NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-601

Analysis of the Computed Torque Drive Method and Comparison With Conventional Position Servo for a Computer-Controlled Manipulator

B. R. Markiewicz

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

March 15, 1973

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-601

Analysis of the Computed Torque Drive Method and Comparison With Conventional Position Servo for a Computer-Controlled Manipulator

B. R. Markiewicz

PREFACE

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.

CONTENTS

CONTENTS (contd)

ABSTRACT

A manipulator and its control system (modeled after a Stanford design) is being developed at JPL as part of an artificial intelligence project. This development includes an analytical study of the control system software. This report presents a comparison of the computed torque method and the conventional position servo. No conclusion is made as to the preference of one system over the other, as it is dependent upon the application and the results of a sampled data analysis.

I.   INTRODUCTION

A manipulator and its control system (modeled after a Stanford design) is being developed at JPL as part of an Artificial Intelligence project. This development includes an analytical study of the control system software. The purpose of this report is to present the results of this study.

The Stanford manipulator control system models the manipulator and its dynamics and computes the motor drive torque for each joint of the manipulator. If the model were exact, the manipulator could be driven open-loop with no error. But because the model is not exact, rate and position feedback is used in addition to the computed drive torque.

The manipulator operation is described from a control analysis viewpoint for the "Computed Torque" method in Section II and for a conventional position servo in Section III. The gains and output response for both systems are derived and discussed in detail. A comparison is made of the two control methods in Section IV. The Stanford gain values are used here for the "Computed Torque" method. But the final values may be different because the JPL application does not require as high turning rates, and the design is slightly different.

II.   COMPUTED TORQUE SYSTEM

In the "Computed Torque" method the trajectory that the manipulator is to follow as a function of time is computed in the planning program. Each link angle (the angle of the manipulator joint) as a function of time, necessary to follow the desired trajectory, is also computed in the planning program. The drive motor torque required to achieve the link angle/time function is then computed based on a model of the manipulator hardware. The motor voltage/torque characteristic is also modeled and the computed torque is converted to applied motor voltage. This voltage is computed at a very high rate, applied to D/A converter, and finally to the motor input.

If the manipulator and motor models were exact, the response would be exact, but this is not possible because of modeling inaccuracies and parameter variations. Rate and position feedback is therefore used to compute correction torques which are summed with the main driving torque. The result is a sampled data control system with performance dependent on the feedback gain values.
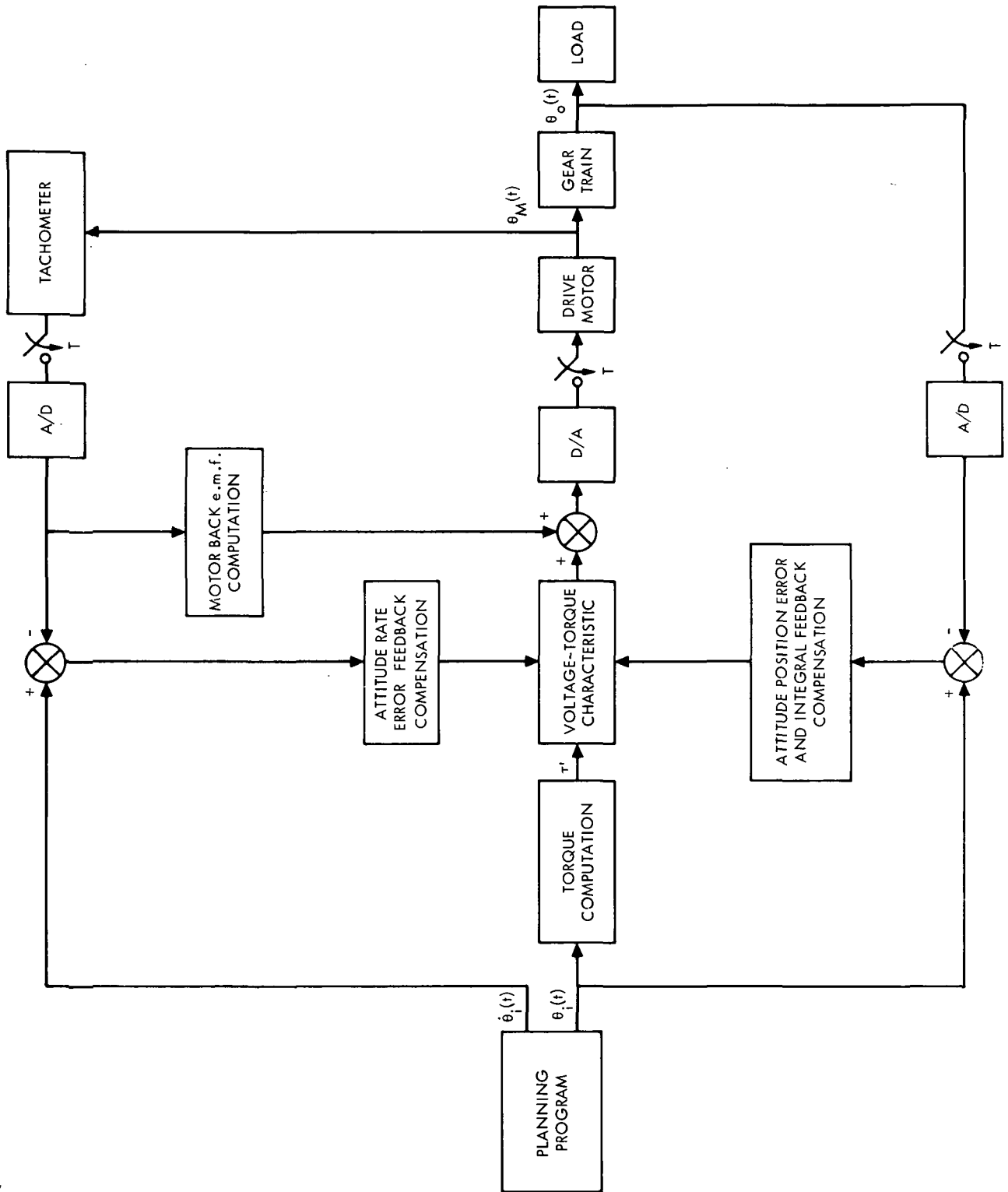
Fig. 1.   Physical Configuration of Computed Torque Method Control Loop
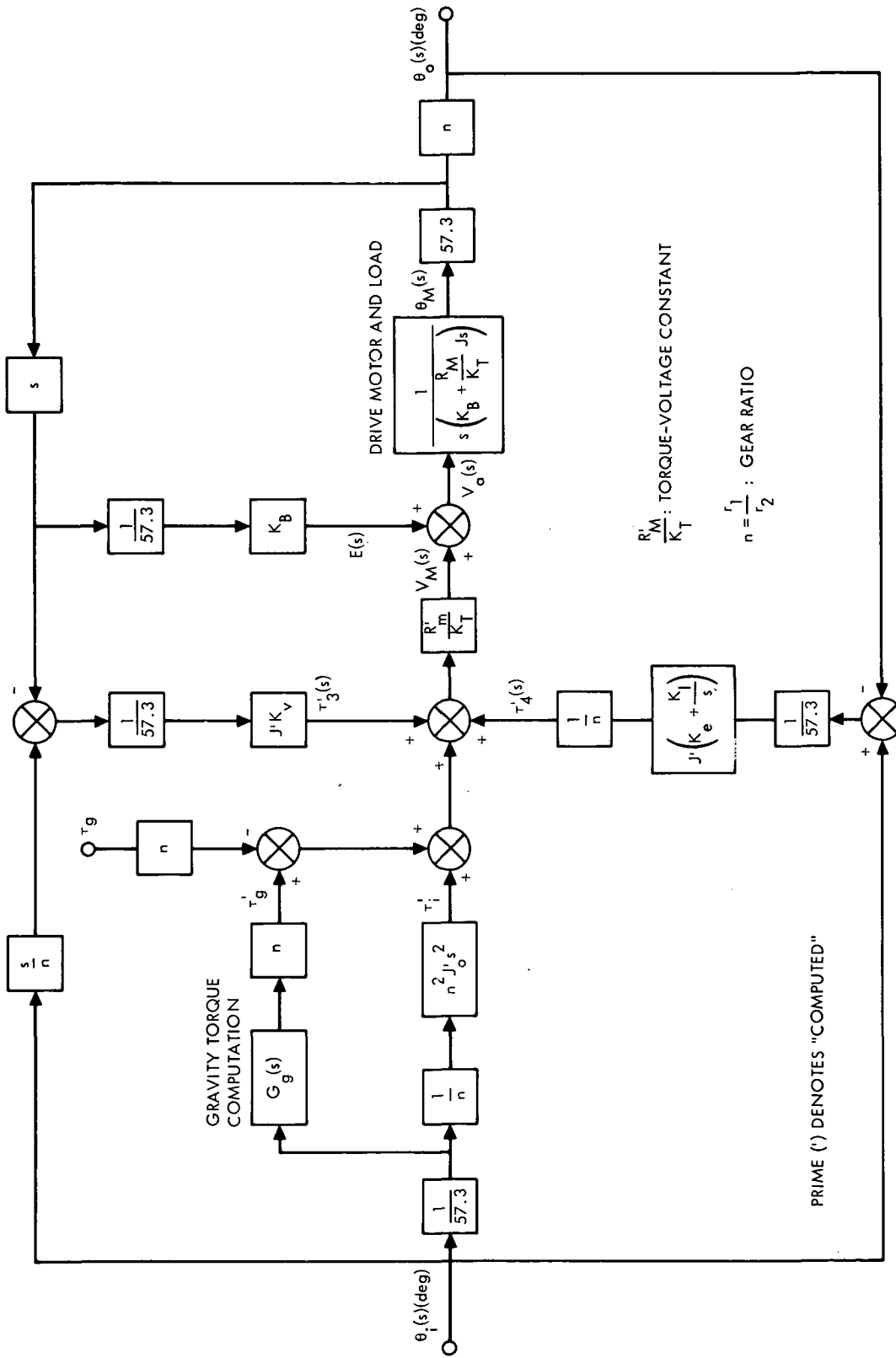
Fig. 2. Transfer Function Control Loop Configuration

1.   Transfer Function Derivation

The interconnection of the physical components and the computer subroutines is shown in Fig. 1, and the detailed flow diagram for control analysis is shown in Fig. 2. The latter does not contain the data samplers because this analysis is for a linear continuous system. A subsequent sampled data analysis is required to determine the effect of the sampling frequency. The motor and gear train nonlinearities are modeled in the voltage torque characteristic which allows the linear analysis approach.

