

STATISTICAL TURBULENCE THEORY AND TURBULENCE PHENOMENOLOGY

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SUMMARY

The question considered here is how deductive turbulence theory can shed light on the validity of turbulence phenomenology at the level of second-order, single-point moments. Chosen for particular consideration are the phenomenological formula relating the dissipation to the turbulence energy and the Rotta-type formula for the return to isotropy. First, methods which deal directly with most or all of the scales of motion explicitly are reviewed briefly. The two discussed here are the spectral technique of Orszag and Patterson (ref. 1) and the subgrid scale parameterization of Smagorinsky (ref. 2) and Lilly (ref. 3). It appears that, at present, the spectral technique can deal with homogeneous turbulence with satisfactory accuracy up to a (micro) Reynolds number of 45. The virtues and faults of the subgrid scale method are briefly pointed out.

The statistical theory of turbulence is presented here as an expansion about randomness. Two concepts are involved: (1) a modeling of the turbulence as nearly multipoint Gaussian and (2) a simultaneous introduction of a generalized eddy viscosity operator. In this context, the direct interaction approximation (DIA) of Kraichnan (ref. 4) and a more recent theory, the test field model (TFM) of Kraichnan (ref. 5), are discussed briefly.

Some results obtained by using the DIA and TFM to predict the energy dissipation relation and the return to isotropy are next presented. For self-similar free decay, the theory gives an energy-dissipation relationship in qualitative agreement with the phenomenology; but the numerical coefficient appears, at large Reynolds numbers, to be about 40 percent smaller than that used, for example, by Donaldson (ref. 6). This result is considered an indication of a lack of universality of the phenomenological formula in question. For the return to isotropy, the results are in satisfactory accord with the phenomenology, but here again there are symptoms of lack of universality of the phenomenology.

Finally, it is observed that the structure of the phenomenology closely resembles, in some respects, the large Reynolds number limiting form of the DIA equations.

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This observation is illustrated by the form of the triple moment used by Hanjalić and Launder (ref. 7), which may be obtained as a limiting form of the DIA equation for the same third-order moment.

INTRODUCTION

The past few years have seen great advances in numerical computations of turbulent flows around a variety of complicated geometrical shapes. One has to be impressed by the volume of these calculations and by their success in predicting mean velocity profiles, etc., as witnessed, for example, by the compilation of comparisons of test calculations with experiments by Reynolds (ref. 8), and as will undoubtedly be demonstrated again at the present conference. Although the general outlook engendered by these calculations is optimistic, the flourishing of empirical constants, which must be determined by appeals to experiments, is somewhat disturbing. This state of affairs should not be indefinitely tolerated; somehow, a "deductive turbulence theory," free of empiricism, can be brought to bear on these problems, and at the very least can determine the phenomenological constants for the present models. Of course, a deductive theory probably will determine much more than is now contained in the phenomenology – for example, the two-point correlation functions.

This paper will describe how statistical turbulence theory can be brought to bear on the problem of determining the parameters of turbulent flow models and how such a theory could help evaluate several of the phenomenological constants entering the treatment of homogeneous flows. The basic philosophy here is that an understanding of turbulence for relatively simple geometries will shed light on the correctness of the phenomenological equations used in the more interesting and complicated flow geometries. In developing such an approach it may be appropriate to alter (or even give up part of) the empiricism.

The word statistical is used here in the sense of statistical mechanics. Several such theories have been offered, most notably Kraichnan's direct interaction approximation (ref. 9) which will be discussed in more detail later. These theories offer closed deductive equations for the statistical parameters of the flow field (that is, the mean value of the velocity field, and the covariance). Alternatively, some procedures employ the distribution function of the velocity (Lundgren, ref. 10) or the distribution for its Fourier transform (Herring, refs. 11 and 12). The theories may be described as expansions about a type of randomness, which in some sense is supposed to be close to turbulence. Most are free of empirical parameters but make somewhat arbitrary assumptions about the statistics of the flow.

Consider some of the equations used to treat turbulent flows. (Symbols are defined in the appendix of this paper.) The equation for the mean Reynolds stress, $R_{ij} = \langle v_i v_j \rangle$, is

$$\begin{aligned}
\left(\partial R_{ij}/\partial t\right) + \left(U_k \partial R_{ij}/\partial x_k\right) = & -\left(R_{ik} \partial U_j/\partial x_k\right) - \left(R_{jk} \partial U_i/\partial x_k\right) - \left(\partial \langle v_i v_j v_k \rangle / \partial x_k\right) \\
& - \left[\left(\partial \langle v_i p \rangle / \partial x_j\right) + \left(\partial \langle v_j p \rangle / \partial x_k\right) \right] \\
& + \left\langle p \left[\left(\partial v_i / \partial x_j\right) + \left(\partial v_j / \partial x_i\right) \right] \right\rangle \\
& + \left(\underline{\partial^2 R_{ij} / \partial x_k \partial x_k} - 2\nu \left\langle \left(\partial v_i / \partial x_k\right) \left(\partial v_j / \partial x_k\right) \right\rangle \right) \quad (1)
\end{aligned}$$

Here U_i is the mean velocity field and p is the pressure field. The (constant) ambient fluid density ρ_0 has been set at 1, restricting the discussion to incompressible flows. The angle brackets denote an average about the instantaneous velocity field, \underline{v} . Equation (1) is needed to "close" the equation for the mean velocity field U_i , which is not given here (see, for example, Reynolds, ref. 8, p. 9). Before closed equations for \underline{U} and \underline{R} can be written, the third, fourth, and sixth terms on the right-hand side of equation (1) must be related back to \underline{U} and \underline{R} . Here attention will be restricted to the fifth and seventh terms (underlined), which alone survive for homogeneous flows. The fifth term is responsible for the return of an initially nonisotropic flow field to isotropy. It is usually modeled by an equation of the form (Rotta, ref. 13):

$$\left\langle p \left[\left(\partial v_i / \partial x_j\right) + \left(\partial v_j / \partial x_i\right) \right] \right\rangle = -C \frac{\epsilon}{E} \left(\langle v_i v_j \rangle - \frac{2}{3} \delta_{ij} E \right) \quad (2)$$

Here $E = \frac{1}{2} R_{ii}$ is the total turbulent kinetic energy and ϵ is the energy dissipated by viscosity:

$$\epsilon = 2\nu \left\langle \left(\partial v_i / \partial x_k\right) \left(\partial v_i / \partial x_k\right) \right\rangle$$

For ϵ it is customary to take

$$\epsilon = AE^{3/2}/L \quad (3)$$

The seventh term on the right-hand side of equation (1) is usually modeled in either of two ways:

$$2\nu \left\langle \left(\frac{\partial v_i}{\partial x_k} \right) \left(\frac{\partial v_j}{\partial x_k} \right) \right\rangle = \frac{1}{3} \delta_{ij} \epsilon$$

or

$$\left\langle \left(\frac{\partial v_i'}{\partial x_k} \right) \left(\frac{\partial v_j'}{\partial x_k} \right) \right\rangle \approx R_{ij} / \lambda^2$$

The first, which seems more appropriate for large Reynolds number flow, will be used here. In equation (3), L is a length scale associated with the large-scale part of the turbulence. In the following discussion, L is taken to be the longitudinal integral scale, following Batchelor (ref. 14, p. 105, Eq. 6.1.2). The quantity A is supposed to be a constant of order unity; $A = 0.707$ according to Donaldson (ref. 6). Note that equation (2) in conjunction with equation (1) (suppressing temporarily the viscous term) states that R_{ij} relaxes to its isotropic value $\frac{2}{3} \delta_{ij} E$ exponentially if C and the turbulence decay rate ϵ/E are constants.

