FREE TURBULENT MIXING IN A COFLOWING STREAM*

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DEVELOPMENT OF THE MODEL

The turbulent transport model used here is of the classical, gradient transport, eddy viscosity type; that is, it is based upon

$$\tau = \rho\epsilon \frac{\partial U}{\partial y} \tag{1}$$

where ϵ is to be expressed in terms of mean flow quantities. (Symbols are defined at the end of the text.) Thus, no turbulence information is used in the functional expression for the eddy viscosity; however, recent work in extending the model has employed turbulence information in the proportionality constant. This work will be discussed further in a later section. The development of the model is described in references 1 to 4, but a short summary of the major points is included here for convenience.

This work grew out of an inquiry into the relation, if any, between various more or less successful eddy viscosity models, each developed for a different flow situation. Consider the following three planar examples:

Prandtl (jet mixing) model (ref. 5)

$$\epsilon_1 = 0.037b_{1/2}\left|U_{max} - U_{min}\right| \tag{2}$$

Schlichting (wake) model (ref. 6)

$$\epsilon_2 = 0.022C_D D U_e \tag{3}$$

Clauser (boundary-layer) model (ref. 7)

$$\epsilon_3 = 0.018U_e\delta^* \tag{4}$$

These apparently bear no direct relation to each other, even though they are all intended to model rather similar flow problems. It is very useful to examine the Schlichting wake model more closely. First,

$$C_D D \equiv 2\theta \tag{5}$$

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so that equation (3) becomes

$$\epsilon_2 = 0.044 U_e \theta \tag{6}$$

Now, equations (4) and (6) appear in actual contradiction, but this discrepancy quickly vanishes when it is observed that Schlichting used a linearized "far wake" analysis in developing his model, where the integrand in the momentum thickness θ was approximated as

$$\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) \approx 1 - \frac{U}{U_e} \tag{7}$$

which makes it the same as that for the displacement thickness δ^*. Thus, the Clauser boundary-layer model and the Schlichting wake model, in reality, differ only in the value of the proportionality constant. This too can be explained, as will be shown later.

A study of the Prandtl jet model in relation to the Clauser model showed that they could not be reduced to the same functional form. However, a simple numerical exercise demonstrates that the actual values of ϵ predicted by each model for a series of reasonable profile shapes (variation of U with y) are virtually identical. This comparison requires a generalization of the Clauser model to let the displacement thickness measure a mass-flow excess with respect to the free stream as well as the more usual mass-flow defect. Also, since a wake or jet has two sides to the mixing layer as opposed to a one-sided boundary layer, the Clauser model must be written as

$$\epsilon_3 = 0.036 U_e \int_0^\infty \left|1 - \frac{U}{U_e}\right| dy \tag{8}$$

This model will provide good predictions of the development of planar, constant-density flows of either the wake type (see test case 14 below) or jet type (see ref. 4).

At this early point in the development, it remained to extend the model in equation (8) to axisymmetric and/or variable-density cases. The constant-density axisymmetric case was considered first. The model for this geometry was obtained by introducing a new, physical interpretation of the Clauser model (applicable to either its original form, eq. (4), or its extended form, eq. (8)). This interpretation is stated, "The turbulent viscosity $\rho\epsilon$ is proportional to the mass flow defect (or excess) per unit width of the mixing region." This can be carried over to the axisymmetric situation as

$$\rho\epsilon_4 = \frac{K\rho_e U_e}{L} \int_0^\infty \left|1 - \frac{U}{U_e}\right| 2\pi r \, dr \tag{9}$$

where L is some characteristic width required to be dimensionally correct. Note that the unit width of the planar case does not appear here. Some studies were made to determine a suitable width for this use. The obvious choice of the local half-radius $r_{1/2}$ was found to be unsuitable for all but jet cases with $U_j/U_e >> 1$. Since such cases are of marginal practical interest, this choice was rejected, and the simple choice of the initial radius a was made. For wakes, $r_{1/2}(0)$, the half-radius at the "initial" station at the end of the "near wake," is used.

It was still necessary to determine the constant K for use in equation (9). It should be emphasized here that a useful turbulent exchange model must contain fixed empirical constants that are determined once and then are not changed from problem to problem! In the present case, this was accomplished by considering the experimental case of Forstall and Shapiro (ref. 8) with $U_j/U_e = 2.0$. The prediction using $K\pi = 0.018$ (the apparent correspondence to Clauser's constant in eq. (4) is pure coincidence) is compared with experiment in figure 1, where the excellent agreement can be noted. This constant has been adopted as universal for use with this model and has not been changed for comparisons with any other experimental case. Two points are worth noting from figure 1. First, a variation in the constant does not change the slope of the predicted velocity decay, it merely moves the curve up and down on the paper. Thus, the correct decay rate predicted is due solely to the functional form of the model. Second, the straightforward extension of Prandtl's planar model, equation (2), to the axisymmetric case

$$\epsilon_5 = 0.025 r_{1/2} \left| U_{max} - U_{min} \right| \tag{10}$$

gives a poor prediction.

The extension of the unified model, equation (8) for planar and equation (9) for axisymmetric cases, to variable-density situations was accomplished by simply using the appropriate definition for the mass-flow defect (or excess). Thus, the final model is:

Planar

$$\rho\epsilon = 0.036 \rho_e U_e \int_0^\infty \left| 1 - \frac{\rho U}{\rho_e U_e} \right| dy \tag{11}$$

Axisymmetric

$$\rho\epsilon = \frac{0.018 \rho_e U_e}{a} \int_0^\infty \left| 1 - \frac{\rho U}{\rho_e U_e} \right| 2r\, dr \tag{12}$$

This is at variance with the suggestions of some workers in the boundary-layer field, who have used a "kinematic displacement thickness" based on $1 - \frac{U}{U_e}$ for variable-density cases in trying to extend Clauser's basic model to such situations. It will be shown below that such a choice is clearly inappropriate for at least free-mixing problems.

