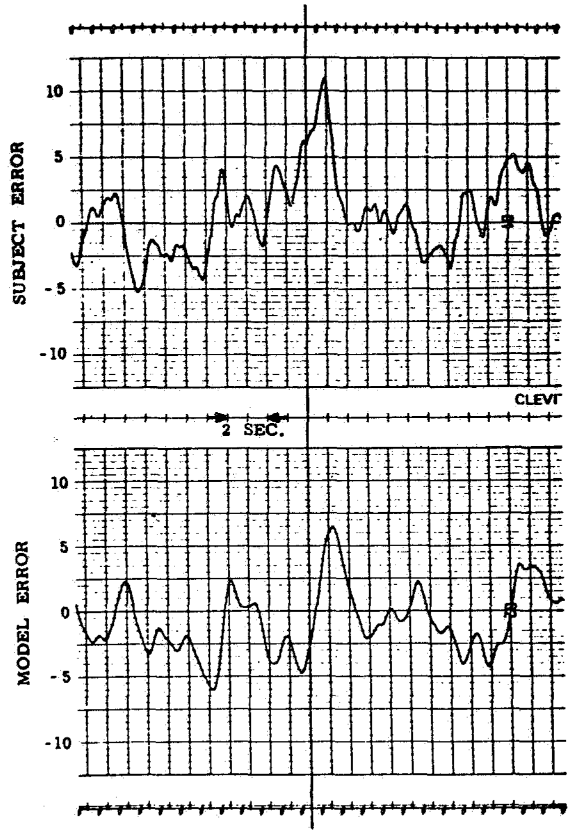

FIGURE 9.—Typical subject and model error time histories. Input signal is 3.14, 4th.

MODEL TRANSIENT BEHAVIOR

Convergence times of $\hat{\alpha}(t)$ and $\hat{\beta}^*(t)$ are functions **of** the convergence rate, initial conditions and the input signal. It is almost meaningless to try to express convergence times in any other context than of the time required to adapt from one signal to another.

Figure 13 shows what occurs during the transition 1.57, 4th to 3.14, 3rd and figure 14 shows the return. The change in signal characteristics is accomplished within a few milliseconds and in such a way that there is no discontinuity in signal position or velocity.

For the transition 1.57, 4th to 3.14, 3rd the "time constant" (most easily seen from the record of $\hat{\beta}^*(t)$) is about 8 sec. However, for the reverse transition the time constant is about 25 sec. This is because relatively more information about the signal is contained in a segment of a

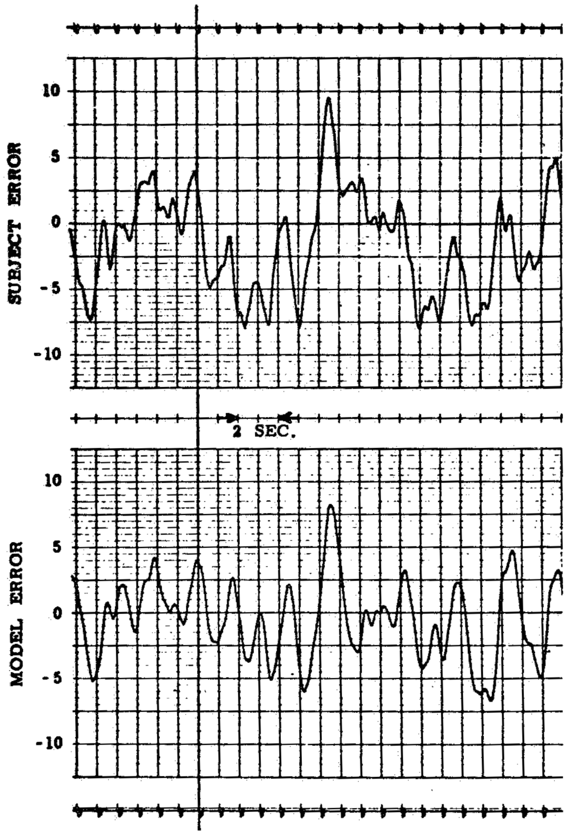

FIGURE 10.—Typical subject and model error time histories. Input signal is 3.14, 3rd.