The control loop configuration can be reduced to the configuration of Fig. 3, with the branch transfer functions defined as follows:
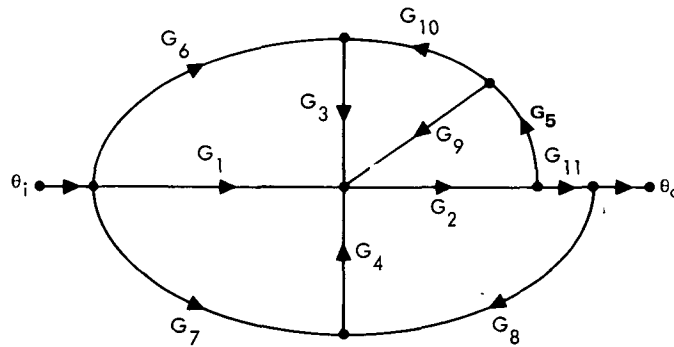
Fig. 3.   Computed Torque System Signal Flow Graph

$$G_1(s) = \frac{1}{57.3n} J' \frac{R'_M}{K_T} s^2$$

$$G_2(s) = \frac{57.3}{s \left(K_B + \frac{R_M}{K_T} J s \right)}$$

$$G_3(s) = \frac{1}{57.3} J' K_V \frac{R'_M}{K_T}$$

$$G_4(s) = \frac{1}{57.3n} J' \left(K_e + \frac{K_I}{s} \right) \frac{R'_M}{K_T}$$

$$G_5(s) = s$$

$$G_6(s) = \frac{1}{n} s$$

$$G_7(s) = 1$$

$$G_8(s) = -1$$

$$G_9(s) = \frac{K_B}{57.3}$$

$$G_{10}(s) = -1$$

$$G_{11}(s) = n$$

From signal flow graph theory

$$\frac{\theta_o}{\theta_i}(s) = \frac{G_1 G_2 G_{11} + G_6 G_3 G_2 G_{11} + G_7 G_4 G_2 G_{11}}{1 - G_5 G_{10} G_3 G_2 - G_8 G_4 G_2 G_{11} - G_5 G_9 G_2} \tag{1}$$

$$= \frac{(G_1 + G_6 G_3 + G_7 G_4) G_{11}}{\dfrac{1}{G_2} - (G_5 G_{10} G_3 + G_8 G_4 G_{11} + G_9 G_5)}$$

$$= \frac{J' \dfrac{R'_M}{K_T} s^2 + J' K_v \dfrac{R'_M}{K_T} s + J'\left(K_e + \dfrac{K_I}{s}\right) \dfrac{R'_M}{K_T}}{J \dfrac{R_M}{K_T} s^2 + s K_B + J' K_v \dfrac{R'_M}{K_T} s + J'\left(K_e + \dfrac{K_T}{s}\right) \dfrac{R'_M}{K_T} - K_B s}$$

$$= \frac{J' \dfrac{R'_M}{K_T} s^2 + J'_o \dfrac{R'_M}{K_T} K_v s + J' \dfrac{R'_M}{K_T}\left(K_e + \dfrac{K_I}{s}\right)}{J \dfrac{R_M}{K_T} s^2 + J' \dfrac{R'_M}{K_T} K_v s + J' \dfrac{R'_M}{K_T}\left(K_e + \dfrac{K_I}{s}\right)} \tag{2}$$

If $J' = J$, $R'_M/K_T = R_M/K_T$.

$$\frac{\theta_o}{\theta_i}(s) = \frac{s^3 + K_v s^2 + K_e s + K_I}{s^3 + K_v s^2 + K_e s + K_I} = 1 \qquad (3)$$

Each of the individual branch transfer functions $(G_1 \to G_{11})$ will now be described in detail.

$\underline{G_2}$: Drive motor and load.

The voltage applied to the motor is

$$V_a = E + i_M R_M \qquad (4)$$

where

E = back e.m.f. voltage   (volts)

i_M = motor current   (DC amps)

R_M = motor resistance   (ohms)

The motor torque (referred to motor shaft) is a function of the current and the torque constant, K_T for a particular motor.

$$\tau_M = K_T i_M \qquad (5)$$

where

K_T = motor torque constant   (oz-in./amp)

The back e.m.f. is a function of another constant for the particular motor.

$$E = K_B \dot{\theta}_M \qquad (6)$$

where

K_B = motor back e.m.f. constant $\left(\dfrac{\text{volts}}{\text{rad/sec}}\right)$

$\dot{\theta}_M$ = motor shaft rotation rate $\qquad$ (rad/sec)

Substituting for E and i_M in Eq. (4)

$$V_a = K_B \dot{\theta}_M + \frac{R_M}{K_T} \tau_M \tag{7}$$

The motor torque will produce an angular acceleration of the motor shaft.

$$\tau_M = J \ddot{\theta}_M \tag{8}$$

where

$$J = J_M + n^2 J_L \qquad (\text{oz-in.}/\text{rad/sec}^2 \equiv \text{oz-in.-sec}^2) \tag{9}$$

This total inertia is referred to the output by dividing by n^2

$$J_o = \frac{1}{n^2} J \tag{10}$$

J_M = motor rotor inertia $\qquad$ (oz-in.-sec^2)

J_L = load inertia $\qquad$ (oz-in.-sec^2)

$n = \dfrac{n_1}{n_2}$ = ratio of motor gear teeth to load gear teeth

$\ddot{\theta}_M$ = motor shaft angular acceleration $\qquad$ (rad/sec^2)

As indicated above, the load inertia is referred to the motor shaft by multiplying by the gear ratio squared. This is simply derived by using the conservation of energy principle.

$$\frac{1}{2} J_{LM} \dot{\theta}_M^2 = \frac{1}{2} J_L \dot{\theta}_o^2 \tag{11}$$

where

$\qquad J_{LM}$ = load inertia referred to motor shaft

$\qquad \dot{\theta}_o$ = load or output rotation rate

Substituting $\dot{\theta}_o = n \dot{\theta}_M$ results in

$$J_{LM} = n^2 J_L \tag{12}$$

Now, to derive the motor transfer function, substitute τ_M from (8) into (7).

$$V_a = K_B \dot{\theta}_M + \frac{R_M}{K_T} J \ddot{\theta}_M \tag{13}$$

Taking the Laplace transform with I.C. = 0

$$G_2(s) = \frac{\theta_M(s)}{V_a(s)} = \frac{57.3}{s \left(K_B + \dfrac{R_M}{K_T} J s \right)} \tag{14}$$

The motor transfer function is in terms of radians output, but the program as presently defined uses degrees. Therefore, the 57.3 factor is introduced in the numerator.

$\underline{G_1}$: Computed torque

The driving torque (not including error torques) is computed in G_1 from known motor and load parameters and then changed to applied motor voltage. For this analysis the gravity torque shown in fig. 2 is not

included in G_1. This is done because it can be handled as a known disturbance input which is then cancelled by the computed gravity torque, τ_g'. As shown in Fig. 2 the gravity torque τ_g is transferred from the output to the input for analysis purposes. It then has to be reduced by the gear ratio, n. The cancellation can be quite accurate except for the effect of an unknown weight being lifted by the manipulator, although the object weight can be determined from the error signal and then included in the computed gravity torque. This will be discussed under disturbance torque.

With the gravity loop eliminated, the remaining branch of G_1 computes the required motor torque. From known manipulator parameters and $\theta_i(t)$, the changing inertia $J'(t)$ can be computed. Referring to Eq. (8), it is seen that the torque is proportional to the angular acceleration and the total inertia referred to the motor shaft. J_o' (referred to output) is computed in the program, therefore the n^2 factor is needed. Angular acceleration can be computed because $\theta_i(t)$ is analytic (third degree polynomial) and is represented by s^2. The $1/n$ factor is needed because θ_i refers to the manipulator joint angle, but we are now referencing to the motor input. Therefore, θ_i (used to compute the motor torque) must be divided by the gear ratio. The final operation here is to change this required torque to applied motor voltage. This could be done as shown by using a constant R_M'/K_T (volts/oz-in.) which is the torque to voltage conversion constant for a particular motor neglecting motor friction and reduction gear losses. With only coulomb friction torque present (given in motor specs), the constant slope voltage torque curve is displaced up for plus $\dot{\theta}_M$ by the voltage to overcome friction and down for minus $\dot{\theta}_M$. But with the reduction gear, the characteristics are not so simple. An approximation to the curves are shown in Fig. 4. They will be determined from lab tests by measuring the output torque, keeping $\dot{\theta}_M$ constant and varying the applied voltage (the ordinate, V_M does not include the back e.m.f. voltage). The solid lines are theoretical for no gear reduction and the dotted lines connect data points from lab data. Note that there are three different slopes or torque constants (K_{M1}, K_{M2}, K_{M3}). K_{M1} differs from the theoretical value (R_M/K_T) by the constant μ which is the additional voltage required due to loss in the gear transmission. For $+\dot{\theta}_M$ and $+\tau_o$, μ is plus because the loss is opposing the desired output τ_o, but for $-\tau_o$ the loss is aiding. The area covered by K_{M3} is somewhat nonlinear and the value K_{M3}
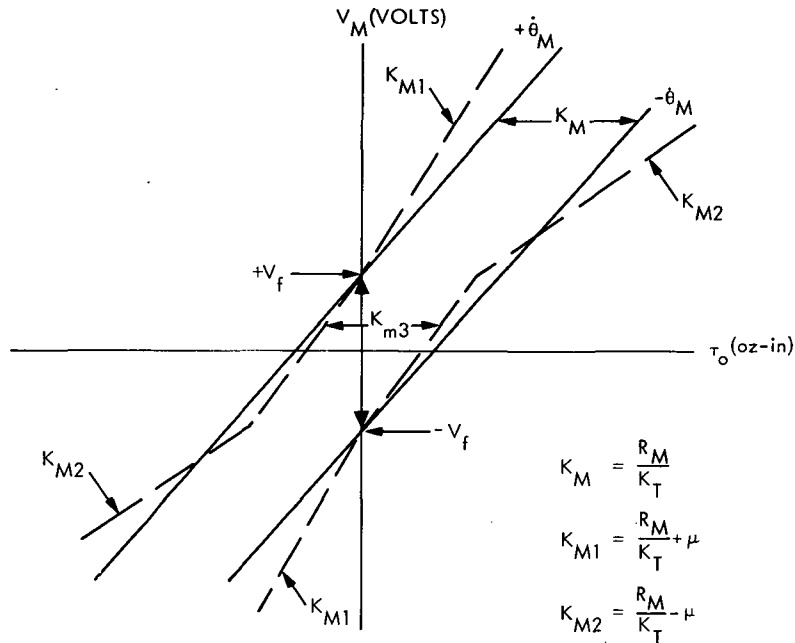
Fig. 4. Typical Drive Motor Voltage - Torque Characteristic

(referring to $+\dot{\theta}_M$) is an approximation joining the ordinate cross point from K_{M1} and a point with an ordinate $-V_f$ on K_{M2}. The point does not seem to have any theoretical relationship but was selected arbitrarily as coinciding with lab data. The final result of all this is that only three torque constants are needed (K_M, μ, K_{M3}), along with the sign of $\dot{\theta}_M$ and the ordinate breakpoints for τ_o, to convert this torque to voltage. Lab tests will be performed to obtain this characteristic.

From Eq. 4 under G_2 the applied motor voltage is

$$V_a = E + i_M R_M = E + V_M \tag{15}$$

With V_M now computed the back e.m.f. (E) has to be computed to get V_a. This is done in G_9 using the motor rate $\dot{\theta}_M$ from the tachometer.