TEST CASES

Calculations were run for test cases 9, 10, 11, 12, 14, 15, 16, and 17, using equation (11) or (12) as appropriate to the particular problem. The equations of motion were solved numerically by using an explicit finite-difference scheme in von Mises (x,ψ) coordinates. The actual computer routine used is a modification of that developed in reference 9. The program is written in FORTRAN II, and the total execution time on an IBM 370/155 for the compressible, two-dimensional wake case (test case 16) was 1 minute, 8 seconds, as an example. Two-dimensional shear layers (test cases 1 to 5) and jets in still air (test cases 6 to 8) were excluded since the concept of a mass-flow defect (or excess) with respect to some "free stream" is unclear for such situations.

The results for the Forstall and Shapiro jet with $U_j/U_e = 4.0$ (test case 9) are shown in figure 2. The rate of decay of the center-line velocity is not as accurately predicted here as for the same experiment with $U_j/U_e = 2.0$ shown in figure 1. The agreement is greatly improved for this case, as well as for all cases with $U_j/U_e \gg 1$, by using $L = r_{1/2}$ in equation (9), but the predictions at the lower, more useful, values of U_j/U_e are worsened.

The hydrogen-air jet of Chriss (test case 10) is considered in figures 3, 4, and 5. The predictions of center-line values of the velocity and hydrogen concentration are quite good, and the predicted profile shape is in good agreement with the data. This is strong support for the utility of the functional form of $\epsilon(y)$ as modeled in equation (12). It is also interesting to consider the use of δ_K^* rather than δ^* $\left(\text{that is, } 1 - \frac{U}{U_e} \text{ rather than } 1 - \frac{\rho U}{\rho_e U_e} \text{ in eq. (12)}\right)$ for this highly variable density problem. The results are shown as dashed curves in figures 3 and 4, where the use of δ_K^* gives much poorer agreement with the data.

The results for the air-air, compressible jet experiment of Eggers and Torrence (test case 11) are shown in figure 6. The calculations are started beyond the end of the "potential core," and the agreement with the data is good.

Results obtained from equation (12) by Eggers for his hydrogen-air jet problem (test case 12) are shown in figure 7. The decay rate of center-line quantities for this low mass-flux ratio $\left(\rho_j U_j/\rho_e U_e = 0.16\right)$ is considerably overestimated.

The low-speed wake cases of Chevray and Kovasznay (test case 14) and Chevray (test case 15) are plotted in figures 8 and 9. The adequacy of the model for such cases is clearly demonstrated by these results. Note that the same constants previously determined for a boundary layer in the planar model and for a jet in the axisymmetric model have been successfully used here for wakes.

The supersonic wake cases of Demetriades (test cases 16 and 17) are given in figures 10, 11, and 12. The planar wake case appears to be rather poorly predicted in figure 10, but it should be noted that the method of presenting the comparison, as dictated by the meeting organizers, is very sensitive to small inaccuracies in either the data or the prediction. A more conventional plot of the same results is shown in figure 11.

The agreement between prediction and data for the axisymmetric wake is good. The prediction in terms of W is given in figure 12. A conventional plot of these results is given in reference 4.

LIMITATIONS OF THE MODEL

It is worthwhile to summarize the limitations of this model as they can be discerned either from the derivation or from the results for the test cases and other experimental cases that have been considered in references 1 to 4.

First, the model makes no attempt to describe flows either in the "near wake" or in the potential core of a jet. Thus, it is always strictly necessary to start with some "initial" profile that is downstream of these regions. If the region of primary interest is long as measured in diameters, however, one can simply assume that the model applies in the potential core and accept the attendant inaccuracy in the near field.

Second, the results show that the accuracy of the predictions obtained, as compared with experimental data, deteriorates for $\left(\rho_j U_j / \rho_e U_e\right) \rightarrow \infty$ and $\left(\rho_j U_j / \rho_e U_e\right) \rightarrow 0$. The best results are obtained in the range $0.4 \leqq \left(\rho_j U_j / \rho_e U_e\right) \leqq 3.0$. Fortunately, essentially all wakes and most jets of practical interest fall in this range.

Finally, in all cases, even those where the overall prediction is good, the area of poorest agreement is in the near field downstream of the "initial" station. It is believed that this is due to the fact that the Clauser model, from which the models presented here are directly descended, was developed for a flow in dynamic "equilibrium." Clearly, the rapidly relaxing flows in the near field are not in such a state.

IMPORTANT PHENOMENA NOT COVERED BY THE TEST CASES

It is unfortunate that none of the wake cases selected as test cases were for the wake behind a bluff body such as a circular cylinder, since it has been known for some time that the proportionality constant in any eddy viscosity model must be increased above a value appropriate for jets or the wake behind a streamline body in order to obtain comparable agreement with the data. This question was examined in detail in reference 4, where it was found that the proportionality constant must be made a function of the turbulence field in order to make a rational choice of an appropriate value. In the constant-density planar case, for example, it was shown that the proportionality constant in the extended Clauser model, equation (8), should be taken as

$$K \propto \frac{\overline{(U')^2}}{|U_c - U_e|^2} \tag{13}$$

A further interesting case which is not adequately handled by conventional eddy viscosity models is the wake behind a self-propelled body where the net momentum defect is zero.