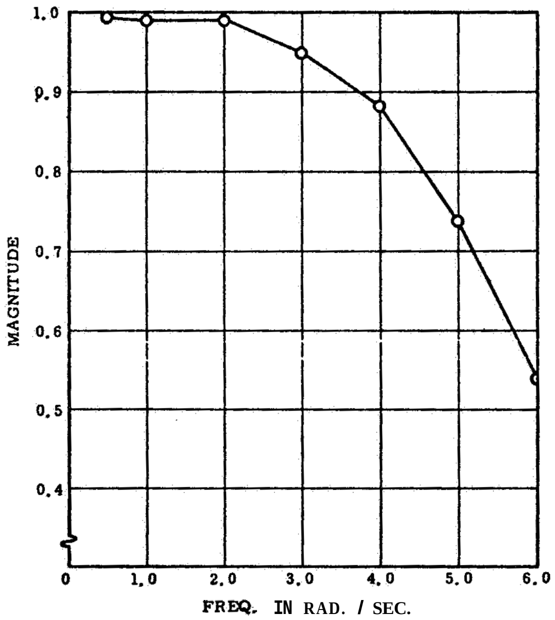

FIGURE 11.—Model response to sine wave inputs.

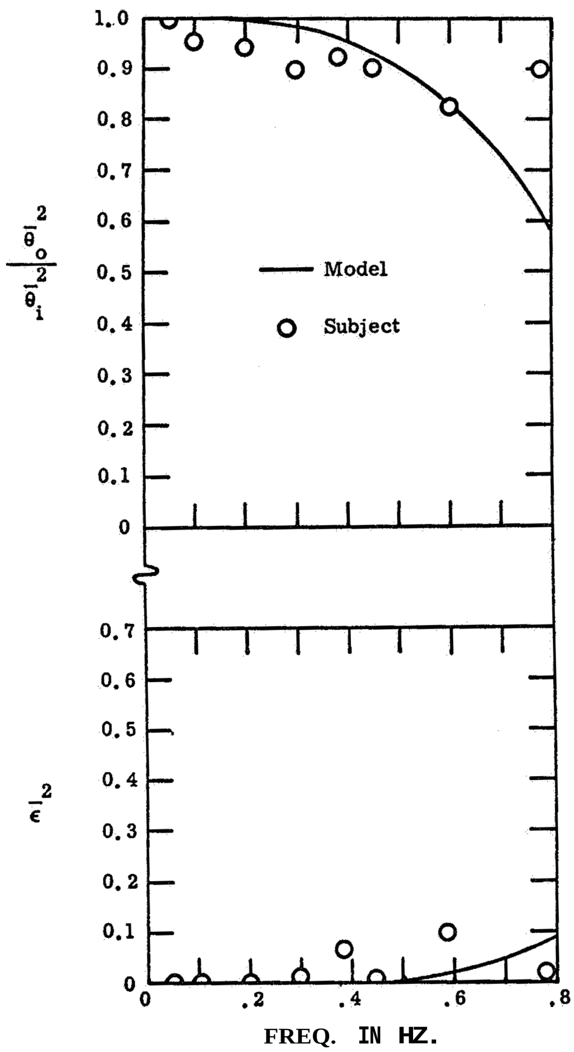

FIGURE 12.—Comparison of subject and model normalized output power and mean square error for sine wave input.

"fast" signal than in the same time segment of a relatively "slow" signal.

The model adapts very quickly to sine waves, as shown in figure **15,** but much more slowly when a transition is reversed as in figure **16.**

Although it is not known how long trained subjects take to adapt to changes in input signal, it is probably less time than the model requires, at least in the case of a change from a relatively fast signal to a relatively slow one. For this reason, the model may not be a valid representation of human behavior during the transition period.