G_6, G_5, G_{10}: Rate error

The motor output rate is obtained from the tachometer (G_5) mounted on the motor shaft. The desired link angular rate ($\dot{\theta}_i$) is computed in the planning program and is multiplied by $1/n$ to reference it to the motor. The two are compared to indicate the rate error. The unity gain, G_{10} is only for transfer function derivation purposes to separate the rate error from motor rate which is used in G_9. The tachometer output is in volts, and an A/D converter is required so the computer can interrogate the motor rate. The interrogation of $\dot{\theta}_M$ is at some finite rate, thus, we have a sampled data system.

G_3: Rate error feedback gain

The rate error is converted to radians/sec and multiplied by J' to remove this changing parameter from the system response characteristic. If it remained, the damping ratio and natural frequency (reducing to a second order system) would change with J'. The system damping ratio is now a function of the constant, K_v and of K_e.

G_7, G_8: Position error

The manipulator link rotation angle (θ_o) is obtained from a potentiometer mounted on the link, and the desired link angle (θ_i) is available from the planning program. The two are compared to indicate the position error which is later multiplied by $1/n$ to reference it to the motor. θ_o is in volts, thus, an A/D converter is required. As with $\dot{\theta}_M$ above, the interrogation is at some finite rate, which makes the system a sampled data system which can be analyzed as such using Z transform analysis methods. Any small delay time for reading in θ_o and $\dot{\theta}_M$ can be incorporated in the analysis also.

G_4: Position error and integral feedback gains

The position error is converted to radians and multiplied by J' as with G_3, but here it is to keep the system undamped natural frequency constant. The integral gain (K_I) which will be discussed in Section II-4 is also now a

constant. It should be noted that the integration (K_I/s) is done numerically in the computer as:

(1) Initialize $\tau_{Ii-1} = 0$

$$\tau_{Ii} = \tau_{Ii-1} + \frac{J'}{57.3n} K_I \Delta t \, \theta_\epsilon \tag{16}$$

$$\tau_{Ii-1} = \tau_{Ii}$$

where τ_{Ii} is the computed torque for the present computation cycle

Δt is the computation cycle time

θ_ϵ is the position error in degrees

$\underline{G_9}$: Motor back e.m.f. voltage

In Fig. 2 G_1, G_3 and G_4 do not include the voltage-torque conversion

$$V_M = \left(\frac{R'_M}{K_T}\right) \tau$$

because in the computer program, the three outputs are summed and only one conversion is necessary. In the analysis the factor is included in each only for simplicity, but the total output voltage is $V_M(s)$.

From the G_2 derivation, the applied motor voltage is

$$V_a = E + i_M R_M = E + V_M$$

Thus, the back e.m.f. voltage, E has to be computed. This is done in G_9 in which the known motor constant K_B is used with the tachometer motor rate.

$$E = K_B \dot{\theta}_o$$

2. Response to Input Forcing Functions

As shown in Eq. (3) the output is identical to the input if the inertia, J' is computed correctly and if the voltage torque curve is exact, i.e.,

$$\frac{R'_M}{K_T} = \frac{R_M}{K_T} \; .$$

Therefore, the output or link movement $\theta_o(t)$ exactly follows the input, $\theta_i(t)$ which was computed in the planning program. But practically, errors due to approximations in modeling and unknown system nonlinearities prevent this ideal response, consequently the error feedback loops were included.

It is obvious that if the motor torque is computed correctly, the output should respond correctly in an open loop manner and there would always be zero error. And the previous derivation of the system transfer function would be unnecessary. However, analyzing the response to disturbance torques or unknown gravity torques does require a transfer function. Also, it is possible to insert an inertia error, $\Delta J'$ and determine the response error. Additional transfer functions are derived in the following sections.

3. Transfer Function for Feedback Position and Rate Errors

The two feedback transducers involved in each link control loop are a potentiometer and a tachometer which could have bias or null errors. The transfer functions for these errors can be derived easily from the system transfer function derived in Section II-1.

Position Feedback Null Error:

The flow diagram is indicated in the following figure:
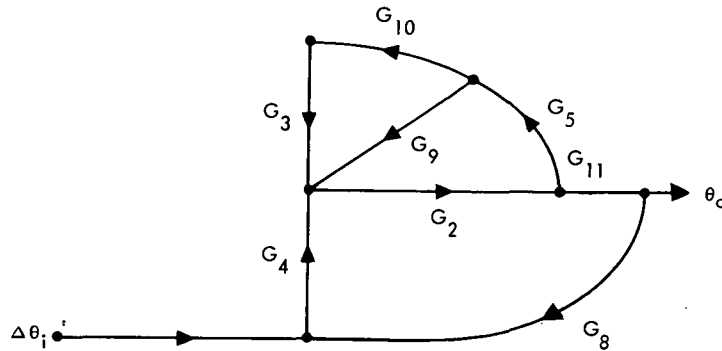
Fig. 5. Signal Flow Diagram for Position Feedback Null Error

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{G_4 G_{11}}{\dfrac{1}{G_2} - (G_5 G_{10} G_3 + G_8 G_4 G_{11} + G_9 G_5)} \tag{17}$$

$$= \frac{J'\left(K_e + \dfrac{K_I}{s}\right)\dfrac{R'_M}{K_T}\, s}{J\dfrac{R_M}{K_T}\, s^3 + J'K_v \dfrac{R'_M}{K_T}\, s^2 + J'\dfrac{R'_M}{K_T} K_e s + J'\dfrac{R'_M}{K_T} K_I} \tag{18}$$

for $R'_M/K_T = R_M/K_T$, $J' = J$

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{(K_e s + K_I)}{s^3 + K_v s^2 + K_e s + K_I} \tag{19}$$

For $\Delta\theta_i(s) = \Delta\theta_i/s$

$$\theta_o(s) = \frac{K_e(s + K_I/K_e)\Delta\theta_i}{s(s^3 + K_v s^2 + K_e s + K_I)} \qquad (20)$$

The steady state error is

$$\begin{array}{cc} \theta_o(t) & = s\,\theta_o(s) \\ t \to \infty & s \to 0 \end{array} = \Delta\theta_i \qquad (21)$$

with $K_I = 0$

$$\theta_o(s) = \frac{K_e\,\Delta\theta_i}{s(s^2 + K_v s + K_e)} \qquad (22)$$

$$\begin{array}{c} \theta_o(t) \\ t \to \infty \end{array} = \Delta\theta_i \qquad (23)$$

Thus, the steady state error is the same with or without the integral gain. The second order transfer function can be in the form

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (24)$$

where ζ is the damping ratio and ω_n the undamped natural frequency. The step function response for $\zeta = 1$ is shown in Fig. 6.

The response with $K_I \neq 0$ is altered very little if the resultant additional pole and zero are close together. Thus, the ratio of K_I/K_e is the determining factor. It should be noted that the above is the system response if the link position is at some different position than θ_i which comes from the planning program.
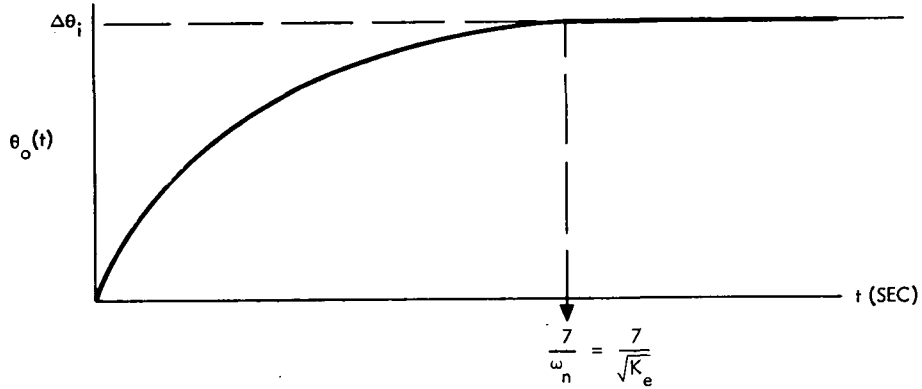
Fig. 6. Step Function Response for Unity Damping Ratio

Rate feedback error:

The signal flow diagram for a rate null error is
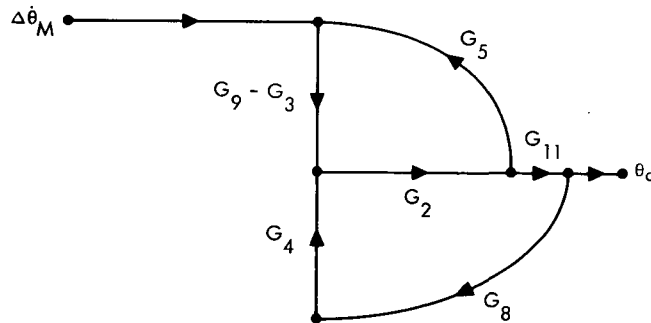
Fig. 7. Signal Flow Diagram for Rate Null Error

$$\frac{\theta_o(s)}{\Delta\dot{\theta}_M(s)} = \frac{(G_9 - G_3) G_{11}}{\dfrac{1}{G_2} - \left[G_5(G_9 - G_3) + G_8 G_4 G_{11}\right]} \tag{25}$$

$$= \frac{n\left(-J' \dfrac{R'_M}{K_T} K_v + K_B\right) s}{J \dfrac{R_M}{K_T} s^3 + J' \dfrac{R'_M}{K_T} K_v s^2 + J' \dfrac{R'_M}{K_T} K_e s + J' \dfrac{R'_M}{K_T} K_I} \tag{26}$$

For $R'_M/K_T = R_M/K_T$, $J' = J$

$$\frac{\theta_O(s)}{\Delta\theta_M(s)} = \frac{n\left(-K_v + \frac{K'_T}{R_M}K_B\right)s}{s^3 + K_v s^2 + K_e s + K_I} \qquad (27)$$

For a step input error, $\Delta\dot{\theta}_M(s) = \Delta\dot{\theta}_M/s$

$$\theta_O(s) = \frac{n\left(-K_v + \frac{K'_T}{R_M}K_B\right)\Delta\dot{\theta}_M}{s^3 + K_v s^2 + K_e s + K_I} \qquad (28)$$

for which

$$\theta_O(t)\Big|_{t\to\infty} = 0$$

If $K_I = 0$

$$\theta_O(s) = \frac{n\left(-K_v + \frac{K'_T}{R_M}K_B\right)\Delta\dot{\theta}_M}{s(s^2 + K_v s + K_e)} \qquad (29)$$

for which

$$\theta_O(t)\Big|_{t\to\infty} = \frac{n\left(-K_v + \frac{K'_T}{R_M}K_B\right)}{K_e} \qquad (30)$$

Thus, there is a steady state error if the integral feedback gain is zero. The time required to bring this error to zero depends on the magnitude of K_I.