The immediate goal is to see what statistical turbulence theory can reveal about the universality (or, indeed, validity) of equations (2) and (3). This question is explored for the restricted case of homogeneous flows. It is admittedly a very restricted context, but by no means an empty question, as shall be demonstrated.

The most direct method of assessing equations (2) and (3) by a turbulence theory is to set up a decay calculation to be done by the particular theory in question, march the theory's covariance equations forward in time, and compare the results of the calculation with equations (2) and (3). The generality of equation (3) could be tested by varying the initial energy spectral shape (or equivalently, the initial shape of $\langle \underline{v}(\underline{x}, 0) \underline{v}(\underline{x}', 0) \rangle$), and perhaps by driving the system at low wave number to simulate the destabilizing effects of shear instability. The same procedure could be used to test equation (2), with the additional freedom of varying the degree and spectral shape of the initial anisotropy. At best, a reliable statistical theory would pin down universal values for A and C ; at worst, it would reveal that A and C as defined by equations (2) and (3) are time and context dependent, so that the equations are useless as universal closure prescriptions. Even if these equations are found to be justified for homogeneous flows, there remains the problem of explaining how they could be valid (with the same values of A and C) for non-homogeneous flows. The success of the models may be taken as some indication of the universality of the equations.

Before proceeding with this task, two methods which are capable (with a large enough computer) of answering these questions directly will be reviewed briefly. These are methods which directly integrate the Navier-Stokes equations. They will have an important role in the next few years in understanding turbulence. A comparison of their predictions at low Reynolds numbers for good statistical initial data will help select the correct statistical theory. Wind-tunnel data is not very helpful here because of the lack of statistical initial data at $t = 0$ (just after the grid bars have generated the turbulence). Moreover, the statistical theories are most easily studied if the initial state of the velocity field is random; this "statistical aspect" of the initial data cannot be matched by wind-tunnel experiments. As these methods are improved, they will replace wind-tunnel data as reliable data for low Reynolds number flow. In addition, they provide much more detail than wind-tunnel data; the numerical integrations have enormous amounts of data concerning the flow. These data can be stored on tapes, and interrogated as desired.

DIRECT INTEGRATION OF NAVIER-STOKES EQUATIONS

The simplest turbulence theory is just the Navier-Stokes equations; since most turbulence calculations are numerical anyway, no insight is lost by considering direct integration of the Navier-Stokes equations forward in time, starting from some suitable initial data (for example, data generated by random numbers). Several years ago, this approach appeared much too time consuming to be feasible. It still is not feasible for the sort of problems considered at this conference. However, for simple boundaries and for moderate Reynolds numbers ($R_\lambda = 30$ or 40), it is now possible to do a creditable job of treating turbulence, including all relevant scales of motion, directly by computer. Orszag and Patterson (ref. 1), using the spectral technique, have succeeded in simulating homogeneous turbulence in a periodic cubical box, with good numerical accuracy, up to a Reynolds number R_λ of 45. Their results, I believe, will soon replace wind-tunnel data in accuracy for the case of homogeneous turbulence. As mentioned, this method employed a spectral technique; the velocity field was Fourier analyzed and each Fourier mode was assigned an initial value according to a set of Gaussian random numbers. These were generated so as to yield a statistically Gaussian, homogeneous, isotropic initial spatial velocity field. They then integrated the equations of motion forward in time, allowing the dynamics to build up correlations out of the initial state of chaos. Of course, the realization of complete initial chaos (and isotropy) is limited by the finite number of degrees of freedom that can be handled by a computer. At present, Orszag and Patterson treat one time step of a $(32)^3$ -mode decay calculation in 7.5 sec of CDC 7600 time, and about 100 time steps are required to evolve the system significantly.

It may appear somewhat surprising that Orszag and Patterson resort to Fourier modes in a day when finite mesh methods have all but taken over as a rapid numerical

integration technique for the Navier-Stokes equations. There are three points, however, which make their method competitive with the grid-point method, especially for problems with simple geometry. First, the estimates of derivatives made by the Fourier transform method are much superior to those made by the finite spatial grid method. Second, the finite trigonometric series approximation to the velocity field (sometimes called an infinite order approximation) converges uniformly and very rapidly to the exact solution as the order of the approximation is increased. Such convergence is not obtained for an arbitrary polynomial or piecewise-polynomial approximation to the flow field, as is, for example, used in ordinary finite mesh or finite element schemes. The third point is that the fast Fourier transform technique can be successfully implemented for this problem so as to greatly speed up the calculation (the evaluation of convolution sums). Orszag estimates that for the same given accuracy, the "Galerkin-Fourier" method is faster by a factor of 3 (for a three-dimensional cubic box of turbulence) than the Arakawa staggered mesh differencing scheme.

So far, the main use of these simulations has been to develop insight into the nature of the turbulence and to check the statistical approximations, such as the direct interaction approximation of Kraichnan, which will be discussed subsequently. One simple turbulence parameter pinned down by these calculations is the (differential) skewness, defined by

$$s = \frac{\langle (\partial v_1 / \partial x_1)^3 \rangle}{\langle (\partial v_1 / \partial x_1)^2 \rangle^{3/2}}$$

It measures the strength of nonlinear transfer in the turbulence microscale region. The simulations give a value for s of ≈ 0.47 (for $20 \lesssim R_\lambda \lesssim 50$), whereas the experimental value in this range is more nearly 0.43 (Frenkiel and Klebanoff, ref. 15).

One present problem with the wave-number spectral technique lies in the difficulty of obtaining good statistical information at low wave numbers without using a formidable number of wave-number points. The method uses a grid of equally spaced points, and in three dimensions the density of such points is proportional to k^2 , so there are problems in accurately representing the large-scale features of the flow. At present, this causes difficulties in simulating free, moderate R_λ , turbulence decay if the initial energy spectrum to be simulated has too much energy at small k (that is, $E(k) \propto k$, $k \rightarrow 0$). The problem is that as the spectrum decays, the larger k regions decay out faster because

viscosity acts on them more strongly, leaving only the low k region excited at later times, and this region has poor statistics. At present no self-similar calculations (which are necessary to obtain eqs. (1) and (2)) have been done because of the problem at low wave numbers. According to Leith (ref. 16), a self-similar spectrum whose decay law is $E \propto t^{-1}$ requires $E(k) \propto k$ at small k .