SYMBOLS

a	initial jet radius
$b_{1/2}$	half-width
C_D	drag coefficient
D	diameter
K	constant
L	characteristic length
r	radial coordinate
$r_{1/2}$	half-radius
U	axial velocity
$\overline{(U')^2}$	time average of square of axial-velocity fluctuation
$W = 1 - \frac{U_c}{U_e}$	
X	axial coordinate
y	normal coordinate
α	mass fraction of hydrogen

δ^*	displacement thickness of boundary layer
δ^*_K	kinematic displacement thickness
ϵ	eddy viscosity
θ	momentum thickness of boundary layer
ρ	density
τ	shear stress

Subscripts:

c	center line
e	free stream
j	initial jet condition
max	maximum
min	minimum
1,2,3,4,5	different eddy viscosity models

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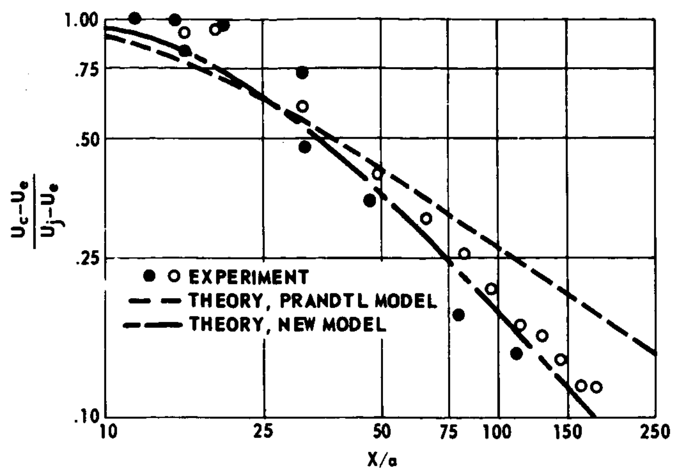

Figure 1.- Prediction and experiment for air-air jet of Forstall and Shapiro (ref. 8). $U_j/U_e = 2.0$.

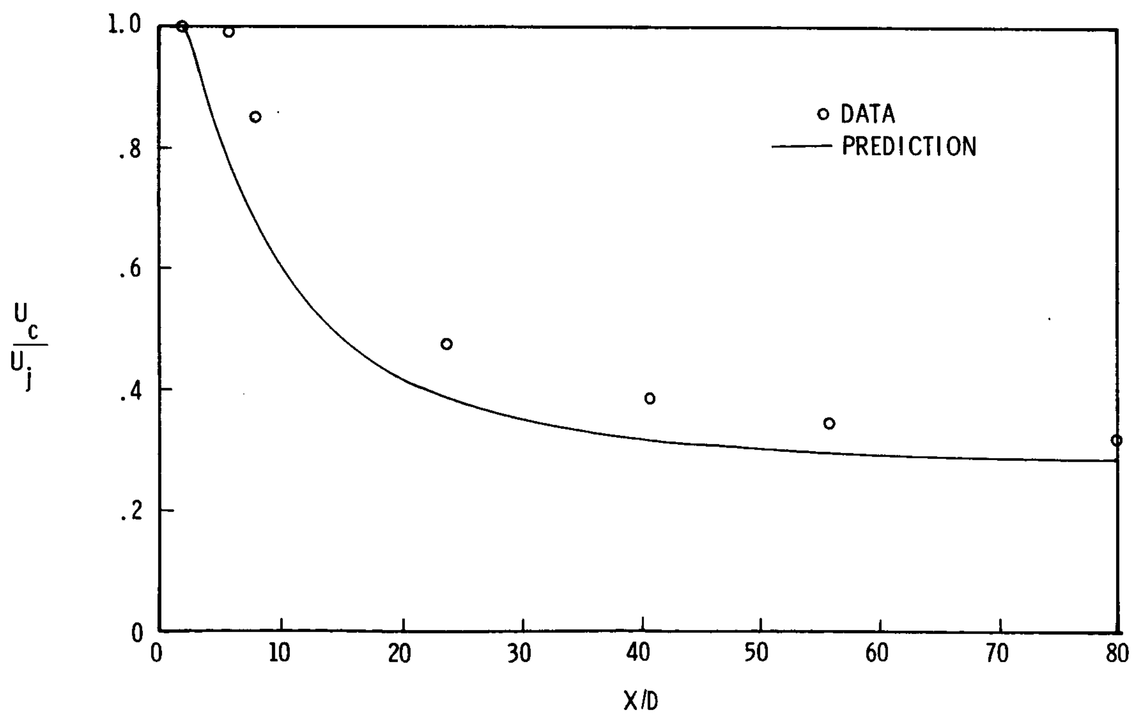

Figure 2.- Prediction and experiment for test case 9 (Forstall and Shapiro jet, $U_j/U_e = 4.0$).