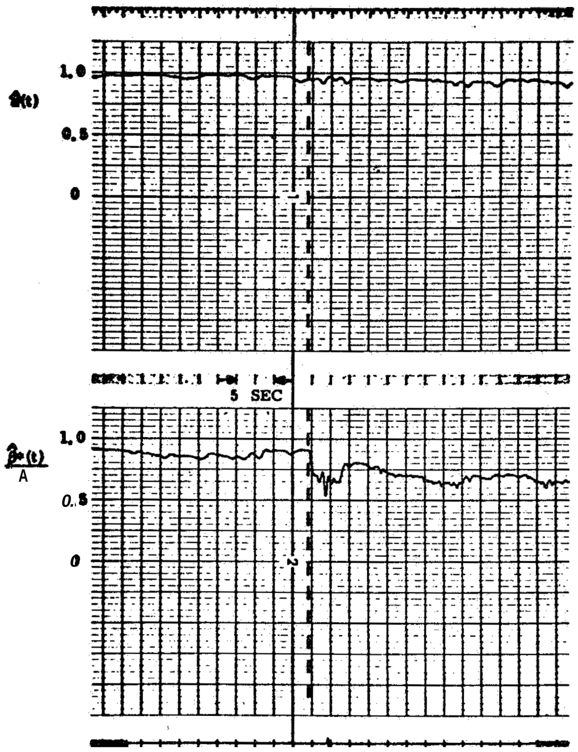

FIGURE 13.—Model parameter response to step change in input signal: 1.57, 4th to 3.14, 3rd.

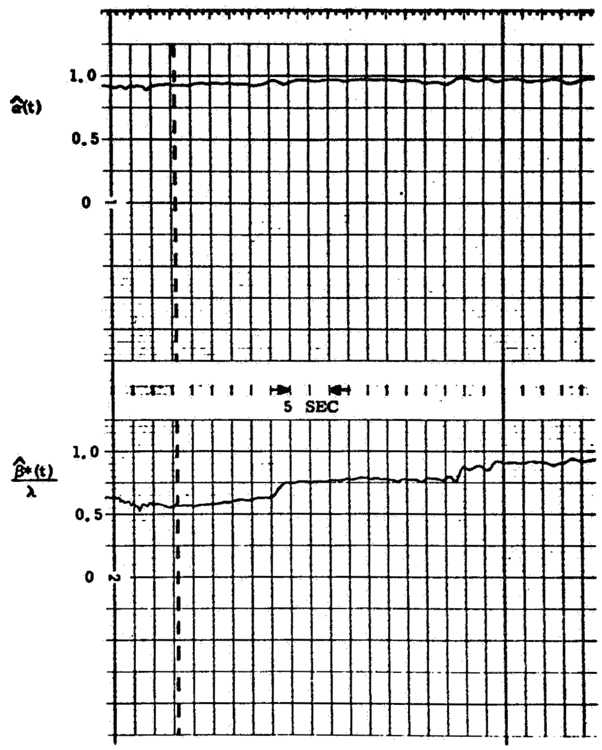

FIGURE 14.—Model parameter response to step change in input signal: 3.14, 3rd to 1.57, 4th.

EFFECTIVE BANDWIDTHS

Since many different forms of input signals are used in the study of human tracking behavior it is desirable to have some measure by which signals may be compared. The most common measure is the "equivalent" or "effective" bandwidth, which compares the nominal cutoff frequency of the input signal to that of a rectangular spectrum.

Magdaleno and Wolkovitch (ref. 8) suggested that the number of axis crossings would be a useful measure. Elkind (ref. **9)** proposed that effective bandwidths be computed by

$$\omega_e=\frac{\left[\int_0^\infty S(\omega)\,d\omega\right]}{\int_0^\infty S^2(\omega)\,d\omega}$$

in which

ω_e =effective bandwidth,
$S(\omega)$ =signal spectrum.

However, this cannot be used if the input signal is composed of the sum of sinusoids. The positive spectrum of this signal is made N delta functions located at $\omega_1, \omega_2, \ldots, \omega_N$. That is, the positive spectrum

$$S(\omega)=\delta(\omega-\omega_1)+\delta(\omega-\omega_2)+\cdots\delta(\omega-\omega_N)$$

in which, for simplicity, all magnitudes are assumed to be unity. Thus when the magnitudes are all unity the effective bandwidth is

$$\omega_e=\frac{N^2}{N}$$

$$\omega_e=N.$$

This result can hardly be considered valid since the effective bandwidth is totally independent of the frequencies of the sine waves summed.