Another method which should be mentioned at this point is the "subgrid scale closure approximation," to which Smagorinsky (ref. 2), Lilly (ref. 3), and Deardorff (refs. 17 and 18) have made substantial contributions. The method is really a phenomenological closure scheme, but the logistics of the approximation is such that as computers become larger the phenomenology can be phased out, leaving in the limit only the exact numerical procedure. The procedure is, very briefly, to average the Navier-Stokes equations over a cubical box of dimensions δ (δ is actually the grid size of the calculation). The dimensions of this box determine the "subgrid scale." Roughly speaking, any scale larger than δ is treated explicitly, and any scale of motion smaller than δ is not treated explicitly but is represented in the averaged equations of motion by mixing-length-type terms. More precisely, the averages of products of fluctuating terms are approximated by suitable functions of average field quantities. The particular functional forms are chosen in accordance with eddy viscosity ideas and are consistent with Kolmogoroff's inertial range assumption (the eddy coefficients are proportional to a scale length δ and to the dissipation rate only). The logic of the procedure implies that the scale size δ is at least as small as the scale sizes in the inertial range. Hence, in using this method, one must be prepared to treat explicitly all scale sizes larger than the scales in the inertial range. As mentioned, the method does contain an empirical constant, but Lilly has shown how it is related in a simple way to Kolmogoroff's constant. The procedure has been applied successfully to thermal convection and shear flow by Deardorff (refs. 17 and 18).

More recently, Lilly (ref. 3) has made an analogous closure for third-order moments. The idea here is to again form averages of the equations of motion over the grid size δ . Now, however, instead of relating the averaged Reynolds stresses to mean gradients, equations of motion for the Reynolds stresses themselves are sought by suitable manipulation of the equations of motion. In this way a relationship is obtained between the time derivatives of the Reynolds stresses and their local sources and sinks which include the averaged triple moments. The latter are then "approximated" by mixing-length-like terms involving the mean gradients of the velocity field and the Reynolds stresses themselves.

The equations obtained this way for the subgrid scale parameterization are very similar to those obtained from other closure procedures (cf. Hanjalić and Launder, ref. 7) with the obvious difference in interpreting and computing the mean fields.

One virtue of the subgrid scale method not possessed by methods parameterizing all scales of motion is that a parameterization of only those scales much smaller than the energy-containing range give the theory a universality not possessed by the other methods. For example, it is hard to accept that $L\epsilon/E^{3/2}$ is the same for any type of flow, regardless of how the turbulence is generated, if E and L pertain to the energy-containing region. One has much less difficulty here if the scales and energy involved are in the inertial range.

Another virtue of the method is its ability to assess its own accuracy. If the actual explicitly treated flux of momentum, for example, turns out to be much larger than that estimated by the subgrid scale closure procedure, the total flux calculation may safely be considered accurate, regardless of the accuracy of the subgrid scale procedure. Deardorff (ref. 18) has found this to be essentially the case for a calculation of a buoyantly active shear layer at heights greater than one or two δ .

One difficulty with this approach, appreciated by Fox and Lilly (ref. 19), is that there appears to be no strictly deductive way whereby scales of motion less than a certain scale size may be treated as statistical and scales larger than this size may be treated deterministically. Because of the nonlinearity of the Navier-Stokes equations, small-scale uncertainties will in time penetrate the large-scale region, thereby contradicting the logical framework of the theory. This difficulty may be cast in terms of statistical turbulence theory by considering a statistical initial value problem in which scales larger than δ are identical from ensemble member to ensemble member but scale sizes less than δ are known only statistically (varying in amplitude and phase from ensemble member to ensemble member). This initial specification corresponds to the logical framework of the "subgrid scale" closure approximation. As time passes, however, destabilizing nonlinear terms in the equations of motion cause the scale sizes larger than δ to be contaminated with that uncertainty initially residing at scale less than δ . This mixing transforms the initial "mean" (or certain) field into an eventual "fluctuation" (or uncertain) field. Of course, the closure procedure takes no cognizance of this fact, and therein lies a problem in calculations of the evolution of a flow. As long as the errors do not penetrate the large-scale energy-containing region, it may safely be assumed that the predicted values of these large scales are correct. However, once the errors penetrate the large-scale region, the solutions of the subgrid scale procedure must be interpreted as a "typical" flow field, and the logical connection with the initial flow field is obscure. It is of some interest in this connection to have estimates of error-growth time scales.

A theory for error growth has recently been developed by Leith and Kraichnan (ref. 20). These investigators use the "test-field approximation," which is a type of statistical approximation that will be discussed subsequently. Their conclusion is that

errors initially confined to infinitely small scales grow and saturate a region of scale sizes δ within a time $\approx 9.5\epsilon^{-1/3}\delta^{2/3}$, which is 11 times the eddy circulation time for length scales δ . Their analysis was limited to the limit of infinite Reynolds number and to δ in the inertial range. Whether or not the errors grow until they penetrate the energy-containing range is not yet known, although if δ is initially sufficiently large, there seems little doubt that the energy region would be contaminated by error.

STATISTICAL TURBULENCE THEORY

The original objective of applying statistical turbulence theory to the practical problems of determining turbulence parameters is pursued next. The physical content of several of the theories is discussed briefly, but the equations are not derived. As mentioned in the introduction, these theories are pivoted on the idea that turbulence is, in some sense, close to a state of randomness. "Completely random" would mean, here, that the averages of products of the velocity field at different space-time points $y = (\tilde{y}, t)$ are joint normal. (A space-time point is indicated by y , the spatial components by \tilde{y} , and the time component by t .) For the third- and fourth-order moments, in particular,

$$\left. \begin{aligned} \langle v_i(y_1)v_j(y_2)v_k(y_3) \rangle &= \langle v_i(y_1) \rangle \langle v_j(y_2)v_k(y_3) \rangle + \langle v_i(y_1)v_j(y_2) \rangle \langle v_k(y_3) \rangle \\ &\quad + \langle v_i(y_1)v_k(y_3) \rangle \langle v_j(y_2) \rangle \\ \langle v_i(y_1)v_j(y_2)v_k(y_3)v_l(y_4) \rangle &= \langle v_i(y_1)v_j(y_2) \rangle \langle v_k(y_3)v_l(y_4) \rangle + \dots \\ &\quad + \langle v_i(y_1) \rangle \langle v_j(y_2)v_k(y_3)v_l(y_4) \rangle + \dots \end{aligned} \right\} \quad (4)$$

where the dots indicate omitted terms in which the indices 1, 2, 3, and 4 are permuted. Here the property of randomness is expressed in terms of two-point (and three-point) functions like $\langle v_i(y)v_j(y') \rangle$, which are more general covariances than those used in equation (1). For deductive turbulence theories the two-point correlation functions are indispensable. It is noted, in this connection, that the problem of viscous decay (flow at very low Reynolds number) is closed with respect to the two-point functions $R_{ij}(x, x', t) = \langle v_i(x, t)v_j(x', t) \rangle$ but not with respect to the single-point function $R_{ij}(x, x, t)$; hence the need for the two-point functions at the very least.