4. Transfer Function for a Disturbance Torque

A disturbance torque on the output can be expressed as a motor input for analysis purposes. It is then treated as a normal input but must be reduced by the gear ratio, n (n < 1). An uncompensated gravity torque such as a load picked up by the manipulator is such a disturbance torque. The signal flow diagram for a disturbance torque is shown in Fig. 8.
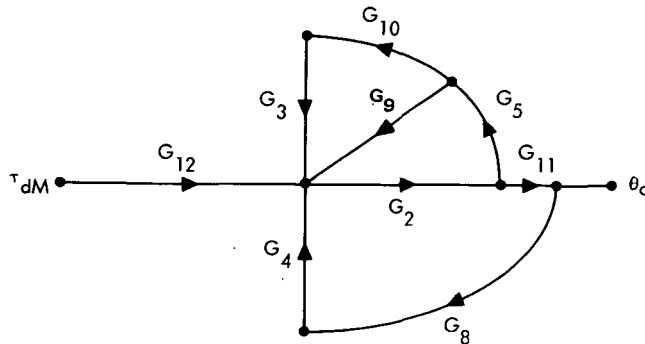
Fig. 8. Signal Flow Diagram for a Disturbance Torque

where $G_{12} = R_M/K_T$

$$\frac{\theta_o(s)}{\tau_{dM}(s)} = \frac{G_{12}G_{11}}{\dfrac{1}{G_2} - (G_5 G_{10} G_3 + G_8 G_4 G_{11} + G_9 G_5)} \qquad (31)$$

$$= \frac{57.3 \dfrac{R_M}{K_T} ns}{J \dfrac{R_M}{K_T} s^3 + J' \dfrac{R_M'}{K_T} K_v s^2 + J' \dfrac{R_M'}{K_T} K_e s + J' K_I \dfrac{R_M'}{K_T}} \qquad (32)$$

Since $\tau_{dM} = n\tau_{do}$

$$\frac{\theta_o(s)}{\tau_{do}(s)} = \frac{57.3\, n^2\, \frac{R_M}{K_T}\, s}{J\frac{R_M}{K_T}s^3 + J'\frac{R'_M}{K_T}K_v s^2 + J'\frac{R'_M}{K_T}K_e s + J'\frac{R'_M}{K_T}K_I} \tag{34}$$

For a step input, $\tau_{do}(s) = \tau_{do}/s$

$$\theta_o(t)\Big|_{t\to\infty} = 0$$

with $K_I = 0$

$$\theta_o(t)\Big|_{t\to\infty} = \frac{57.3\, n^2\, \frac{R_M}{K_T}\, \tau_{do}}{J'\frac{R'_M}{K_T}K_e} \tag{35}$$

The torque conversion factor was left in because in this case the R_M/K_T does not cancel out if the computed value, R'_M/K_T (from the curves) is not equal to the real value, R_M/K_T. If they are equal.

$$\theta_o(t)\Big|_{t\to\infty} = \frac{57.3\, n^2\, \tau_{do}}{J'K_e} \tag{36}$$

where τ_{do} is the disturbance torque in oz-in., and θ_o is in degrees.

As might be expected, the steady state error is not a function of the changing inertia (J), but with the J' multiplier the error is reduced by the resultant gain, $J'K_e$.

5. Feedback Gain Values and Resultant Response

The purpose of this section is to insert gain values in the various transfer functions and steady state error expressions to indicate realistic output error values. The gains used are the <u>Stanford values</u> but will be converted to eliminate the Δt (jiffy)[*] element. This is necessary to use them in the transfer functions.

First, we will look at the characteristic equation.

$$s^3 + K_v s^2 + K_e s + K_I = 0 \tag{37}$$

If $K_I = 0$, we have

$$s^2 + K_v s + K_e = 0 \tag{38}$$

which is a second order system and can be expressed as

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{39}$$

in which

ζ = the system damping ratio

ω_n = the system undamped natural frequency (rad/sec)

Common practice is to select $\zeta = 0.7$ and ω_n as high as practical to increase the response or tightness of the servo loop. Stanford has selected $\zeta = 1$ which is reasonable since the overshoot for a step input is then reduced from 7% ($\zeta = 0.7$) to zero. Doing this we have

$$2\omega_n = K_v$$

$$\omega_n^2 = K_e$$

---

[*]The term "jiffy" refers to the 1/60 second sampling interval used at Stanford.

or

$$K_e = \left(\frac{K_v}{2}\right)^2 \qquad (40)$$

Now, selection of K_e or K_v determines ω_n. Remember from Fig. 6 that the response has reached its final value (practically speaking) at $t = 7/\omega_n = 7/\sqrt{K_e}$.

This is all very nice, but as shown above the characteristic with $K_I \neq 0$ is third order. Fortunately, (depending on the value of K_I) the system can still be considered second order to get a measure of the system transient response. This will be elaborated upon further.

The gains used by Stanford for joint 1 are

$$K_e = 0.038$$

$$K_v = 0.39 \qquad (41)$$

$$K_I = 0.005$$

These have the time element of 1/60 sec in them, therefore to eliminate this,

$$K_v = 0.39 \times 60 = 24.4$$

$$K_e = 0.038 \times 60^2 = 132 \qquad (42)$$

$$K_I = 0.005 \times 60^3 = 1080$$

Substituting these values in the transfer function for a step displacement from Eq. 19

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{132(s+8)}{s^3 + 24s^2 + 132s + 1080} \qquad (43)$$

factoring (for Stanford gains)

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{132(s + 8)}{(s + 20.2)(s^2 + 3.9s + 52)} \quad \begin{cases} K_I = 1080 \\ \zeta = 0.27 \\ \omega_n = 7.2 \end{cases} \qquad (44)$$

If K_I is doubled

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{132(s + 16.3)}{(s + 22.4)(s^2 + 1.6s + 96)} \quad \begin{cases} K_I = 2160 \\ \zeta = 0.08 \\ \omega_n = 9.8 \end{cases} \qquad (45)$$

for $K_I = 0$

$$\frac{\theta_o(s)}{\Delta\theta_i(s)} = \frac{K_e}{s^2 + 24.4s + 132} \quad \begin{cases} K_I = 0 \\ \zeta = 1.0 \\ \omega_n = 11.5 \end{cases} \qquad (46)$$

As K_I increases from $0 \rightarrow 1080$ the transient response is altered very little (although the steady state error will now go to zero over a longer time than the initial transient). At $K_I = 1080$ the zero and pole added are not too close together, the ζ is quite low, but the pole has a stabilizing effect to cancel the destabilizing effect of the zero and the decreased ζ. The result is a response similar to when $K_I = 0$. But as K_I approaches 2160, the pole and zero approach each other and can be cancelled leaving just a second order system with very low damping. As K_I increases further the system becomes marginally stable, i.e., $\zeta = 0$. Thus K_I cannot be increased much beyond 1000.

For stability purposes K_I has to be controlled as above, but to eliminate rate and position null errors and to reduce the output from a torque disturbance

to zero as rapidly as possible, K_I should be large as feasible. The restriction for at least marginal stability is that

$$K_I \leq K_v K_e \tag{47}$$

This is derived from the characteristic equation in Eq. (19). Let $K_I = K_v K_e$ in

$$s^3 + K_v s^2 + K_e s + K_I = 0$$

which then factors into

$$(s + K_v)(s^2 + K_e) = 0$$

This indicates that the two complex roots are on the imaginary axis of the complex s plane, therefore the system is marginally stable.

If K_v and K_e are selected on the basis of considering only the second order response for $\zeta = 1$ (Eq. 40), then

$$K_I \leq \frac{K_v^3}{4} \tag{48}$$

The above restriction on K_I applies as well to a torque disturbance since the characteristic equation (Eq. 34) is the same. It is of some interest to consider the steady state error when $K_I = 0$.

$$\theta_o(t) \Big|_{t \to \infty} = \frac{57.3 \, n^2 \, \tau_{do}}{J' K_e} \tag{49}$$

In all the previous discussion the value of J' did not matter since it cancels out if all gains and operations are as shown in Fig. 2. Here we have assumed that J' not J'_o is used as the gain multiplier, otherwise the n factors would be different. Consequently, the above error is a function of J', the computed inertia referred to the motor. This is of course $n^2 J'_o$.

A value of J'_o for the first link from the Stanford arm is derived as follows (the prime denoting computed is dropped).

Total inertia: $J_o = 698,000$ oz-in. jiffy^2

Motor inertia: $J_{Mo} = 285,000$ oz-in. jiffy^2

Load inertia: $J_{Lo} = 413,000$ oz-in. jiffy^2

To eliminate the jiffy^2 time element requires dividing by 60^2

$$J_{Lo} = 115 \qquad \text{oz-in.-sec}^2$$

$$J_{Mo} = 79 \qquad \text{oz-in.-sec}^2$$

The total inertia referred to the motor is

$$J = n^2(J_{Mo} + J_{Lo}) \tag{50}$$

$$= \frac{1}{(120)^2}(79 + 115)$$

$$= 0.005 + 0.008 = 0.013 \ \text{oz-in. sec}^2$$

The value of n for the JPL arm (link 1) is 100 and the inertia values will be larger.

Substituting values in (49)

$$\theta_o(t) \Big|_{t \to \infty} = \frac{57.3 \ \tau_{do}}{(120)^2 (0.013)(132)} = 0.002 \ \tau_{do} \tag{51}$$

If a force of 16 oz is applied with a lever arm of 30 inches, the resulting output angular error is 0.96 degrees. Thus if this magnitude of error cannot be tolerated, K_e would have to be increased, keeping in mind that J' is a variable. The other alternative is to introduce the integral gain, K_I which would reduce the error to zero.

III. CONVENTIONAL POSITION SERVO

The positional servo described here uses the same sensors or transducers as the computed torque system but feeds this information directly to a summing amplifier which drives the motor, rather than to the computer.

Position, rate and integral feedback are used. A linear analysis is used which ignores the nonlinearity of the motor and gear train. A further analysis using nonlinear techniques is required to determine if and what type of compensation is needed to get the required performance despite these nonlinearities.

1. Transfer Function Derivation

The components in a control loop for one manipulator link are shown in Fig. 9 and the form for analysis in Fig. 10.