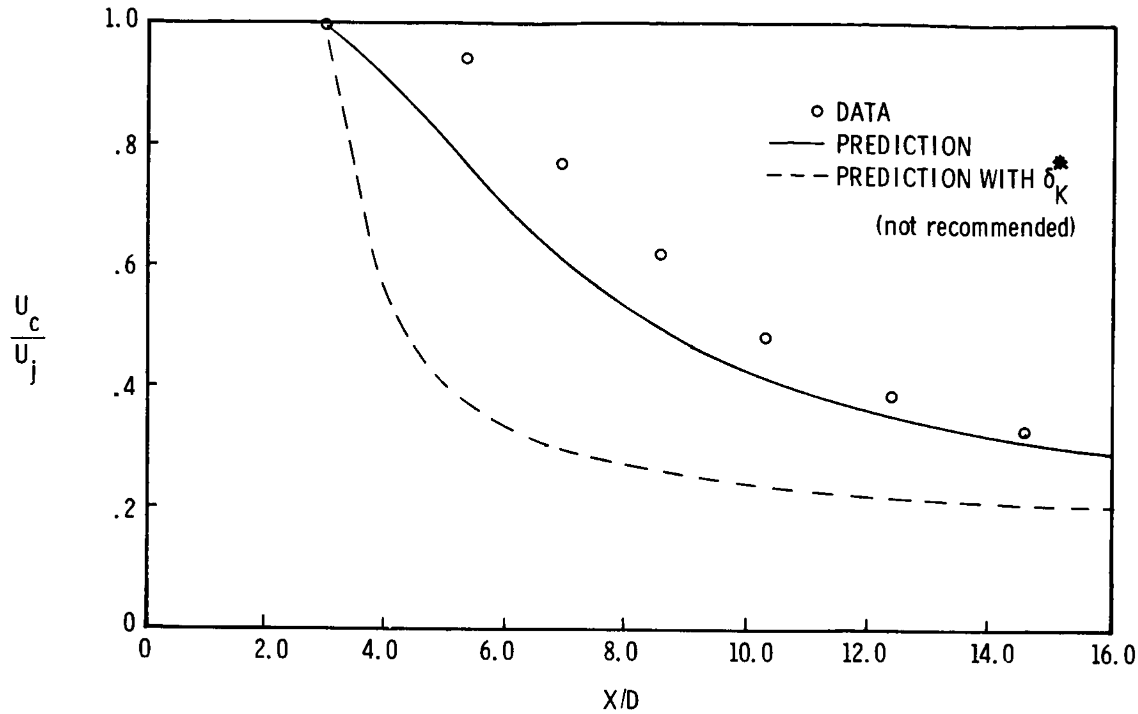

Figure 3.- Predicted and experimental center-line velocity for test case 10 (Chriss hydrogen-air jet, $\rho_j U_j / \rho_e U_e = 0.56$).

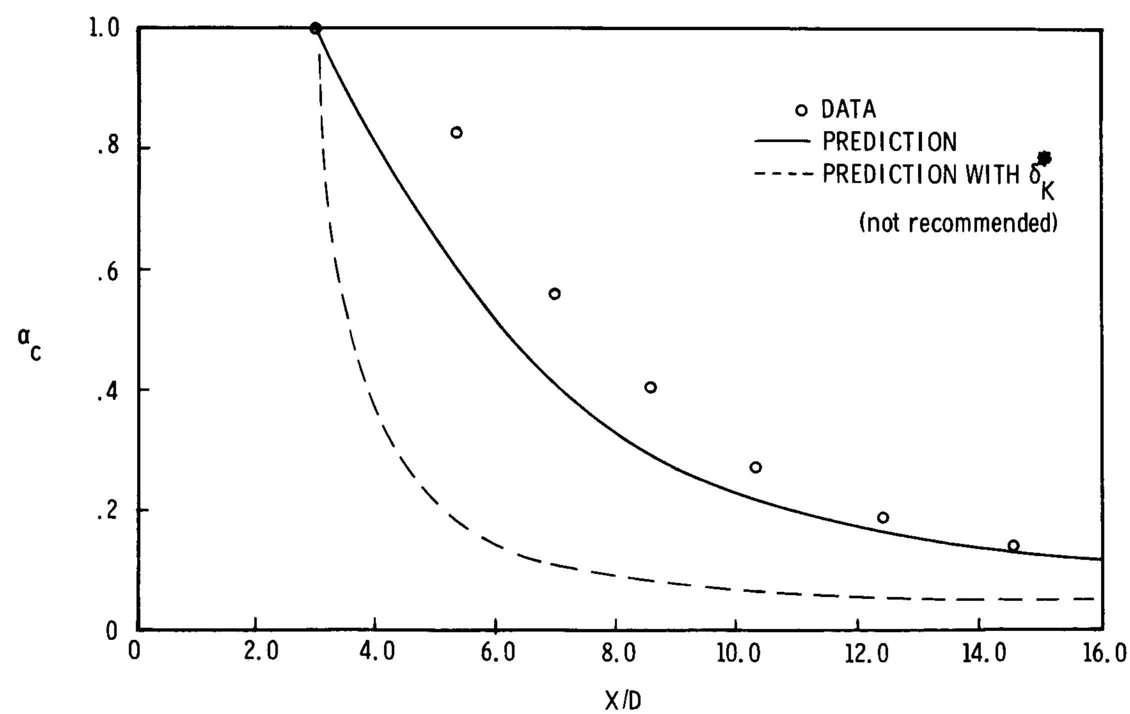

Figure 4.- Predicted and experimental center-line mass fraction of hydrogen for test case 10 (Chriss hydrogen-air jet, $\rho_j U_j / \rho_e U_e = 0.56$).