The hypothetical consequences of complete randomness for the Navier-Stokes equations are examined now. These equations are written in a compact notation (summation convention understood):

$$\mathcal{L}_{in}^0(\mathbf{y})\mathbf{u}_n(\mathbf{y}) = P_{ij}(\tilde{\mathbf{y}})\frac{\partial}{\partial \tilde{y}_l}\left(u_l u_j - \langle u_l u_j \rangle\right) \quad (5)$$

where the operator \mathcal{L}^0 is

$$\mathcal{L}_{in}^0 = \delta_{in}\left(\frac{\partial}{\partial t} - \nu \nabla_{\tilde{\mathbf{y}}}^2\right) - P_{in}(\tilde{\mathbf{y}})U_l(\mathbf{y})\frac{\partial}{\partial \tilde{y}_l} - P_{ij}(\tilde{\mathbf{y}})\frac{\partial}{\partial \tilde{y}_n} U_j$$

and the operator $P_{ij}(\tilde{\mathbf{y}})$ is specified by

$$\nabla_{\tilde{\mathbf{y}}}^2 P_{ij}(\tilde{\mathbf{y}}) = \delta_{ij} - \frac{\partial^2}{\partial \tilde{y}_i \partial \tilde{y}_j}$$

Here, \underline{u} is the departure of \underline{v} from its ensemble mean, $\underline{U} = \langle \underline{v} \rangle$. To obtain P_{ij} Poisson's equation must be inverted for the boundary conditions appropriate to the problem at hand (a cubic box of homogeneous turbulence). The operator $P(\tilde{\mathbf{y}})$ simply suppresses the compressible part of $(\underline{v} \cdot \nabla)\underline{v}$, so that $\nabla \cdot \underline{v}(\underline{x}, t) = 0$, and results from an elimination of the pressure term from the equations of motion.

In equation (5) the terms which give rise to the closure problem are isolated on the right-hand side. These are the terms that may build up multipoint non-joint-normal correlations out of multipoint normal initial data. That is to say, relations (4) are consistent with the Navier-Stokes equations if the right-hand side of equation (5) is suppressed. The simplest nontrivial closure procedure is to discard the right-hand side of equation (5) entirely. Such a theory, sometimes called the quasi-linear or mean-field theory (also called by Deissler "weak turbulence" approximation in ref. 21), has met with some success for thermal convection (Herring, refs. 22 and 23). It has also been applied to shear flows by Deissler (ref. 21).

First the assumption of complete randomness, as embodied by equations (4), is tried on equation (5). Closed equations for $\langle u_i(\mathbf{y})u_j(\mathbf{y}') \rangle$ are then obtained simply by multiplying equation (5) evaluated at \mathbf{y} by itself at \mathbf{y}' . The result is

$$\begin{aligned}
\mathcal{L}_{in}^0(y)\mathcal{L}_{i'n'}^0(y')\langle u_n(y)u_{n'}(y') \rangle &= P_{ij}(\tilde{y})P_{i'j'}(\tilde{y}')\frac{\partial^2}{\partial \tilde{y}_l \partial \tilde{y}'_{l'}} \left[\langle u_l(y)u_{l'}(y') \rangle \langle u_j(y)u_{j'}(y') \rangle \right. \\
&\quad \left. + \langle u_l(y)u_{j'}(y') \rangle \langle u_j(y)u_{l'}(y') \rangle \right] \\
&\equiv F_{ii'}(y,y')
\end{aligned} \tag{6a}$$

It is convenient to rewrite equation (6a) as a first-order equation in time by multiplying it by $(\mathcal{L}^0)^{-1}(y)$, with the result

$$\mathcal{L}_{in}^0(y)\langle u_n(y)u_{n'}(y) \rangle = \int G_{n'i'}^0(y',z)dz F_{ii'}(y,z) \tag{6b}$$

where

$$\tilde{\mathcal{L}}_{in}^0(y)G_{np}^0(y,y') = P_{ip}(\tilde{y})\delta(y - y')$$

Here, $\tilde{\mathcal{L}}$ is the adjoint of \mathcal{L} .

The dz -integration here is over all the spatial part of z , but the time integration extends only over the past up to the time argument of y',t' . It is recalled here that basic interest is in the simultaneous moments, $\langle v_i(y)v_j(y) \rangle$. An equation of motion for these may be obtained from equation (6b) by forming the limit,

$$\lim_{y \rightarrow y'} \left[\mathcal{L}_{in}^0(y)\langle u_n(y')u_i(y) \rangle + \mathcal{L}_{in}^0(y')\langle u_n(y')u_i(y) \rangle \right] \equiv N(y) \tag{7}$$

There is a fundamental difficulty with equations (6a) and (6b); namely, they do not conserve kinetic energy (for closed systems $N \neq 0$). This difficulty stems from the fact that the "turbulence force" $P(\underline{v} \cdot \nabla)\underline{v}$ has been assumed to be a completely random stirring force, and random forces are known to increase the kinetic energy of systems to which they are applied. Hence, the assumption of complete randomness must be abandoned.

The statistical theories modify $\mathcal{L}^0(y)$ used in equations (6a) and (6b) to a new operator $\mathcal{L}(y)$ so as to restore energy conservation. The modification consists of including on the left-hand side of equations (6) a generalized eddy viscosity term so as to make $N = 0$ for closed systems. Formally, $\mathcal{L}^0(y)$ is replaced by $\mathcal{L}(y)$, and $G^0(y,z)$ ($\equiv (\mathcal{L}^0)^{-1}$) by $G(y,z)$ ($\equiv \mathcal{L}^{-1}$). The modified equations are

$$\mathcal{L} = \mathcal{L}^0 + \eta$$

$$\mathcal{L}_{in}^0(y) \langle u_n(y) u_n(y') \rangle + \int \eta_{in}(y,z) dz \langle u_n(z) u_n(y') \rangle = \int G_{n,j}(y',z) dz F_{ij}(y,z) \quad (8a)$$

where $F_{ij}(y,z)$ is defined by equation (6a). The operator $\eta(y,z)$ is defined by

$$\begin{aligned} \frac{1}{2} \eta_{in}(y,z) = & -P_{ij}(y) \frac{\partial}{\partial \tilde{y}_l} P_{pn}(z) \frac{\partial}{\partial \tilde{z}_s} G_{lp}(y,z) \langle u_s(z) u_j(y) \rangle \\ & - P_{ij}(y) \frac{\partial}{\partial \tilde{y}_l} P_{ps}(z) \frac{\partial}{\partial \tilde{z}_n} G_{lp}(y,z) \langle u_s(z) u_j(y) \rangle \end{aligned} \quad (8b)$$

The new term on the left-hand side of equation (8a) cancels the right-hand side upon forming the equation for the total energy, $N = 0$.