This difficulty cannot be avoided by assuming that the summed sine waves approximate a continuous function because the spacing and the number of sine waves used are of importance. In order to show this consider another measure of

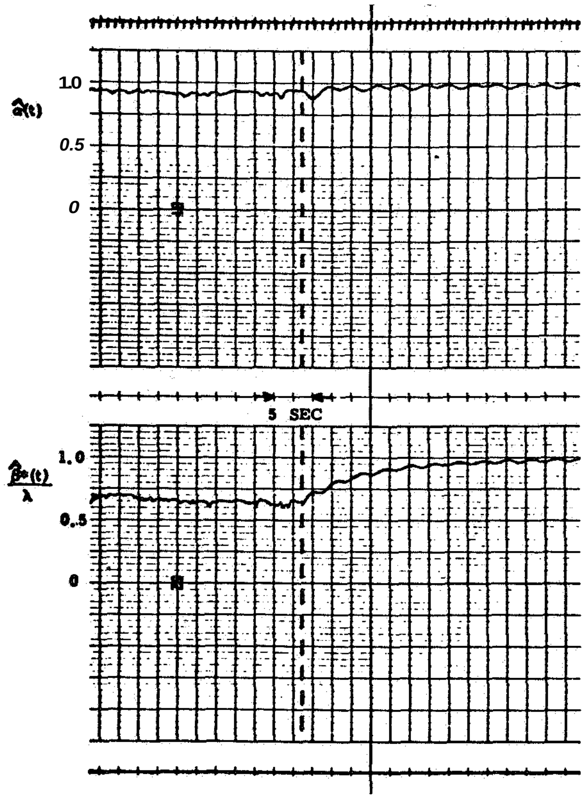

FIGURE 15.—Model parameter response to step change in input signal: 3.14, 3rd to 1.0 rad/sec sine wave,

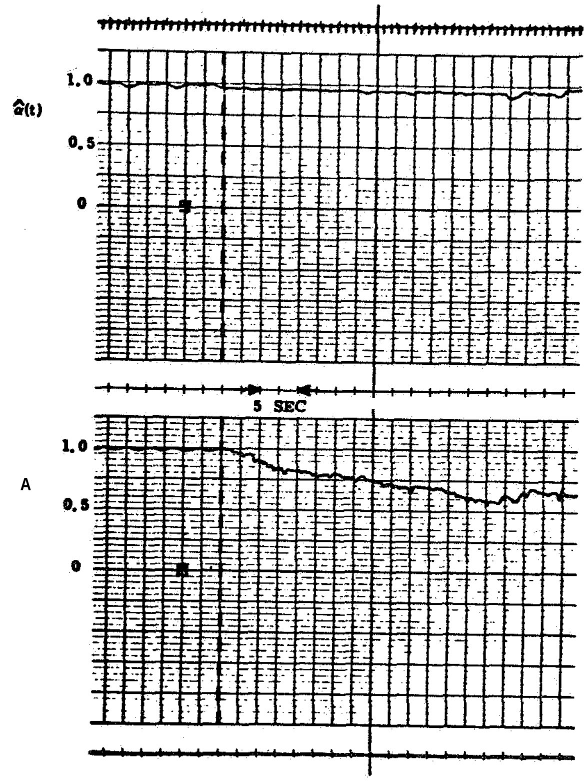

FIGURE 16.—Model parameter response to step change in input signal: 1.0 rad/sec sine wave to 3.14, 3rd.

effective bandwidth. Since the human operator must predict the input signal one time delay ahead the relation between the signal at time t and at time $t+h$ should serve as a useful measure of bandwidth. This number is, of course, the optimal position weighting defined earlier:

$$\alpha=\frac{R(\lambda)}{R(0)}$$

Table 4 shows the effective bandwidths for signals composed of white noise filtered by

$$G(j\omega)=\frac{\omega_0{}^N}{(\omega_0+j\omega)^N},\ N=1,2,3,4.$$

As can be seen the correlation technique agrees quite well with the other proposed measures. The correlation technique does not give a linear relationship between nominal cutoff frequency and equivalent bandwidth as the others do. See ref. 4 for details.) Figure 17 shows the result of attempting to approximate a rectangular spectrum by summing sinusoids of equal magnitude and random phase. The effective bandwidth, as computed using the correlation technique, is plotted versus the number of sinusoids summed for both equal and logarithmic frequency spacing. The significant observation is that while the effective bandwidth approaches the desired level for equal spacing, it decreases continually for logarithmic spacing. This is due to the concentration of power at low frequencies and sharply points out the possible pitfalls of logarithmic spacing.