The forward gain function is

$$G(s) = \frac{nK_a(s + K_I)}{s^2\left(K_B + \frac{R_M}{K_T} s\right)}$$

(52)

The feedback gain function is

$$H(s) = \frac{K_v}{n} s + K_e$$

(53)
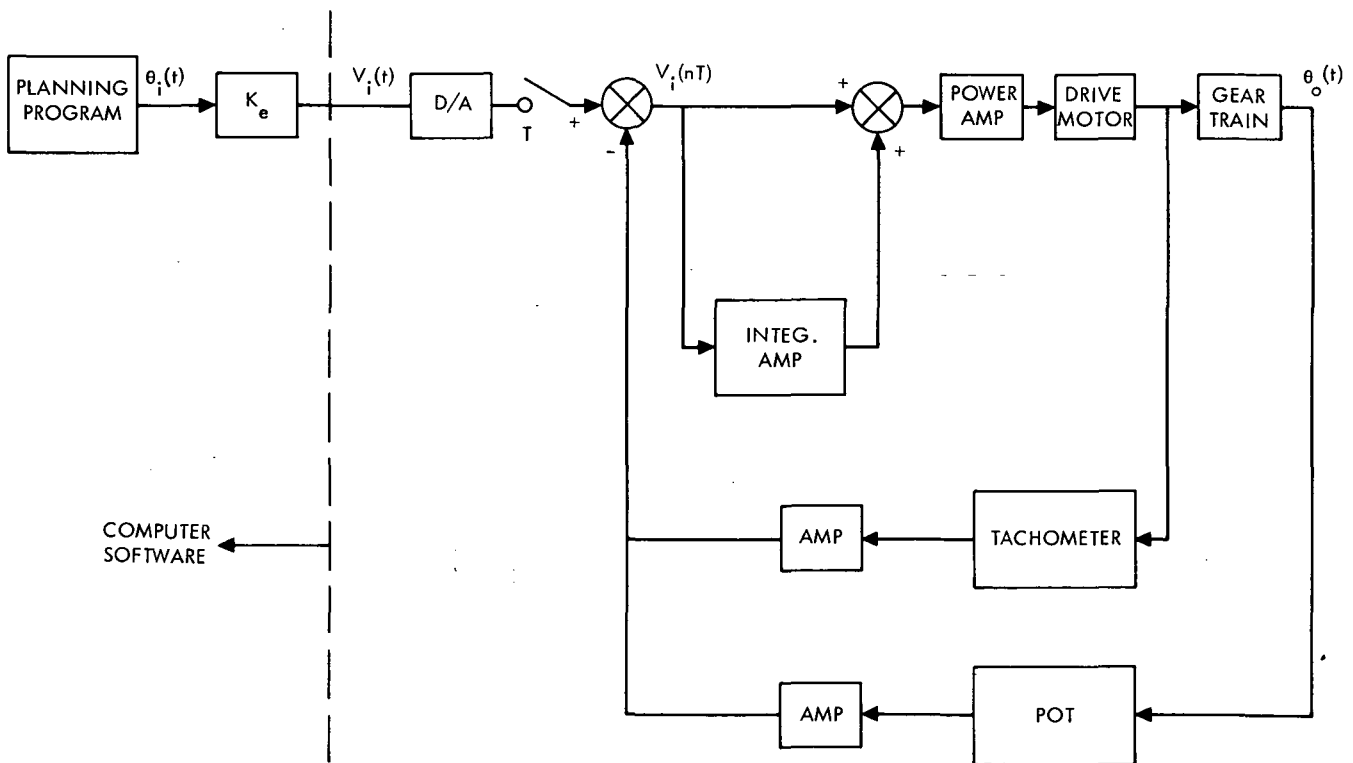
Fig. 9. Physical Configuration of a Control Loop
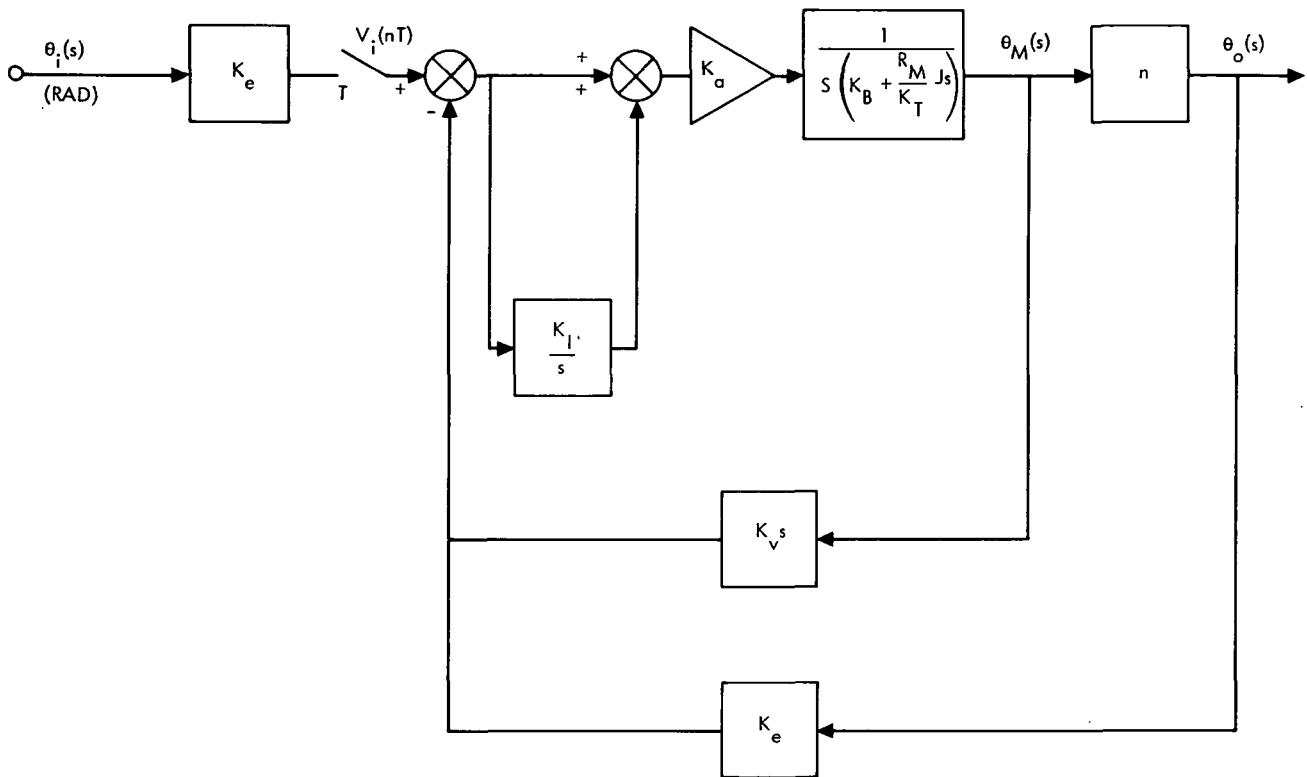
Fig. 10. Transfer Function Control Loop Configuration

Hence, the transfer function is

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_e G(s)}{1 + G(s)\,H(s)}$$

$$= \frac{nK_a K_e (s + K_I)}{\dfrac{R_M}{K_T}\, J\, s^3 + (K_B + K_a K_v)s^2 + K_a(nK_e + K_v K_I)s + nK_a K_e K_I} \tag{54}$$

2. Stability Criterion

If the gains are selected as in the computed torque system, then K_I is first assumed to be zero, reducing the system to second order.

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{nK_a K_e \Big/ J\frac{R_M}{K_T}}{s^2 + \dfrac{(K_B + K_a K_v)}{J\frac{R_M}{K_T}} s + \dfrac{n K_a K_e}{J\frac{R_M}{K_T}}} \tag{55}$$

where

$$\omega_n = \left(n K_a K_e \Big/ J\frac{R_M}{K_T}\right)^{1/2} \tag{56}$$

$$\zeta = \frac{K_B + K_a K_v}{2\left(nK_a K_e \, J\frac{R_M}{K_T}\right)^{1/2}} \tag{57}$$

selecting $\zeta = 1$ and solving for K_a and K_v

$$K_a = \frac{-K_B K_v + 2nK_e J\frac{R_M}{K_T} \pm 2\sqrt{nK_e J\frac{R_M}{K_T}\left(nK_e J\frac{R_M}{K_T} - K_B K_v\right)}}{K_v^2} \tag{58}$$

$$K_v = \frac{-K_B + 2\sqrt{nK_a K_e J\frac{R_M}{K_T}}}{K_a} \tag{59}$$

In the above equations, all parameters except K_e are known, and a maximum value for this gain will be developed in Section 5.

From Eqs. 27, 47 and 54 the stability criterion with $K_I \neq 0$ is

$$\frac{nK_a K_e K_I}{J\frac{R_M}{K_T}} \leq \frac{K_a(K_B + K_v K_a)(nK_e + K_v K_I)}{\left(J\frac{R_M}{K_T}\right)^2} \tag{60}$$

or

$$K_I \leq \frac{nK_e(K_B + K_aK_v)}{n J \frac{R_M}{K_T} K_e - K_v(K_B + K_aK_v)}$$ (61)

Generally the second denominator term is very small and

$$K_I \leq \frac{K_B + K_aK_v}{J \frac{R_M}{K_T}}$$ (62)

This criterion only specifies a maximum K_I for conditional stability. Of course K_a and K_v did not have to be computed from Eqs. 58 and 59 for $\zeta = 1$ but that is a preferable way. The point to consider is the proper value of $K_I << K_{Imax}$ to produce a transient response similar to that expected from the second order equation.

3.   Response to Input Forcing Functions

The response to an input from the planning program, unlike the computed torque case, depends entirely on the feedback gains. The input is $\theta_i(t)$ and the ability of $\theta_o(t)$ to follow depends on the amplification of the resultant error signal which then drives the motor. A typical input function is shown in Fig. 11. As indicated the JPL requirement is for a slower response, at least for the present.

A measure of the error in $\theta_o(t)$ can be obtained by approximating the input function with a sinusoid as shown, and then finding the frequency response to this input frequency, ω.

Substituting $j\omega$ for s in Eq. (54).

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \frac{nK_aK_e(j\omega + K_I)}{nK_aK_eK_I - (K_B + K_aK_v)\omega^2 + j\left[K_a(nK_e + K_vK_I)\omega - \frac{R_M}{K_T}\omega^3\right]}$$ (63)
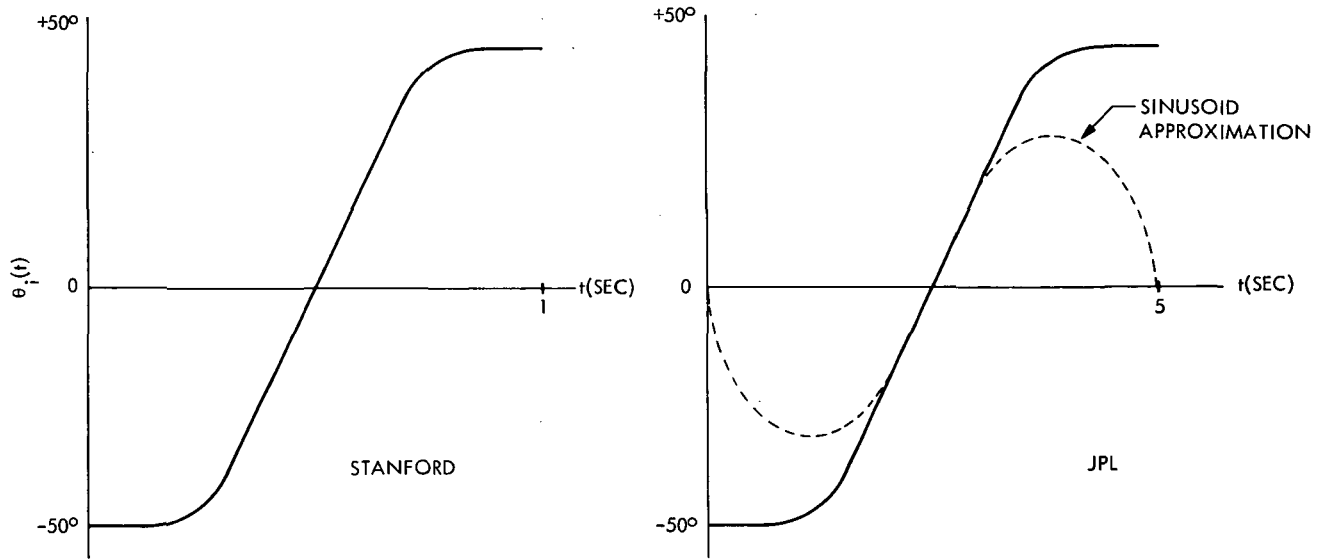
Fig. 11. Typical Input Function, $\theta_i(t)$