Equations (8a) and (8b) constitute a complete statistical theory for the two-point covariances $\langle \underline{v}(y) \underline{v}(y') \rangle$. The ingredients embodied in it are: (1) the modeling of the turbulence force $P(\tilde{y})(\underline{v} \cdot \nabla) \underline{v}$ as Gaussian-multivariate and (2) the simultaneous introduction of a generalized eddy viscosity operator $\eta(y,z)$, so that energy is conserved within the context of the Gaussian assumption. The type of "modeling" done here involves a qualitative characterization of the statistics of the flow rather than any explicit quantitative assumption about relationships between the various terms in the theory.

Still to be specified is $G(y,z)$. On this point the theories differ. Generally $G(y,z)$ specifies the mechanisms whereby the flow at different space-time points becomes decorrelated and decays away under the joint action of viscosity and turbulence. At very low Reynolds numbers G is expected to be entirely viscous, so that

$$\mathcal{L}_{in}^0(y) G_{nj}(y,y') = P_{ij}(\tilde{y}) \delta(y - y') \quad (R_\lambda \rightarrow 0) \quad (9a)$$

At larger Reynolds numbers the turbulence itself contributes an eddy viscosity, so that in the determination of $G(y,z)$ the generalized eddy viscosity operator $\eta(y,z)$ is included:

$$\tilde{\mathcal{L}}_{in}^0(y)G_{nj}(y,y') + \int \eta_{in}(y,z)dz G_{nj}(z,y') = P_{ij}(\tilde{y})\delta(y - y') \quad (9b)$$

The choice of equation (9a) is sometimes called the "two-time quasi-normal approximation," and is sensible only at very small Reynolds numbers ($R_\lambda \leq 10$). The choice of equation (9b) together with equation (8a) constitutes the direct interaction approximation (Kraichnan, ref. 4). Note that equation (9b) lapses over into equation (9a) for small Reynolds number flow.

Another approach in determining $G(y,y')$ is to select the most appropriate physical mechanism responsible for producing decorrelation, work out the corresponding G-function, and use it in equations (8a) and (8b). Such a procedure was used by Kraichnan (ref. 5) in deriving the test-field model, a theory which uses a G-function that incorporates "pressure scrambling" and viscous effects alone. The test-field model also makes the more severe approximation of modeling the turbulence force on white noise, which is a simplification introduced to avoid the time-history integrals in equations (8a) and (8b). Yet another method (to my knowledge yet untried) is to base G on the evolution of a pair of particles whose relative position at $t = t'$ is $\tilde{y} - \tilde{y}'$.

At this point the objection may be raised that the comments of the last paragraph have introduced an arbitrariness into the theory, which was to be avoided. Expressed differently, why worry about specific physical mechanisms for decorrelating the flow, if the DIA is a complete theory. The answer is that the DIA, though complete, cannot be correct at large Reynolds numbers because it does not behave properly under random uniform velocity translation (Kraichnan, ref. 24). As a consequence, it does not have a proper inertial range ($E(k) \propto k^{-3/2}$ instead of $E(k) \propto k^{-5/3}$). On the other hand, the two mechanisms mentioned above do produce a properly invariant theory.

The arbitrariness alluded to in discussing the test-field model is just the (arbitrarily assumed) strength parameter which couples the "test field" to the actual velocity field. The (statistical) deviation of this (compressive) test field from the actual velocity field measures the decorrelation effects expressed by g . This theory, with a single adjustable parameter, is capable of treating nonisotropic and nonhomogeneous flows.

The accuracy of the direct interaction and test-field models has recently been assessed by comparing their predictions with the Orszag-Patterson type of simulations at low to moderate Reynolds numbers (Herring and Kraichnan, ref. 25). These comparisons were made for homogeneous and isotropic flow fields with $R_\lambda \leq 50$. A detailed

comparison was made for the Fourier transform of the two-point function $R_{ij}(\underline{x}, \underline{x}', t)$, as well as for the total energy decay and skewness. The direct interaction approximation gave entirely satisfactory results for all the above spectra, including the dissipation and energy transfer spectra. The test-field model gave satisfactory results for all the above spectra if the parameter mentioned previously was adjusted so that the predicted and simulated skewness agreed. At much larger R the test-field model with the same parameter gave the Kolmogoroff spectrum, with the Kolmogoroff constant $C = 1.8$.

COMPARISON OF STATISTICAL THEORY WITH PARAMETERIZED EQUATIONS

Assuming that they are valid, what do these theories indicate about the universality of equations (2) and (3)? Consider first equation (3), which may be examined within the context of homogeneous, isotropic flows. On general grounds, an equation like (3) can be expected if the decay is self-similar; that is, if

$$\left\langle v_i(\underline{x}', t) v_j(\underline{x}' - \underline{x}, t) \right\rangle \equiv R_{ij}(\underline{x}, t) = \text{Function of } (x/\lambda, x'/\lambda, t)$$

where λ is the Taylor microscale. Equivalently, the energy spectrum $E(k)$, defined to be the Fourier transform of R_{ii} with respect to $\underline{x} - \underline{x}'$, should be of the form $F(k\lambda, t)$ for k in both the energy containing range and the dissipation range. Figure 1 shows $\epsilon L/E^{3/2}$ as a function of t for such a self-similar calculation. Here the initial energy spectrum is that found experimentally by Ling and Huang (ref. 26). The theory, incidentally, does not confirm that this initial spectrum is self-similar during the decay; the spectrum predicted by the theory is a good deal more peaked (especially at large R_λ) than this. The final asymptotic values of R_λ label the curves. In all cases studied, it was found that $R_\lambda(t)$ became independent of t for $t \geq 1$ and that the $E(k)$, $k^2 E(k)$, and energy transfer spectrum became self-similar. (Note that $E(k\lambda(t))/E_0\lambda(t)$ is a universal function of R_λ and $k\lambda$.) For a given initial R_λ , $A = \epsilon L/E^{3/2}$ becomes very nearly constant at large t , but the value of A appears to be a function of R_λ . At the larger values of R_λ , $A \leq 0.5$ is indicated. This may be compared to the value $A = 0.7$ (Donaldson, ref. 6), used in the phenomenological approach.

The difference (≈ 40 percent) between the computed value of A and the phenomenological value of ≈ 0.7 probably indicates the kind of departure from universality to be expected in using the phenomenological approach. The lack of universality in the value of A probably is due to the lack of universality in the spectral shape of $E(k)$ in the energy containing region.