TABLE 4.—*Effective Bandwidths as a Function of Nominal Cutoff Frequency*

Filter order	Axis crossings	Elkind	Correlation $\lambda = .200$
1	$1.73\ \omega_0$	$3.14\ \omega_0$	$2.4\ \omega_0$
2	$1.73\ \omega_0$	$1.25\ \omega_0$	$1.34\ \omega_0$
3	$.78\ \omega_0$	$0.90\ \omega_0$	$0.95\ \omega_0$
4			

The major advantage to using the correlation

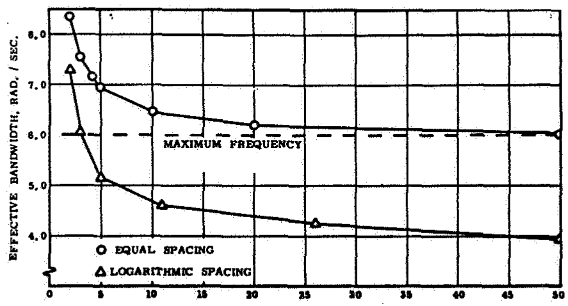

FIGURE 17.—Effective bandwidth vs number of equal amplitude sinusoids summed for different spacing techniques.

technique is that the calculations are easily made given only a time history of the input signal and it is possible to use the recursive estimation procedures outlined earlier to make on-line, quantitative measurements of signal characteristics and changes in signal characteristics.

SUMMARY

For pursuit mode, signal tracking tasks in which the input is low pass filtered white noise it is postulated that the human operator behaves approximately as an optimal predictor attempting to overcome his inherent time delay. Because humans appear to use only position and velocity information the predictor is of the form

$$G(j\omega) = (\alpha + \beta j\omega)e^{-\lambda j\omega}.$$

Test results indicate that the ability to use velocity information decreases with increasing signal velocity indicating a biased estimator of the optimal velocity weighting should be used. A suitable approximation is derived which is equivalent to approximating velocity on the basis of the difference of two successive samples.

A recursive procedure is derived for estimating the position and velocity weighting based on observations of the input signals and applied to an analog simulation. The model thus derived is shown to be a good approximation to human operator steady state behavior and adapts quickly to input signal changes. Since human operators may be very fast in their adaptation the model may not be valid during the transition period.

The model can also be used to track deterministic signals up to approximately the frequency at which human operators would ordinarily begin to track precognitively. The ability of the model to adapt to both random and deterministic signals appears to be a step toward a more general model of tracking behavior.

The optimal position weighting was shown Bo be a simple and convenient measure of effective signal bandwidth.

Since the optimal position weighting is a measure of effective bandwidth, it can be thought of as a measure of signal "coherence". It may be possible to use this measure, perhaps with longer delay times, as an indicator of the sampling rate in a multiaxis task. For example, if an operator is presented with two displays, one consisting of a rapidly moving target and the other consisting of a relatively slow moving target, and asked to track both he must of necessity spend more time observing the faster moving target. Thus an inverse relationship exists between signal coherence and sampling time, and there is a strong likelihood that this relationship could be used for adaptive modelling of this type of system.

The predictive portion of the model could also be used as a signal preprocessor relieving the operator of the burden of prediction and thus reducing his workload, especially in control of systems with delay significantly greater than the operator's.

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APPENDIX

OPTIMAL PREDICTION

Suppose $x(t)$ is a real, stationary random process with zero mean whose first derivative exists. That is

(1) $E\{x(t)+\tau)x(t)\} = R_{xx}(t+\tau,t) = R_{xx}(\tau)$
(2) $E\{x(t)\} = 0$
(3) $\frac{d}{dt}(x(t))$ exists.