The magnitude of this function is

$$\left|\frac{\theta_o(j\omega)}{\theta_i(j\omega)}\right|^2 = \frac{nK_aK_e\left(\omega^2 + K_I^2\right)}{\left[nK_aK_eK_I - (K_B + K_aK_v)\omega^2\right]^2 + \left[K_a(nK_e + K_vK_I)\omega - \frac{R_M}{K_T}J\omega^3\right]^2}$$

(64)

For $f \leq 1$ cps and for anticipated values of gains, this function is so close to unity that it can be considered as such, and only the phase angle, ϕ need be considered.

$$\phi = \tan^{-1}(\omega/K_I) - \tan^{-1}\left(\frac{K_a(nK_e + K_vK_I)\omega - \frac{R_M}{K_T}J\omega^3}{nK_aK_eK_I - (K_e + K_aK_v)\omega^2}\right)$$

(65)

For f = 0.2 (ω = 1.356), K_v = 0.001, K_a = 200, K_B = 0.05, J = 0.013, R_M/K_T = 0.123, which are nominal values, this reduces to

$$\phi = \tan^{-1}(\omega/K_I) - \tan^{-1}\left[\frac{\omega}{K_I}\left(1 + \frac{K_v K_I}{nK_e}\right)\right]$$

$$\phi = \tan^{-1}\left(\frac{-\omega K_v K_I^2}{nK_e\left(K_I^2 + \omega^2\right) + \omega^2 K_v K_I}\right)$$

for $K_I^2 >> \omega^2$ ($K_I \approx 10$)

$$\phi = \tan^{-1}\left(\frac{-\omega K_v K_I}{nK_e K_I + \omega^2 K_v}\right)$$

For n = 0.01, K_e = 7, K_I = 10, K_v = 0.001, ω = 1.356, which are nominal values,

$$\phi = -\tan^{-1}\left(\frac{\omega K_v}{nK_e}\right) \qquad \text{(rad)} \qquad\qquad (66)$$

This is the phase lag in $\theta_o(t)$ for a sinusoidal input, $\theta_i(t) = \theta_p \sin \omega t$. The error in $\theta_o(t)$ can be expressed as

$$\Delta\theta_o(t) = -\theta_p \sin \omega t + \theta_p \sin(\omega t + \phi), \quad \phi \text{ is neg.}$$

$$= \theta_p\left[\sin \omega t (\cos \phi - 1) + \cos \omega t \sin \phi\right] \qquad\qquad (67)$$

For small ϕ

$$\Delta\theta_o(t) = \theta_p \phi \cos \omega t \qquad\qquad (68)$$

where

θ_p is the peak angular sinusoidal rotation

ϕ is the phase lag in radians from Eq. (66).

Another way of determining the response error is to assume a ramp input for $\theta_i(t)$.

$$\theta_i(t) = kt \tag{69}$$

$$\theta_i(s) = \frac{k}{s^2}$$

From Eq. (55) for $\zeta = 1$, we have

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{1}{\dfrac{s^2}{\omega_n^2} + \dfrac{2}{\omega_n} s + 1} = \frac{\dot{\theta}_o(s)}{s\theta_i(s)} \tag{70}$$

Therefore

$$\dot{\theta}_o(s) = \frac{k}{s\left(\dfrac{s}{\omega_n} + 1\right)^2} \tag{71}$$

$$\dot{\theta}_o(t) = k\left[1 - (1 + \omega_n t)\, e^{-\omega_n t}\right] \tag{72}$$

The output error is

$$\Delta\theta_O(t) = \int_0^t \left[\dot{\theta}_i(t) - \dot{\theta}_O(t) \right] dt$$

$$= k \left[\frac{2}{\omega_n} \left(1 - e^{-\omega_n t} \right) - t \, e^{-\omega_n t} \right] \tag{73}$$

where ω_n can be computed from the system parameters, and k is the ramp slope in degrees or radians per second.

4.   Transfer Function for a Disturbance Torque

An output disturbance torque, τ_{do} can be transferred to the motor input by multiplying it by the gear ratio.

$$\tau_{dM} = n \, \tau_{do} \tag{74}$$

Therefore, applying τ_{dM} at the motor input

$$G(s) = \frac{n}{s \left(K_B + \frac{R_M}{K_T} s \right)} \tag{75}$$

$$H(s) = K_a \left(K_e + \frac{K_v}{n} s + \frac{K_I}{s} \right) \tag{76}$$

$$\frac{\theta_O(s)}{\tau_{do}(s)} = \frac{\frac{R_M}{K_T} n \, G(s)}{1 + G(s)H(s)}$$

$$= \frac{\frac{R_M}{K_T} n^2 s}{\frac{R_M}{K_T} Js^3 + (K_B + K_a K_v)s^2 + nK_a K_e s + nK_a K_I} \tag{77}$$

For a step disturbance torque, the output error is

$$\theta_o(s) = \frac{\frac{R_M}{K_T} \tau_{do} n^2}{\frac{R_M}{K_T} Js^3 + (K_B + K_a K_v)s^2 + nK_a K_e s + nK_a K_I}$$ (78)

The steady state output error is

$$\theta_o(t) \bigg|_{t \to \infty} = s\, \theta_o(s) \bigg|_{s \to 0} = 0$$

5. Gain Values and Resultant Response

The determination of gains for this system is more complicated than for the computed torque system since more variables are involved and physical limitations come into play. Selecting gains based on a second order response as mentioned before means arbitrarily letting the damping ratio, $\zeta = 1$, and selecting an arbitrary undamped natural frequency, ω_n. The latter can be done by considering the response desired as shown in Fig. 6.

The equations for the gains ($\zeta = 1$) are

$$K_v = \frac{nK_e}{\omega_n} \left(2 - \frac{K_B}{J \frac{R_M}{K_T} \omega_n} \right)$$ (79)

$$K_a = J \frac{R_M}{K_T} \frac{\omega_n^2}{nK_e}$$ (80)

Nominal values of the above parameters are

$$
\begin{array}{llll}
J & = 0.013 & & (\text{oz-in. sec}^2) \\
R_M/K_T & = 0.123 & & (\text{v/oz-in.}) \\
K_B & = 0.05 & & (\text{v/rad/sec}) \\
n & = 0.01 & & (N_1/N_2)\ \text{gear ratio}
\end{array}
$$ (81)

The value of K_e is dependent on the angular range which is $\pm 180°$ and the supply voltage = 24. Thus, the pot sensitivity is $\pm 12/\pi$ (volts/rad). Adding some amplification with the same D.C. voltage supply can increase this to $\pm 24/\pi$. A value of $K_e = 7$ will be used here. Inserting all these values in (79) and (80).

$$K_v = \frac{0.07}{\omega_n} \left(2 - \frac{31}{\omega_n}\right) \tag{82}$$

$$K_a = 0.023\, \omega_n^2$$

Selecting $\omega_n = 100$ for a very fast step response (Fig. 6).

$$K_v = 0.0012$$

$$\tag{83}$$

$$K_a = 230$$

Now K_I has to be selected

$$K_I < \frac{\frac{K_B + K_a K_v}{R_M}}{J \frac{R_M}{K_T}} = 204 \tag{84}$$

Obviously, to get a response approaching $\zeta = 1$, K_I has to be considerably less than this maximum value. Selecting $K_I = 10$, the system transfer function is:

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{10,062(s + 10)}{s^3 + 203.7s^2 + 11,787s + 100,625}$$

Factoring

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{10,062(s + 10)}{(s + 10.2)(s^2 + 193.5s + 9813)} \tag{85}$$

Obviously, the pole and zero essentially cancel and the response is second order with

$$\omega_n = 99$$

$$\zeta = 0.98$$

(86)

The high values of K_v and K_a and the relatively low K_I has left the response the same as without K_I. This is fine except that high amplifier gains create a noise problem. The transient response is much better than needed, but to control the error for a very rapidly changing input forcing function such as shown in Fig. 11, requires a high gain. From the sinusoidal input error function (Eq. 68) and Eq. 66,

$$\Delta\theta_o(t) = \theta_p \phi \cos \omega t$$

$$= \theta_p \tan^{-1}\left(\frac{\omega K_v}{n K_e}\right) \cos \omega t$$

(87)

Substituting the previously given parameter values

$$\Delta\theta_o(t) = \theta_p \tan^{-1}(0.017\omega) \cos \omega t$$

(88)

For $\theta_p = 50$ deg, $\omega = 1.356$

$$\Delta\theta_{op} = 50(0.023) = 1.15 \text{ degrees}$$

(89)

For $\omega = 6.28$ which is a nominal input for the computed torque system at Stanford, this error increases to 5 degrees. Thus, it is obvious that, if the system is to compete with the Stanford computed torque system, the gains would have to be increased which is feasible.

A check on the above response error is to use error Eq. (73) for a ramp input. For the sinusoid used, the maximum $\dot{\theta}_0$ is about 60 deg/sec. The natural frequency from Eq. (86) is 100 rad/sec.

$$\Delta\theta_0(t) = k\left[\frac{2}{\omega_n}\left(1 - e^{-\omega_n t}\right) - t\, e^{-\omega_n t}\right]$$

$$= 60\left[\frac{2}{100}\left(1 - e^{-100t}\right) - t\, e^{-100t}\right]$$

(90)

Thus, for $t > 0.1$ seconds, the error equation reduces to

$$\Delta\theta_0 = \frac{2k}{\omega_n} = 1.2 \text{ degrees} \tag{91}$$

which compares well with the previous value of 1.15 degrees from (89).

6. Effect of Changing System Inertia

The inertia changes as the manipulator moves links which extend or retract the arm. This is of no consequence for the computed torque system since it is computed and the computed value is used to eliminate this parameter from the system response. This is essentially computing a variable feedback gain. For the conventional system the natural frequency and damping are

$$\omega_n = \left(nK_a K_e / J\, \frac{R_M}{K_T}\right)^{1/2} \tag{92}$$

$$\zeta = \frac{K_B + K_a K_v}{2\left(nK_a K_e J\, \frac{R_M}{K_T}\right)^{1/2}}$$

As discussed, this is the case if the gains are as shown earlier which essentially reduces the system to second order.

The inertias of the JPL arm are not yet known. They will be larger than the Stanford values which have been used in this report, but using Stanford values is most likely sufficient for this discussion. Supplied data indicates that the inertia value can about double for link No. 1. This means ω_n and ζ are decreased by $\sqrt{2}$ which would not change the transient performance or decrease the stability any significant amount. This is even less significant for other links, except when a large inertia object is being manipulated.