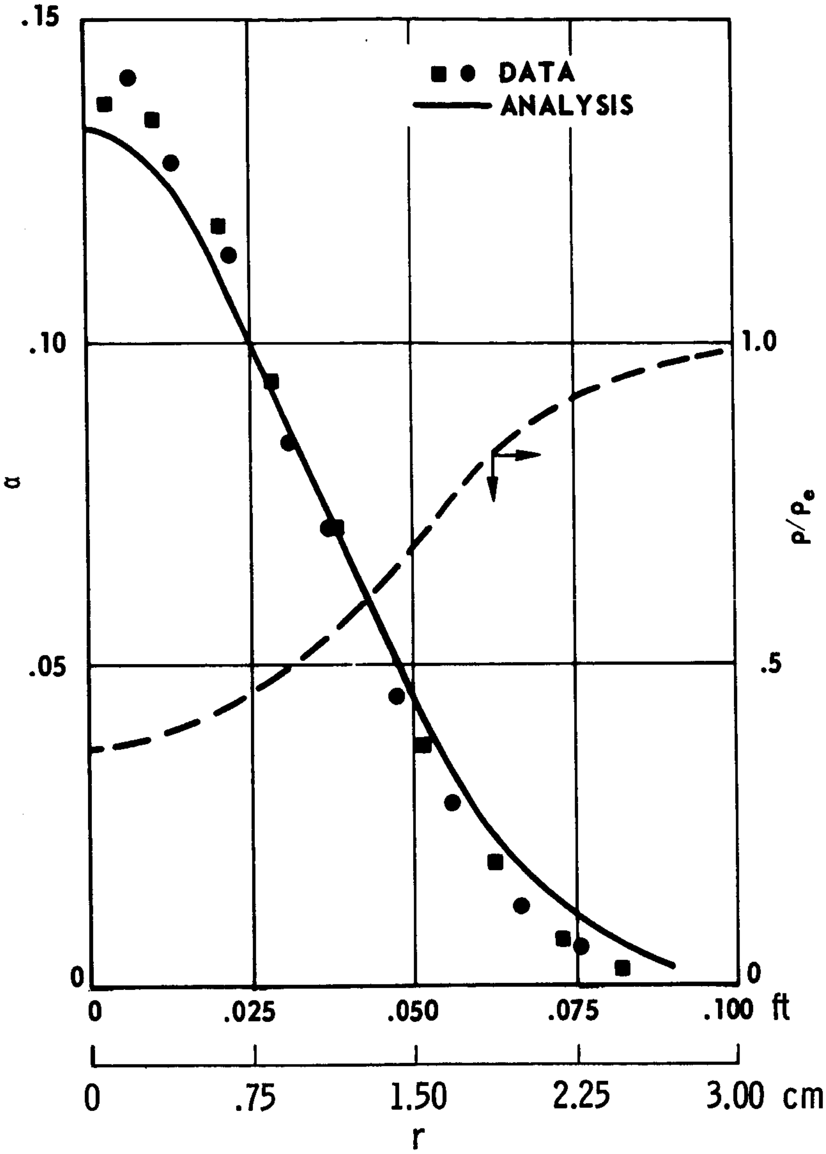

Figure 5.- Radial profiles at X/D = 14.6 for test case 10 (Chriss hydrogen-air jet, $\rho_j U_j / \rho_e U_e = 0.56$).

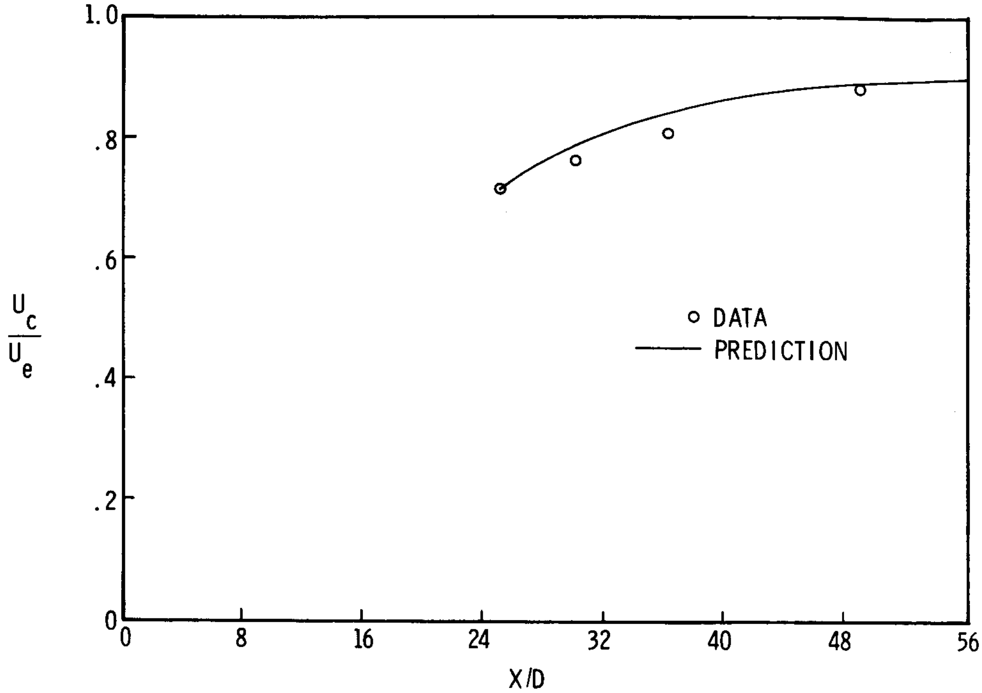

Figure 6.- Prediction and experiment for test case 11 (Eggers and Torrence jet).

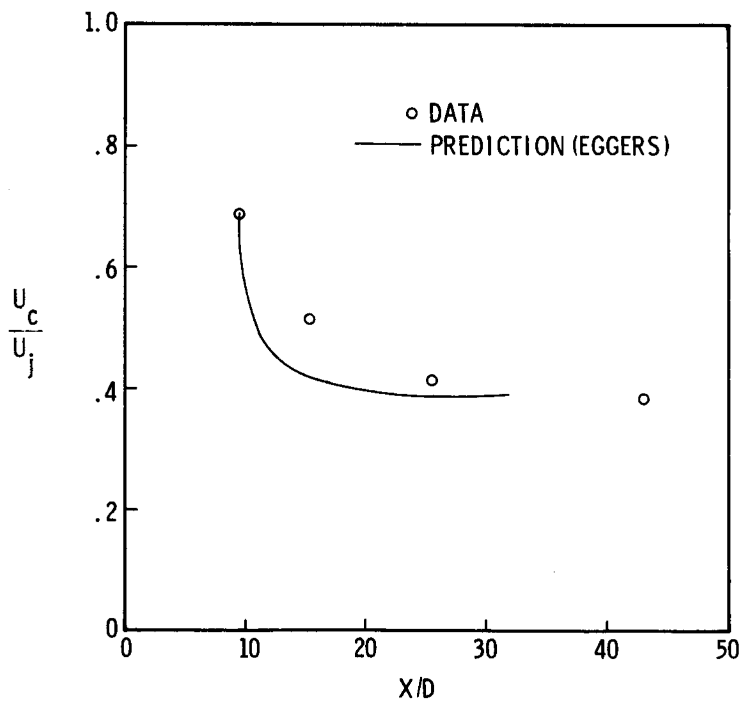

Figure 7.- Prediction and experiment for test case 12 (Eggers hydrogen-air jet, $\rho_j U_j / \rho_e U_e = 0.16$).