(Note: since no confusion is possible the subscript xx will be dropped which will greatly simplify notation; i.e., $R(\cdot) = R_{xx}(\cdot)$). Select a and β so that

$$\hat{x}(t+\lambda) = \alpha x(t) + \beta \dot{x}(t)$$

$$\dot{} = \frac{d}{dt}$$

minimizes

$$\bar{\epsilon}^2 = E((x(t+\lambda) - \hat{x}(t+\lambda))^2\}$$

$$\bar{\epsilon}^2 = E\{(x(t+\lambda) - \alpha x(t) - \beta \dot{x}(t))^2\}.$$

Since the process has zero mean it is possible to apply the orthogonality principle which states, roughly speaking, that in order to minimize mean square error the error vector must be orthogonal to the estimation vectors. Thus the following relationships hold true:

$$E\{(x(t+\lambda) - \alpha x(t) - \beta \dot{x}(t))x(t)\} = 0 \quad \text{(A.l)}$$

$$E((x(t+\lambda) - \alpha x(t) - \beta \dot{x}(t))\dot{x}(t)\} = 0 \quad \text{(A.2)}$$

since (see ref. 10)

$$E\{x(t+\tau)\dot{x}(t)\} = -R'(\tau)$$

$$E\{\dot{x}(t+\tau)\dot{x}(t)\} = -R''(\tau)$$

$$' = \frac{d}{d\tau}$$

Equations (A.l) and (A.2) can be simply expressed as

$$R(\lambda) - \alpha R(0) + \beta R'(0) = 0$$

$$-R'(\lambda) - \alpha R'(0) + \beta R''(0) = 0.$$

Since $R(\tau)$ is symmetric and since the derivative $\dot{x}(t)$ exists then

$$R'(0) = 0$$

and the result is the simple relations

$$\alpha = \frac{R(\lambda)}{R(0)}$$

$$\beta = \frac{R'(\lambda)}{R''(0)}.$$

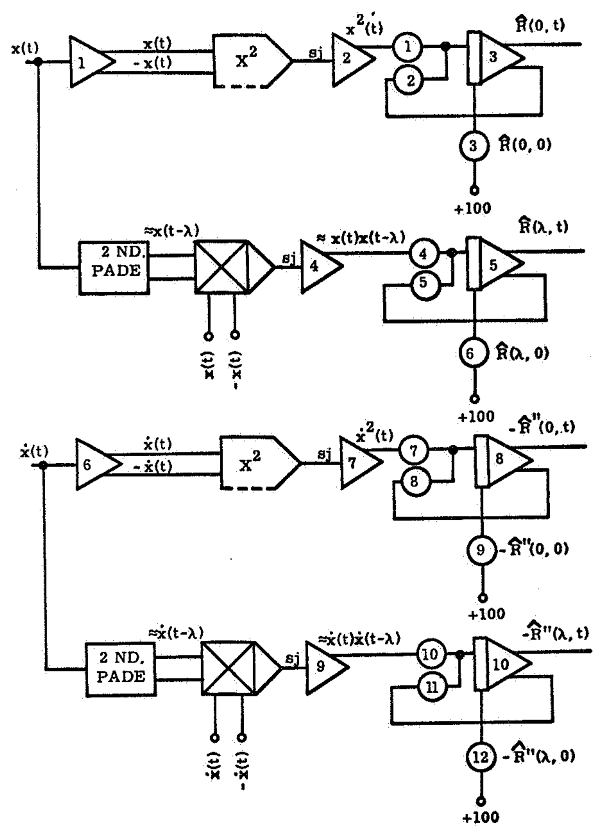

FIGURE A-1.—Circuit for computing estimators of autocorrelation function.