Consider next equation (2) for the deviation of R_{ij} from its isotropic value, $2E\delta_{ij}/3$. The statistical theories may also be applied to this case. To solve this problem requires an order of magnitude more work than the homogeneous isotropic problem, even for the homogeneous case. This is because the energy-correlation function must be specified at all angles. The simplest way to do this is to assume that departures from isotropy may be parameterized by $E_1(k)$ and $E_2(k)$, where E_1 and E_2 are the energy perpendicular and parallel to the axis of symmetry. The departure from isotropy is then represented simply by $\Delta = E_1 - E_2$, and the phenomenology of equation (2) is

$$d\Delta/dt = -C(E/\epsilon)\Delta$$

where

$$\Delta = \int_0^\infty \Delta(k)dk$$

An application of the DIA theory to homogeneous, axisymmetric turbulence has been completed, the details of which will be published elsewhere. Only those points bearing on the phenomenology (that is, eq. (2)) are reported here.

In general, the DIA theory gives equations for $d\Delta(k)/dt$ and $dE(k)/dt \equiv \frac{d}{dt}(E_1(k) + E_2(k))$, which quadratically couple $\Delta(k)$ and $E(k)$ for all wave numbers k with the corresponding G-functions. A careful examination of these equations, which are similar to the Fourier transform of equations (7) and (8), shows that the equation for $\Delta(k)$ is nearly linear. This result is, a priori, somewhat surprising since terms contributing to $d\Delta(k)/dt$ proportional to $\Delta(p)\Delta(\underline{k}-\underline{p})$ are not easily seen to be small. This result tends to lend credence to equation (2). A particularly simple equation results if the energy and $\Delta(k)$ spectrum may be regarded as very sharp ($E(k)$ and $\Delta(k) \propto \delta(k - k_0)$) and if "memory effects" are neglected. In this case,

$$d\Delta(k,t)/dt = -2\eta(k)\Delta(k)$$

For the case in which $E(k)$ is sharply peaked about k_0 , it may be shown, under the restrictions stated above, that

$$L = \frac{3\pi}{4k_0} \quad \eta(k_0) = \frac{0.858E_0^{1/2}}{L}$$

which gives

$$d\Delta/dt = \frac{-1.71E^{1/2}\Delta}{L} \quad (10)$$

where η is the eddy viscosity coefficient, which is just the "Fourier representative" of the operator $\eta(y,z)$ as given by equation (8b).

Equation (10) should not be taken too seriously, especially in view of the rather restrictive assumptions involved in its derivation. It is, nevertheless, an equation agreeing with the phenomenology as to type and order of magnitude and originating in a deductive theory.

Of course, for a particular initial value problem it is possible to do much better than equation (10) by constructing numerical solutions for the direct interaction approxi-

mation. Such results are presented in figure 2. This shows $C = (E/\epsilon) \left(d \ln |E_1 - E_2| / dt \right)$

as a function of t for the case $E_1(k) = \frac{2\pi k^2 \exp(-2k)}{k + 0.5}$, $E_2(k) = 0$, initially, and $R_\lambda(0)$

$= 47.6$. It can be seen that C levels off for the later stages of decay, and then begins to decrease slowly. The initial value of this spectrum was chosen because for isotropic turbulence at the same value of R_λ , the decay of $E(k)$ appeared to be self-similar. The values of $C(t)$ appear consistent with Rotta's suggested value (ref. 27). This comparison should not, however, lead to any generalizations since different initial spectral shapes give somewhat different values for C and since the computed values here are not strictly constant. For example, if the deviation of E_1 from E_2 is made more pronounced at larger k than in the above example, larger values of C result.

So far in confronting statistical theory with phenomenology, certain terms contributing to the evolution of the Reynolds stress have been dealt with in isolation, and only for homogeneous flows. Although such calculations are instructive, they can hardly be thought adequate to deal with shear flow in which mean fields \underline{U} as well as boundaries are present. In my opinion, it will be a few (2 or 3) years before direct computation using the DIA on simple geometries at very large Reynolds numbers will be done. David Leslie, in his forthcoming book "Developments in the Theory of Turbulence," estimates that it would take about 10 man-years to program DIA equations like equations (8a) and (8b) to deal realistically with large R_λ shear flows. I think this estimate is a bit pessimistic, but of the correct order of magnitude.

An alternate to a frontal attack is to try to simplify the DIA equations analytically, using asymptotic functional forms appropriate to large Reynolds numbers, and perhaps large distances from boundaries. Some progress that has already been made in this direction by Leslie (ref. 28) is described in detail in Chapter 15 of his book cited above. Perhaps the severe analytic complexities of the DIA can be in some sense reduced so as to approach the complication level of phenomenological theories discussed at this conference. Consider, for example, the triple moment term in equation (2). Hanjalić and Launder (ref. 7) approximate this term by

$$\langle u_m(y)u_l(y)u_j(y) \rangle = c_s \frac{E}{\epsilon} \left(R_{js} \frac{\partial}{\partial \tilde{y}_s} R_{lm} + R_{ls} \frac{\partial}{\partial \tilde{y}_s} R_{jm} + R_{ms} \frac{\partial}{\partial \tilde{y}_s} R_{lj} \right) \quad (11)$$

where

$$R_{ij} = \langle u_i(y)u_j(y) \rangle$$

The DIA equation for equation (11) is

$$\begin{aligned} \langle u_m(y)u_l(y)u_j(y) \rangle = \lim_{y' \rightarrow y} \int dz \left\{ G_{mp}(y',z)P_{pr}(z) \frac{\partial}{\partial z_s} \left[R_{lr}(y,z)R_{js}(y,z) + R_{ls}(y,z)R_{js}(y,z) \right] \right. \\ + G_{lp}(y,z)P_{pr}(z) \frac{\partial}{\partial z_s} \left[R_{ms}(y,z)R_{rj}(z,y) + R_{mr}(y,z)R_{sj}(z,y) \right] \\ \left. + G_{jp}(y,z)P_{pr}(z) \frac{\partial}{\partial z_s} \left[R_{mr}(y',z)R_{ls}(y',z) + R_{ms}(y',z)R_{lr}(y,z) \right] \right\} \quad (12) \end{aligned}$$

Equation (12) reduces to equation (11) if (arbitrarily)

$$G_{ij}(y,y') = \frac{1}{2} c_s \frac{E}{\epsilon} \delta(t - t') \delta(\tilde{y} - \tilde{y}') \delta_{ij}$$

$$P_{ij} = \delta_{ij}$$

No claim is made here that these expressions for G and P become valid at very large Reynolds numbers and far from boundaries. In fact, quite the contrary is suggested by

an examination of the G-equation. However, perhaps an examination of the complete equations (8a) and (8b), along lines begun by Leslie (ref. 28), will show whether anything close to these can be valid or, better yet, will suggest how to improve equation (11), especially near boundaries and at moderate Reynolds numbers. One point that is clear from even a cursory examination of equations (11) and (12) is that equation (12) is nonlocal in both space and time, whereas equation (11) is not.

Of course the preceding discussion presumes that the statistical theory discussed here is valid. On this point, the reader may examine the comparison of theory with simulation (for isotropic flow) by Herring and Kraichnan (ref. 25) and judge for himself. It is hoped that a more detailed comparison of theory with computer experiment for more general types of flow will soon tell whether this type of theory is on the right track.