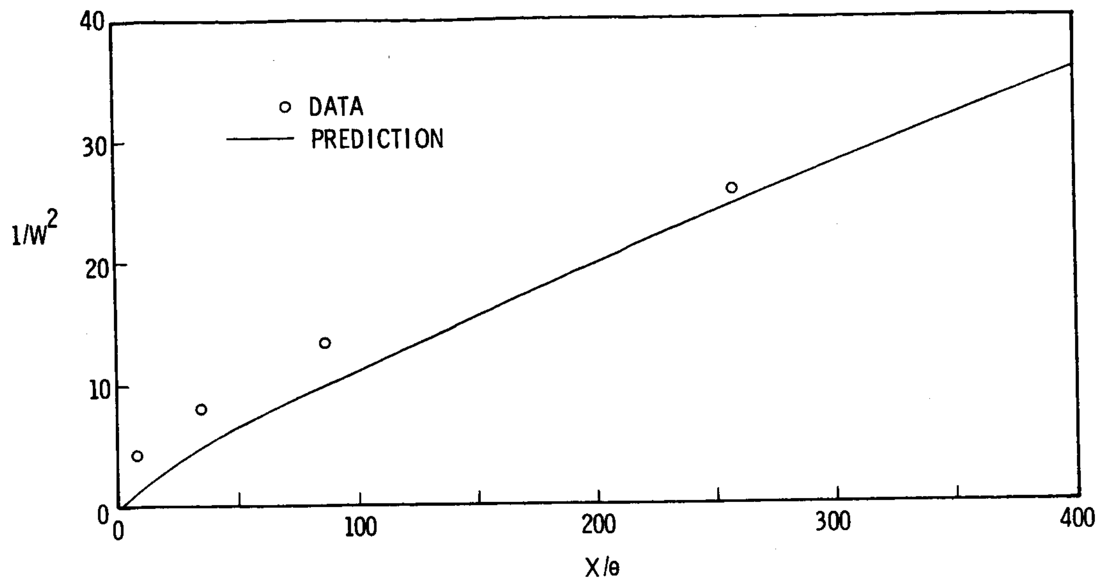

Figure 8.- Prediction and experiment for test case 14 (Chevray and Kovasznay, two-dimensional wake).

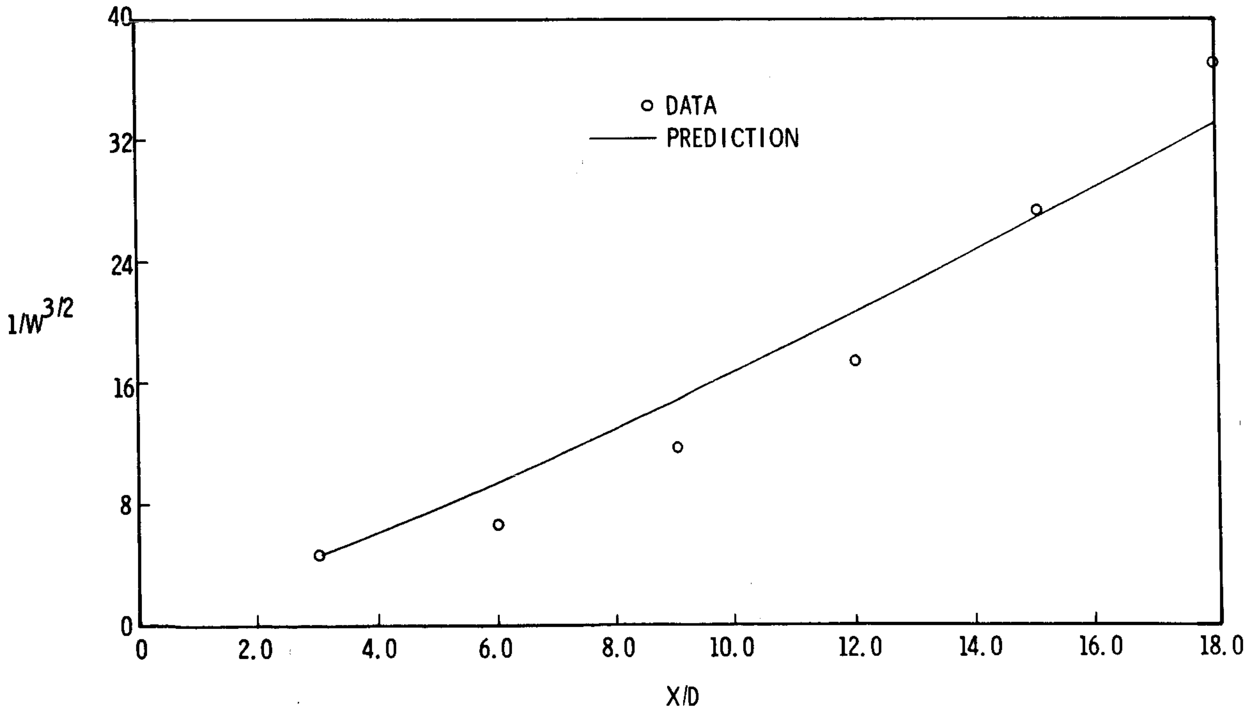

Figure 9.- Prediction and experiment for test case 15 (Chevray axisymmetric wake).