IV. COMPARISON OF THE TWO DRIVE METHODS

The following paragraphs are a summary of the more important characteristics of the computed torque (CT) and conventional position servo (CS) systems. However, no final conclusion is made as to which system should be used because this depends on the particular application and on the results of a sampled data analysis.

1) Input Forcing Functions

CT: There is theoretically no output position error if the torque is computed correctly to produce the response computed in the planning program. This assumes the inertias are computed correctly. The advantage here is that very fast motions can be accomplished without depending on a feedback error.

CS: There is always a transient error which subsides to zero as the motion rate decreases to zero. This error is minimized by selecting the largest possible forward and feedback gains.

2) Disturbance Torque

CT: An unknown external force on the manipulator is a disturbance torque. Gravity torques are computed and compensated along with reaction torques from rotations in other non-perpendicular links. Lifting an object produces an unknown torque which results in an angular position error. The integral gain term then zeros this position error due to any unknown torque. The resulting output of the integral gain term indicates the torque value and thus the weight of the object.

CS: The feedback error signal from the integrating amplifier must compensate for all disturbance torques including gravity and reactions from other links. This is not too serious a problem with high enough gains. The reaction torques are only about 5% of the applied torque, and the gravity torques would not pose any problem since the output error is driven to zero by the integral term. Very fast motions with a large changing gravity and reaction torque would produce an error dependent on the gain values.

3) Gains

CT: The feedback gains (link 1) are such to produce a damping ratio (ζ) of 1 and an undamped natural frequency of about 10. This is not a very tight loop, but the only reason for possibly increasing ω_n is for better response to disturbance torques, although the integral term eventually reduces the error to zero. There appears to be no reason for not making the gains for all the links the same or certainly not much different. These gains and their use take place in the computer, therefore can be easily changed with no effect on the hardware.

CS: The gains are considerably higher to produce an $\omega_n = 100$ rather than 10. This does not appear to be a problem as far as the hardware is concerned. The gains here are constant but the inertia changes which then changes ω_n and ζ. This maximum change factor is only about $1/\sqrt{2}$ (for the manipulator inertia alone). Changing gains means a physical adjustment on the amplifiers. A set of gains would have to be derived for each link since the inertias and motor torque constants are different.

4) Nonlinearities

CT: The nonlinearities in the drive motor, mainly the coulomb friction torque, and the gear train nonlinearity effects are all handled easily in the computed torque program. This is a distinct advantage of this method, assuming the nonlinearities can be modeled accurately. This does, of course, involve precise lab testing.

CS: The linear analysis presented in this memo does not consider these nonlinearities. To be sure of proper performance a nonlinear analysis is required and compensation introduced. This can probably be done with some limitation on performance.

5) Sampling and Delay

CT: The feedback position and rate are sampled at some rate depending on the time span of the fastest input function from the planning program. The input to the drive motor amplifier is also sampled at the same rate. The drive torque is computed which takes a finite time. Thus, the system is sampled data with a delay. If a longer time span is used for the JPL system (5 → 10 seconds for a large angular movement) the sampled time is much greater than the computation delay, and the latter can be ignored. But a sampled data analysis with one period delay in the feedback data is probably needed.

CS: Since the feedback takes place external to the computer, the system is continuous except for sampling of the input driving function from the planning program. Some consideration to this sampling rate is all that is required beyond the linear analysis, complemented by the nonlinear analysis.

6) Complexity

CT: Obviously, the large program for computing the drive torque is the negative aspect of this method. But if the effort of developing the program, debugging it, and the computation time and computer capacity are acceptable, considering the increased performance over a conventional servo, the system appears very attractive. Two A/D converters per link are required to get the position and rate information to the computer. A D/A converter provides the motor drive signal. Another factor is determination of very accurate inertia data and motor torque-voltage characteristics for essentially modeling the manipulator in the computed torque program.

CS: No torque program is needed, therefore, only approximate information on inertias is needed. Nonlinearities need to be known approximately so that the compensation and performance can be determined. The two A/D converters for feedback are not required. One or two additional amplifiers are needed per link.

7) Data Transfer

CT: Three data transfers per link are required. The rate of this transfer depends on how fast the arm is required to move and the desired accuracy. For a 5 second full angular movement of 180 degrees, a sampling rate of 20 to 30 per second is probably sufficient, but a detailed study is desirable using the results of this analysis and some sampled data analysis. At each sampling 3 words per link are transferred.

CS: Only the input function is transferred every sampling interval. This can probably be at the rate mentioned above, although here it is not so critical because the servo loop does all the control rather than the input.

8) Additional Analysis Required

CT: A sampled data analysis with one sampling interval delay plus a computation delay, although the computation delay is probably negligible.

CS: A Nonlinear analysis considering the motor friction torque and gear train effects.

REFERENCE

R. P. C. Paul, "Modelling, Trajectory Calculation and Servoing of a Computer Controlled Arm", Ph.D. Thesis, Dept. Computer Science, Stanford University, August 1972.

APPENDIX A

USE OF A CURRENT SOURCE FOR THE MOTOR DRIVE

The analysis in the body of this report assumes a voltage gain amplifier prior to the drive motor for both the computed torque system and the conventional position servo. A disadvantage of the voltage source is that a change in motor input resistance will affect system performance, whereas with a current source, the input current and thus torque is not affected. The effect of this resistance change on system performance is determined by examining the relevant equations in this report.

The first concern is with stability. This is determined by investigating the characteristic equation which is derived from (2) by assuming $J = J'$ and that the torque constant (K_T) remains constant as the motor resistance (R_M) varies from the nominal value (R_M').

$$\frac{R_M}{R_M'} s^3 + K_v s^2 + K_e s + K_I = 0 \tag{A-1}$$

A ratio change in R_M/R_M' from 1 to 1.2 which is the maximum anticipated change in R_M results in

$$1.2 s^3 + 24 s^2 + 132 s + 1080 = 0 \tag{A-2}$$

or

$$s^3 + 20 s^2 + 110 s + 900 = 0$$

$$K_I \leq 2200.$$

Whereas with $R_M/R_M' = 1.0$

$$K_I \leq 3220.$$

Factoring (A-1)

$$(s + 16.6)(s^2 + 3.4 s + 54.6) = 0 \tag{A-3}$$

The two complex roots have $\omega_n = 7.3$ and $\zeta = 0.23$ which compares to $\omega_n = 7.2$ and $\zeta = 0.27$ for no change in R_M (i.e., $R_M/R_M' = 1$). Thus, the change in the system stability due to a 20% change in motor resistance, (and using a voltage drive) is very small. This effect can be reduced further by using a value of R_M midway between the 20% variation. This actually means altering the voltage-torque characteristic slightly, assuming it is taken at the low end of the resistance change.

The effect of a change in R_M is much the same for a conventional system whose characteristic equation is

$$\frac{R_M}{K_T} J s^3 + (K_B + K_a K_v) s^2 + K_a (n K_e + K_v K_I) s + n K_a K_e K_I = 0 \tag{A-4}$$

For the gains used previously and introducing R_M' as the nominal motor resistance this becomes

$$\frac{R_M}{R_M'} s^3 + 203.7 s^2 + 11,787 s + 100,625 = 0 \tag{A-5}$$

As indicated in (85) and (86) the gains here allow the system to be considered second order (to get the complex roots).

$$\frac{R_M}{R_M'} s^2 + 203.7 s + 11,787 = 0 \tag{A-6}$$

Introducing ζ' and ω_n' as the nominal damping and natural frequency $(R_M/R_M' = 1)$

$$\frac{R_M}{R_M'} s^2 + 2 \zeta' \omega_n' s + \omega_n'^2 = 0 \tag{A-7}$$

Thus, the altered values, ζ and ω_n are

$$\omega_n = \left(\frac{R_M'}{R_M}\right)^{1/2} \omega_n'$$

$$\zeta = \left(\frac{R_M'}{R_M}\right)^{1/2} \zeta' \tag{A-8}$$

The slight increase in ω_n for the computed torque system from (A-3), rather than a decrease, is due to the lower gains which result in a system which cannot be considered second order as above. But if R_M were increased by a larger factor, ω_n would decrease even for that case.

With stability evidently not a serious problem the next area to investigate is transient response and steady state errors.

Driving Input Response

The transfer function is

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{s^3 + K_v s^2 + K_e s + K_I}{\dfrac{R_M}{R_M'} s^3 + K_v s^2 + K_e s + K_I} \tag{A-9}$$

No attempt is made here to investigate transient response errors, since the inversion of Eq. (A-9) is very difficult for any analytic input function. Furthermore, the ratio R_M/R_M' is ≤ 1.2 which is very likely negligible.

Disturbance Torque

The transfer function is

$$\frac{\theta_o(s)}{\tau_{do}(s)} = \frac{57.3\, n^2 s}{J s^3 + J' \dfrac{R_M'}{R_M} K_v s^2 + J' \dfrac{R_M'}{R_M} K_e s + J' \dfrac{R_M'}{R_M} K_I} \tag{A-10}$$

$$\theta_o(t) \Big|_{t \to \infty} = 0$$

But for $K_I = 0$

$$\theta_o(t) \Big|_{t \to \infty} = 57.3 \frac{n^2 \tau_{do}}{J' K_e} \frac{R_M}{R_M'} \cdot \qquad \text{(A-11)}$$

With increased R_M, the steady state error increases proportionately.

Position Feedback Null Error

The transient response is altered slightly but the steady state error is unaltered from Eqs. (21) and (23) regardless of the value of K_I and change in R_M.

Rate Feedback Error

The transient response is altered slightly but the steady state error is unaltered from Eqs. (28) and (30) regardless of change in R_M.

If a current drive is used a slight change in the system occurs. The loop transfer function, G(9) in figure 3 is eliminated because the input is current, not voltage. The voltage-torque characteristic represented by R_M'/K_T now becomes $1/K_T$ which is oz-in/amp. This means the characteristic, obtained from lab tests, relates current rather than voltage to torque. Consequently, the motor speed does not have to be monitored during the test because the back e.m.f. does not have to be computed and subtracted from the applied voltage to get the motor voltage (V_M).

APPENDIX B

EFFECT OF MANIPULATING A LARGE INERTIA OBJECT

It has been shown that a change in system inertia has no effect on the system response (for the CT system), because this change in inertia is continuously computed and is used in the feedback loops to multiply the gains. This is a very desirable situation, but Stanford experience has shown that, if this gain becomes very high, the noise amplification becomes a problem and the system tends to become unstable. Joints (links) 4 and 5 are mostly affected because they experience a very large inertia change when a large inertia object is being manipulated. This assumes that the inertia of the object is known or is determined by the system prior to manipulation.

We are considering two problems here: (1) the noise problem with high inertia and therefore high gain, (2) the change in system response with an incorrect inertia due to an unknown object inertia.