LIMITATIONS OF THEORY

In closing, some of the limitations and defects of the statistical moment theory discussed here will be pointed out. First, there are limitations connected with the fact that the theory is statistical. In comparing theory with experiment, this leads to problems of how to specify initial data. Suppose, for example, the decay of turbulence generated by a wind-tunnel grid is to be predicted. The two-point velocity correlation just behind the grid bars can be measured, and from this data an energy spectrum can be obtained which the statistical theory uses as $E(k,0)$. However, the use of this spectrum alone as initial data in the statistical theory cannot produce the observed subsequent decay even if the theory used is exact. This is because the experiment (or, ideally, an ensemble of experiments) has higher order (statistical) correlations built up at the initial time, whereas the theory presumes these to be zero. Therefore, if only the energy spectrum can be specified initially, the theory can be expected to deviate from experiment while the higher order moments in the statistical theory are being built up from zero. The theory could be worked out for specified initial values for higher order moments, but I do not know that these are measurable. The same dilemma may also afflict the phenomenological theories also in connecting the mean velocity and Reynolds stresses to the "preturbulent state."

That the DIA does not give the proper inertial range has already been mentioned. It is not known, however, how serious this is with regard to the behavior of the single-point Reynolds stresses, even at large Reynolds numbers. In this connection, it may be shown — under rather weak assumptions — that an equation like equation (3) is obtained for the DIA at large R_λ . The value of A has, however, not been computed. In any case the test-field model could be used, but it is a less complete theory than the DIA. Alternatively, there is the Lagrangian history direct interaction theory (Kraichnan, ref. 9), which com-

parens very well indeed with large Reynolds number flows; but this is a very complicated theory, whose logical foundations, in my opinion, are not as secure as those of the DIA.

Finally, there are problems for which the statistical-moment approach discussed here is not very profitable; that is, extremely intermittent flows consisting of very small regions of intense shear separated by quiescent volumes. In this case, the statistical moment theory requires a specification of very high order moments to adequately describe the physics of the problem. Hence, closure at low order moments would be inappropriate.

APPENDIX

DEFINITIONS OF SYMBOLS

- A empirical constant relating energy integral scale and energy dissipation
(see eq. (3))
- C constant
- c_s triple moment constant introduced by Hanjalić and Launder (ref. 7, eq. (2.3))
- E total turbulent kinetic energy, $\frac{1}{2}\langle u^2 \rangle$
- F_{ij} turbulent functional defined in equation (6a)
- G Green's function which includes eddy viscosity effects (see eq. (9b))
- G^0 Green's function which does not include eddy viscosity effects (see eq. (6b))
- g dimensionless scaling parameter entering test field model (see ref. 25 for further details)
- k wave number
- L longitudinal integral scale
- $\mathcal{L} = \mathcal{L}^0 + \eta$
- \mathcal{L}^0 operator (see eq. (5))
- N defined by equation (7)
- P pressure "projection" operator, defined just after equation (5)
- p pressure field
- R ensemble mean Reynolds stress tensor
- R_{ij} ensemble mean Reynolds stress tensor components
- ω

R_λ	Taylor microscale Reynolds number
s	differential skewness, $\frac{\langle (\partial v_1 / \partial x_1)^3 \rangle}{\langle (\partial v_1 / \partial x_1)^2 \rangle^{3/2}}$
t	time
U_i	i th component of ensemble mean velocity field \underline{U}
\underline{U}	ensemble mean velocity field
\underline{u}	deviation of velocity field from its ensemble mean
\underline{v}	instantaneous velocity field vector
\underline{x}	vector coordinate
y	space-time point (\tilde{y}, t)
\tilde{y}	three-vector part of space-time point y
z	space-time point (\tilde{z}, t)
\tilde{z}	three-vector part of space-time point z
$\Delta(\underline{k})$	deviation from isotropy spectrum, $\equiv 2\pi k^2 \langle u_3(\underline{k}, t) ^2 - u_1(\underline{k}, t) ^2 \rangle$
$\Delta = \int_0^\infty \Delta(\underline{k}) d\underline{k}$	
δ	grid-scale length in subgrid-scale method
δ_{ij}	Kronecker delta ($= 0$ if $i \neq j$; $= 1$ if $i = j$)
$\delta(\underline{y} - \underline{y}')$	Dirac δ -function
ϵ	energy dissipation rate, $\equiv -2\nu \langle \underline{v}_i \nabla^2 \underline{v}_i \rangle$

- η eddy viscosity operator (see eq. (8b))
- λ Taylor microscale
- ν kinematic viscosity coefficient
- ρ_0 constant fluid density, set equal to 1
- ' used to distinguish one space point from another
- $\langle \rangle$ indicates ensemble mean

Subscripts:

$\left. \begin{matrix} i, j, k, l, \\ m, n, p, s \end{matrix} \right\}$ vector indices (= 1, 2, 3); summation convention holds, unless otherwise stated

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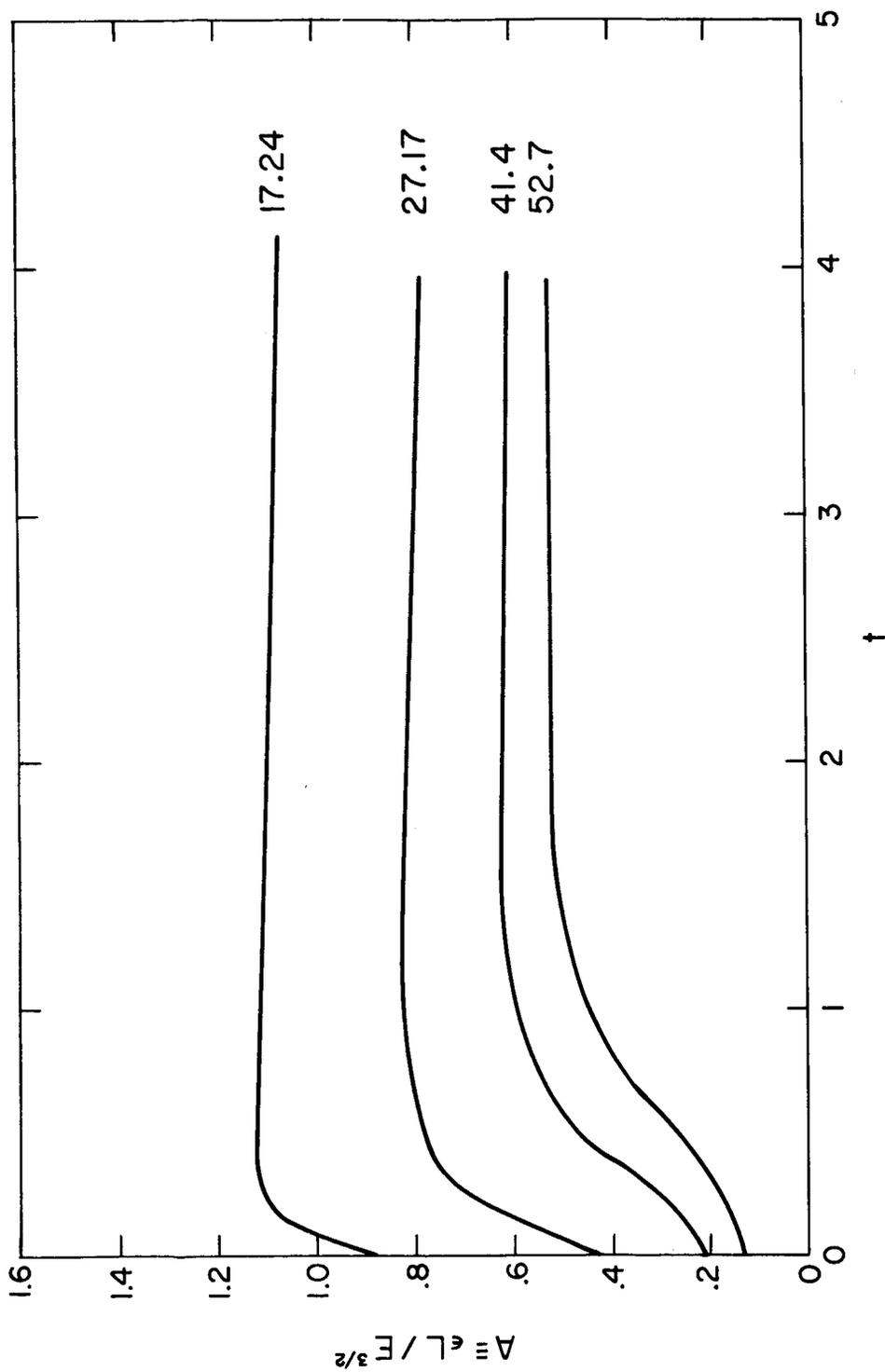