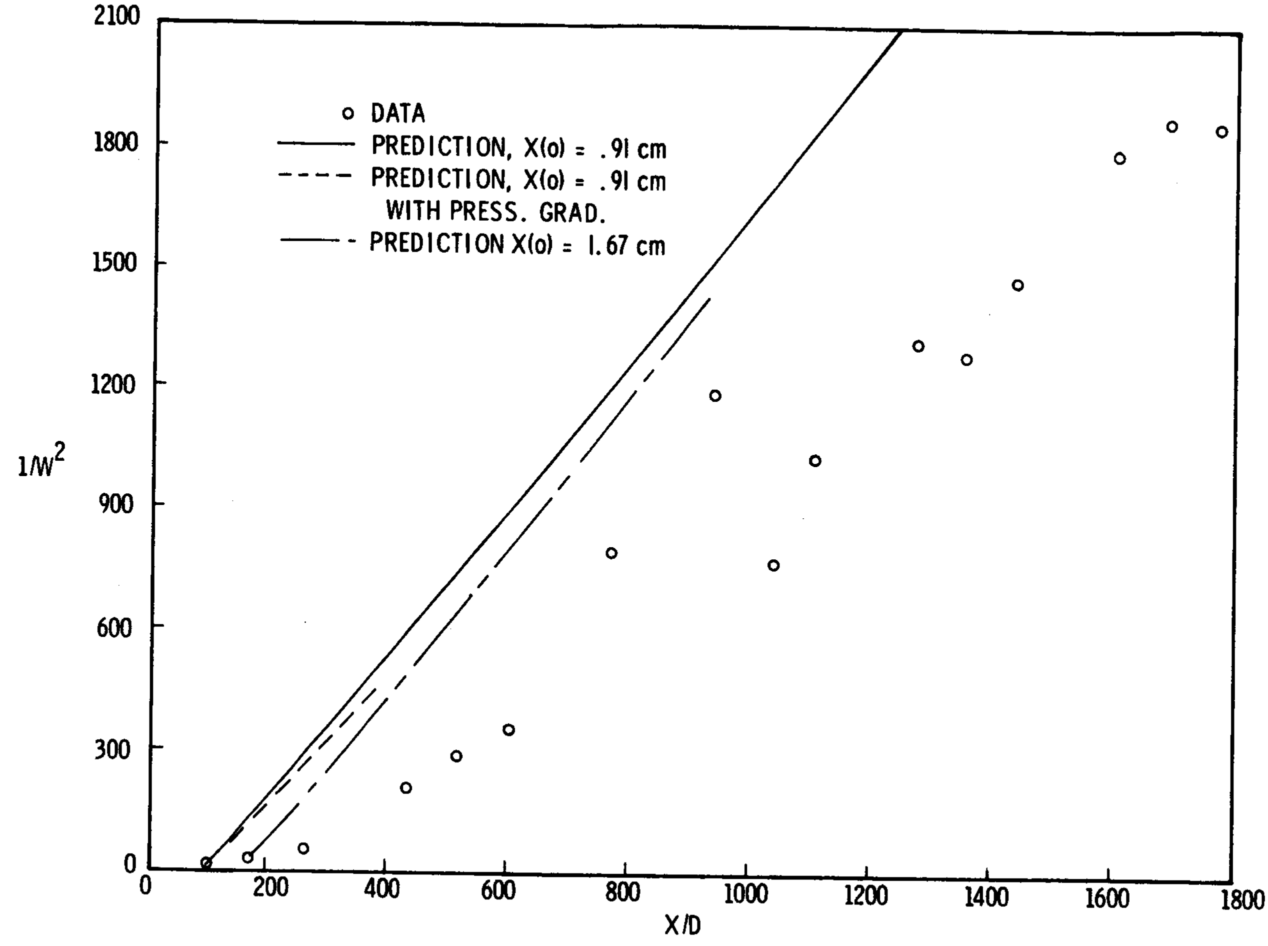

Figure 10.- Prediction and experiment for test case 16 (Demetriades two-dimensional wake).

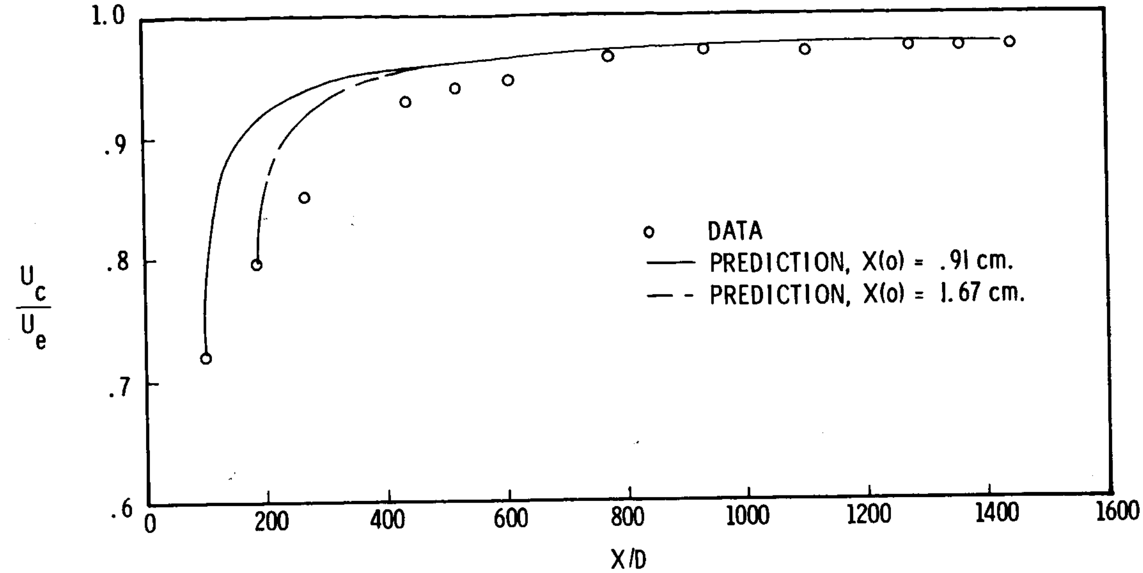

Figure 11.- Center-line velocity plot for test case 16 (Demetriades two-dimensional wake).