The system characteristic equation is

$$s^2 + \frac{J'}{J} K_v s^2 + \frac{J'}{J} K_e s + \frac{J'}{J} K_I = 0 \tag{B-2}$$

Therefore, multiplying the gains K_v, K_e and K_I by the computed inertia, J' assures a system with uniform response. As the inertia changes the stability margin should theoretically remain the same. But if it seems desirable to reduce the response (and keep the damping constant at 1), a number of ways are available. One method is to not multiply K_e as before by J' and to divide K_I by J' rather than multiplying. Assuming that $J' = J$ this results in

$$s^3 + \frac{J' K_v}{J} s^2 + \frac{K_e}{J} s + \frac{K_I}{JJ'} = 0 \tag{B-3}$$

Assuming a second order system and $\zeta = 1$,

$$\omega_n = (K_e/J)^{1/2} \tag{B-4}$$

Thus, as J increases, ω_n will decrease. But first, a desired or nominal frequency ω_n'' has to be selected, which corresponds to some nominal inertia, J''. The gains can then be computed (for $\zeta = 1$)

$$K_v'' = 2\omega_n'' \tag{B-5}$$

and

$$K_e'' = \frac{J'' K_v''^2}{4} \tag{B-6}$$

Now, as J changes, K_e is constant but K_v has to be recomputed each cycle to keep $\zeta = 1$.

$$K_v = 2\left(K_e''/J\right)^{1/2} \tag{B-7}$$

The system natural frequency is

$$\omega_n = \left(\frac{J}{J''}\right)^{1/2} \omega_n'' \tag{B-8}$$

The result is a decrease in K_v and the ratio of the rate error loop gain to nominal is

$$\frac{J' K_v'}{J'' K_v''} = \left(\frac{J'}{J''}\right)^{1/2} \tag{B-9}$$

The result of this gain procedure is that the above gain increases by the square root of the inertia change ratio rather than directly as before. The response is slower but the noise is not amplified as much. It is quite possible that this method which involves recomputing the gain each cycle may not be necessary with tachometers on links 4 and 5 because the noise is much less than if rate data is computed from position feedback data.

The above makes no mention of K_I, but its value is dependent on the other gains. From (47) the maximum value for K_I is

$$K_I \leq K_v K_e$$

In this case it is

$$\frac{K_I}{J^2} \leq \frac{K_v K_e}{J} \quad \text{or} \quad K_I \leq J K_v K_e \qquad (B-10)$$

Therefore if J increases from the nominal J'' its value can theoretically increase. Its nominal value at J'' is some fraction of (B-10).

The second problem being discussed here is the possibility of not knowing the inertia of an object to be manipulated, and its effect on system performance.

For the gain method used in the body of this report (ω_n and ζ independent of a change in J), the transfer function, for an input function from the planning program, is

$$\frac{\theta_o(s)}{\theta_2(s)} = \frac{s^3 + K_v s^2 + K_e s + K_I}{\dfrac{JR_M}{J'R_M'} s^3 + K_v s^2 + K_e s + K_I} \qquad (B-11)$$

This is to show that an unknown change in motor resistance (R_M) can be lumped with a change in inertia (J). But the much greater change in J when an object is manipulated makes the small 20% change in R_M negligible. Therefore only J will be used in this discussion.

The natural frequency and damping, assuming the gains are such that the system can be considered second order, are

$$\omega_n = \left(\frac{J'}{J} K_e\right)^{1/2} = \left(\frac{J'}{J}\right)^{1/2} \omega_n'$$

$$\zeta = \frac{K_v}{2} \left(\frac{J'}{JK_e}\right)^{1/2} = \left(\frac{J'}{J}\right)^{1/2} \zeta'$$

(B-12)

where the primed values are the computed and assumed values, whereas the unprimed are the actual values of the parameters.

For the gain computation method discussed previously in this appendix the parameters are

$$\omega_n = (K_e/J)^{1/2} = \omega_n'$$

$$\zeta = \frac{K_v}{2} \frac{J'}{(JK_e)^{1/2}} = \frac{J'}{J} \zeta'$$

(B-13)

As shown in (B-13) the damping ratio decreases directly with increase in J. Thus, this method of computing gains may accomplish its purpose if J is always known, but if not, the first method as shown in (B-12) does not result in as large a decrease in ζ. Obviously, other methods are easily incorporated, but they will not be discussed here.

APPENDIX C

TEST FOR MAXIMUM MOTOR VOLTAGE

The JPL procedure at present is to use only third degree polynomials to fit the desired trajectory in the planning program. The polynomials are:

$$\theta_i(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\dot{\theta}_i(t) = a_1 + 2a_2 t + 3a_3 t^2 \qquad \text{(C-1)}$$

$$\ddot{\theta}_i(t) = 2a_2 + 6a_3 t$$

There is some concern that the applied motor voltage necessary to generate the above functions may exceed the allowable maximum. This appendix is a very brief discussion of one method of checking this possibility.

The applied voltage is

$$V_a = E + i_M R_M$$

$$\qquad \text{(C-2)}$$

$$= K_B \dot{\theta}_M + \tau_M \frac{R_M}{K_T}$$

Since

$$\dot{\theta}_M = \frac{1}{n} \dot{\theta}_i, \text{ and } \tau_M = J \ddot{\theta}_M$$

$$V_a = \frac{K_B}{n} \dot{\theta}_i + J \frac{R_M}{nK_T} \ddot{\theta}_i \qquad \text{(C-3)}$$

Substituting from (C-1)

$$V_a = \frac{1}{n} \left\{ K_B\left(a_1 + 2a_2 t + 3a_3 t^2\right) + J \frac{R_M}{K_T} (2a_2 + 6a_3 t) \right\} \qquad \text{(C-4)}$$

Taking the derivative and setting it equal to zero, and solving for t results in

$$t = -\frac{J}{K_B}\left(\frac{R_M}{K_T}\right) - \frac{a_2}{3a_3} \qquad (C-5)$$

Now, compute V_a from (C-4) using t from (C-5). If its value is larger than a given maximum, the trajectory is undesirable because the motor torque will not be as large as required to follow the trajectory in (C-1). Of course, the error signals will eventually bring the error to zero, but the resultant deviation before this may be undesirable. In any respect, depending on the error signals for large errors defeats the purpose of the computed torque system, and the operation is then much like a conventional position servo.

LIST OF SYMBOLS

J_L' Computed inertia of load (not including motor) referred to output

J_L Actual inertia of load (not including motor) referred to output

J' Total computed inertia referred to motor shaft

J Total actual inertia referred to motor shaft

J_o' Total computed inertia referred to output

J_M Motor inertia referred to motor shaft

K_v Rate error feedback gain

K_e Position error feedback gain

K_I Integral of position error feedback gain

K_B Motor back e.m.f. constant

n Gear speed reduction ratio (motor to output)

R_M Resistance of motor input winding

K_T Motor current - torque constant

R_M/K_T Motor voltage - torque constant (characteristic)

G_g Program for computing the gravitational torques

τ_I Computed torque from the integral gain feedback

τ_g Actual gravitational torque

τ_1' Computed input driving function torque

V_M Motor voltage for driving torque and error torque

V_a Total applied motor voltage including back e.m.f.

θ_M Motor shaft output angle

θ_i Input angle from planning program

θ_o Output angle

s Laplace operator

E Motor back e.m.f.

i_M Motor input current

τ_M	Motor torque
$\Delta\theta_i$	Bias error in position feedback signal
$\Delta\dot{\theta}_M$	Bias error in rate feedback signal
τ_{do}	Disturbance torque referred to output
τ_{dM}	Disturbance torque referred to motor
Δt	Computation cycle time
ζ	Damping ratio
ω_n	Undamped natural frequency
K_a	Amplifier gain of conventional position servo
$j\omega$	Complex frequency
θ_p	Peak angular excursion of a sinusoidal input
$\Delta\theta_o$	Output angular error for sinusoidal input
$\Delta\theta_{op}$	Peak output error for sinusoidal input

1. Report No. 33-601	2. Government Accession No.	3. Recipient's Catalog No.

4. Title and Subtitle ANALYSIS OF THE COMPUTED TORQUE DRIVE METHOD AND COMPARISON WITH CONVENTIONAL POSITION SERVO FOR A COMPUTER-CONTROLLED MANIPULATOR	5. Report Date March 15, 1973
	6. Performing Organization Code

7. Author(s) B. R. Markiewicz	8. Performing Organization Report No.

9. Performing Organization Name and Address JET PROPULSION LABORATORY California Institute of Technology 4800 Oak Grove Drive Pasadena, California 91103	10. Work Unit No.
	11. Contract or Grant No. NAS 7-100
12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION Washington, D.C. 20546	13. Type of Report and Period Covered Technical Memorandum
	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

A manipulator and its control system (modeled after a Stanford design) is being developed at JPL as part of an artificial intelligence project. This development includes an analytical study of the control system software. This report presents a comparison of the computed torque method and the conventional position servo. No conclusion is made as to the preference of one system over the other, as it is dependent upon the application and the results of a sampled data analysis.

17. Key Words (Selected by Author(s)) Control and Guidance	18. Distribution Statement Unclassified -- Unlimited

19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 52	22. Price

HOW TO FILL OUT THE TECHNICAL REPORT STANDARD TITLE PAGE

Make items 1, 4, 5, 9, 12, and 13 agree with the corresponding information on the report cover. Use all capital letters for title (item 4). Leave items 2, 6, and 14 blank. Complete the remaining items as follows:

3. Recipient's Catalog No. Reserved for use by report recipients.

7. Author(s). Include corresponding information from the report cover. In addition, list the affiliation of an author if it differs from that of the performing organization.

8. Performing Organization Report No. Insert if performing organization wishes to assign this number.

10. Work Unit No. Use the agency-wide code (for example, 923-50-10-06-72), which uniquely identifies the work unit under which the work was authorized. Non-NASA performing organizations will leave this blank.

11. Insert the number of the contract or grant under which the report was prepared.

15. Supplementary Notes. Enter information not included elsewhere but useful, such as: Prepared in cooperation with... Translation of (or by)... Presented at conference of... To be published in...

16. Abstract. Include a brief (not to exceed 200 words) factual summary of the most significant information contained in the report. If possible, the abstract of a classified report should be unclassified. If the report contains a significant bibliography or literature survey, mention it here.

17. Key Words. Insert terms or short phrases selected by the author that identify the principal subjects covered in the report, and that are sufficiently specific and precise to be used for cataloging.

18. Distribution Statement. Enter one of the authorized statements used to denote releasability to the public or a limitation on dissemination for reasons other than security of defense information. Authorized statements are "Unclassified—Unlimited, " "U. S. Government and Contractors only, " "U. S. Government Agencies only, " and "NASA and NASA Contractors only. "

19. Security Classification (of report). NOTE: Reports carrying a security classification will require additional markings giving security and down-grading information as specified by the Security Requirements Checklist and the DoD Industrial Security Manual (DoD 5220.22-M).

20. Security Classification (of this page). NOTE: Because this page may be used in preparing announcements, bibliographies, and data banks, it should be unclassified if possible. If a classification is required, indicate separately the classification of the title and the abstract by following these items with either "(U)" for unclassified, or "(C)" or "(S)" as applicable for classified items.

21. No. of Pages. Insert the number of pages.

22. Price. Insert the price set by the Clearinghouse for Federal Scientific and Technical Information or the Government Printing Office, if known.