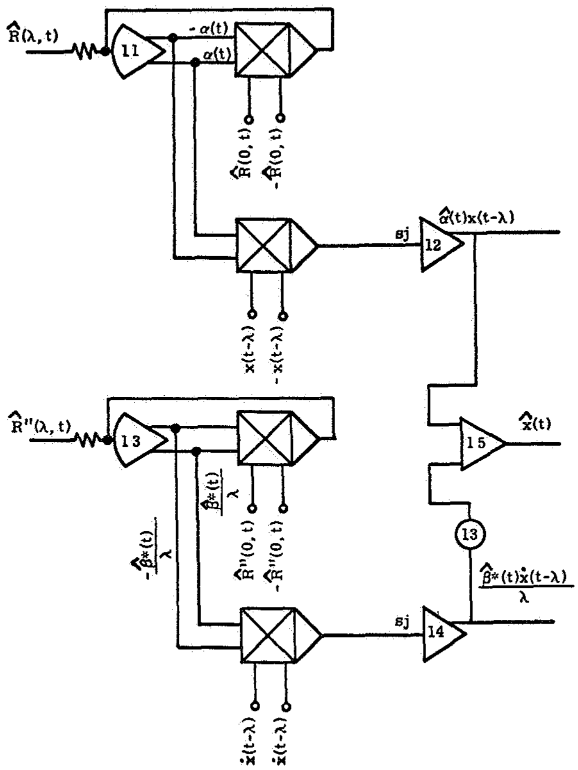

FIGURE A-2.—Circuit **for computing position and velocity weighting.**

CONTINUOUS ESTIMATION OF AUTOCORRELATION

A very simple method for estimating the autocorrelation function and its derivatives for an analog simulation is given below.

Suppose $x(t)$ is a sample function of a stationary random process. It is easy to show that the output of the simple analog circuit of figure **4** is given by

$$\hat{R}(0,t)=\hat{R}(0,0)e^{-\delta t}+\delta e^{-\delta t}\int_0^t x^2(u)e^{\delta u}\,du.$$

Taking expected values

$$E\{\hat{R}(0,t)\}=\hat{R}(0,0)e^{-\delta t}+\delta e^{-\delta t}E\left\{\int_0^t x^2(u)e^{\delta u}\,du\right\}.$$

On interchanging expectation and integration this becomes

TABLE A-1.—*Suggested Coefficient Settings*

Coefficient device	Setting
1	.3000
2	.1000
3	1.0000
4	.3000
5	.1000
6	1.0000
7	,3000
8	.1000
9	1.0000
10	.3000
11	.1000
12	1.0000
13	.2000

$$E\{\hat{R}(0,t)\}=\hat{R}(0,0)e^{-\delta t}+\delta e^{-\delta t}\int_0^t E\{x^2(u)\}e^{\delta u}\,du$$

$$E\{\hat{R}(0,t)\}=\hat{R}(0,0)e^{-\delta t}+R(0)[1-e^{-\delta t}]$$

$$\lim_{t\to\infty} E\{\hat{R}(0,t)\}=R(0).$$

In a similar manner it is possible to find simple analog circuits (fig. A.1) for estimating the other necessary values of the autocorrelation function and its derivatives. (N.B. All amplifiers are bi-polar.) These will yield the equations given below:

$$\hat{R}(\lambda,t)=\hat{R}(\lambda,0)e^{-\delta t}+\delta e^{-\delta t}\int_0^t x(u)x(u-\lambda)e^{\delta u}\,du$$

$$\hat{R}''(\lambda,t)=\hat{R}''(\lambda,0)e^{-\delta t}-\delta e^{-\delta t}\int_0^t \dot{x}(u)\dot{x}(u-\lambda)e^{\delta u}\,du$$

$$\hat{R}''(0,t)=\hat{R}''(0,0)e^{-\delta t}-\delta e^{-\delta t}\int_0^t x^2(u)e^{\delta u}\,du.$$

These are used to form estimators of ***a*** and β by forming:

$$\hat{\alpha}(t)=\frac{\hat{R}(\lambda,t)}{\hat{R}(0,t)} \qquad \frac{\hat{\beta}^*(t)}{\lambda}=\frac{\hat{R}''(\lambda,t)}{\hat{R}''(0,t)}$$

Figure **A.2** is a circuit diagram for computing position and velocity weighting.

Table **A-1** gives suggested coefficient settings.