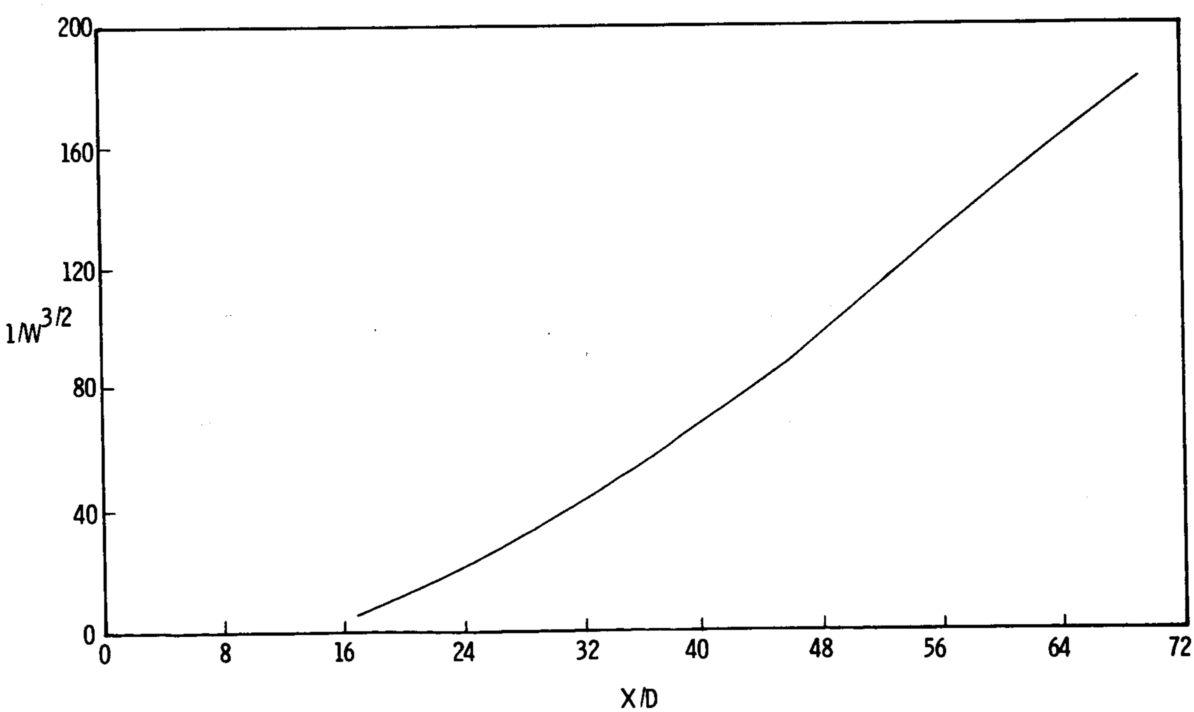

Figure 12.- Prediction for test case 17 (Demetriades axisymmetric wake).

DISCUSSION

S. W. Zelazny: Your model as applied to axisymmetric free layers uses the initial jet radius a as the characteristic length. I have shown* that using the velocity half-radius $r_{1/2}$ rather than the initial jet radius results in an eddy viscosity model that accurately models quiescent jets, which your model cannot. Comparisons between predictions and experiment using both $r_{1/2}$ and a for coflowing streams showed that the two models give about the same results. Where have you shown that using $r_{1/2}$ "was found to be unsuitable for all but jet cases with $U_j/U_e >> 1$"?

J. A. Schetz: If you experiment with the choice of this length scale which must be introduced, you can improve the comparison in different regimes of the data. It is true that an improvement is obtained by using the half-radius in the region of high mass-flux ratios which corresponds to a jet in a quiescent medium. One could adopt that choice if he were interested in problems mostly in that regime. However, considering comparisons with data in the regime of greatest practical interest, the use of the initial radius is definitely superior.

H. McDonald: I don't know if there is that much controversy over the selection of the kinematic displacement thickness as far as boundary-layer methods are going. I think that Cebeci and Mellor both used the kinematic definition and both achieved very good agreement in their predictions. We are then faced with the dilemma that in boundary layer one uses the kinematic displacement thickness, and manifestly from your results, we have to use the normal definition. It would seem to me, in light of Rudy and Bushnell's paper (paper no. 4), one should use mixing-length formulation.

J. A. Schetz: No, there is not much of a controversy. The situation is that, I think, in boundary layers you don't usually get the tremendous density variations that we have in a hydrogen jet into an air free stream. We have made calculations for boundary layers using a kinematic or the real displacement thickness and the effect is generally not very large.

J. Laufer: I noticed in your axisymmetric formulation, when you take the formulation for the limiting case of constant density, and very far downstream where you assume similarity, that you end up with an ϵ that varies as the square of the width of your shear region rather than the usual function of linear variation. Have you worried about that situation?

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*Zelazny, Stephen W.: Eddy Viscosity in Quiescent and Coflowing Axisymmetric Jets. AIAA J., vol. 9, no. 11, Nov. 1971, pp. 2292-2294.

J. A. Schetz: Not directly. I think it is well to recall that in the case of a moving external stream, the exact equations do not admit a similar solution of the classical type. Only the linearized equations accept such a solution.

M. V. Morkovin: I noticed that you avoided the first three test cases.

J. A. Schetz: Yes, the shear-layer cases. I'm talking about a mass-flow defect or excess with respect to some main stream, and if you have two streams in which $+y$ has one velocity and $-y$ has another, that does not make much sense.

B. J. Audeh: You said that this model is not to be used in the potential core, but we have a problem of where to use it. You have shown concentration profiles, and did an excellent job, but if I missed starting at the right place would my concentrations be off considerably?

J. A. Schetz: Calculations can be started at any station beyond the potential core. You could patch in a potential-core prediction, but it is insensitive to where you start as long as you are beyond the potential core. You either have to have a prediction model which you believe in for the potential core and start at the end of that, or start at some measured profile which is clearly beyond the potential core.