Figure 1.- Curves of $A = \epsilon L/E^{3/2}$ as a function of time t for initial value problem, $E(k,0) = \lambda^2(1 + \sqrt{2k\lambda})k \exp(-\sqrt{2k\lambda})$. $\lambda = 0.1, 0.2, 0.3$, and 0.8 , corresponding to $R_\lambda(0) = 20, 40, 60$, and 80 . Curves are labeled by final asymptotic value of R_λ . Theory used is TFM; $g = 1.5$.

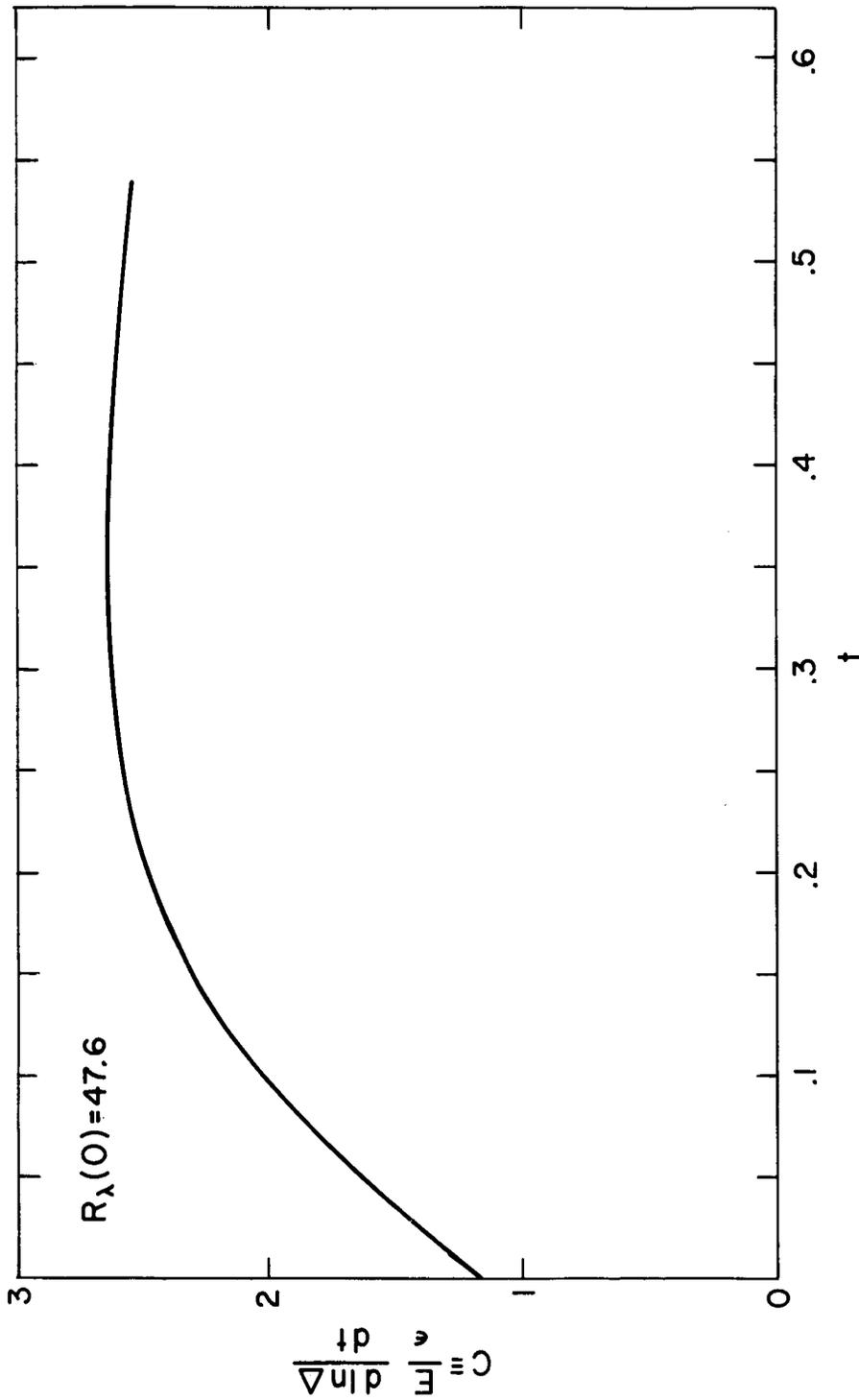


Figure 2.- Return-to-isotropy coefficient C (eq. (2)) as a function of time, according to the DIA theory. Curve depicts $\frac{E}{\epsilon} \frac{d \ln \Delta}{dt}$ for initial spectrum $E_1(k,0) = \frac{2\pi k^2 \exp(-2k)}{k + 0.5}$, $E_2(k,0) = 0$, where $E_1(k,0) = 2\pi k^2 \langle |u_1(\underline{k},t)|^2 + |u_2(\underline{k},t)|^2 \rangle$ and $E_2(k,0) = 2\pi k^2 \langle |u_3(\underline{k},t)|^2 \rangle$.