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Matrix Perturbation Techniques in Structural Dynamics

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PREFACE

The work described in this report was performed by the author
while he was a consultant to the Jet Propulsion Laboratory, under the
cognizance of the Engineering Mechanics Division.   The author is
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ABSTRACT

The purpose of the present report is to develop certain matrix perturbation techniques which can be used in the dynamical analysis of structures where the range of numerical values in the matrices is extreme or where the nature of the damping matrix requires that complex valued eigenvalues and eigenvectors be used. The techniques can be advantageously used in a variety of fields such as earthquake engineering, ocean engineering, aerospace engineering and other fields concerned with the dynamical analysis of large complex structures or systems of second order differential equations.

A number of simple examples are included to illustrate the techniques.

I. INTRODUCTION

In the dynamic analysis of large complex structures the numerical computations are frequently complicated by the fact that the range of numerical values of the matrices is much larger than today's computers can comfortably handle. In other problems the nature of the damping matrix may be such that the structure does not possess classical normal modes and in such a case the additional complications of having to deal with complex valued eigenvalues and eigenvectors may overtax the capacity of today's digital computers. In still other problems the engineer may be interested in the effects of small changes in parameters on the response of a large complex structure whose response is known for one set of parameters. In all these cases it may be possible to employ matrix perturbation techniques to good advantage.

In a previous report (TM 33-484) the author showed how certain matrix perturbation techniques could be profitably applied to the problem of the design of subsystems in large complex structures. The purpose of the present report is to develop certain matrix perturbation techniques which can be used to advantage in the class of problems discussed above. The techniques developed can be used to advantage in a variety of fields such as earthquake engineering, ocean engineering, aerospace engineering and other fields which are concerned with the dynamical analysis of large complex structures or systems of second-order differential equations.

II. PRELIMINARIES

In this report we shall be concerned with the dynamical analysis of large discrete structures. All linear problems in discrete structural dynamics may be expressed in the following form

$$\left. \begin{array}{l} M\ddot{\underline{x}} + D\dot{\underline{x}} + K\underline{x} = \underline{f}(t) \\[2mm] \underline{x}(0) = \underline{a} \quad , \quad \dot{\underline{x}}(0) = \underline{b} \end{array} \right\} \tag{1.0}$$

If the problem is formulated in an inertial frame, then the $N \times N$ matrix M is symmetric and positive definite, while if the system is passive the matrices D and K are at least symmetric and non-negative definite. The N vector, $\underline{x}$, may contain both displacements and rotations, while the N vector, $\underline{f}$, may contain both forces and moments

A. Reduction to Canonical Form

Since M, D and K are symmetric and M is positive definite, the transformation $y = M^{\frac{1}{2}}\underline{x}$, reduces (1.0) to the canonical form

$$I\ddot{\underline{y}} + B\dot{\underline{y}} + C\underline{y} = \underline{q}(t) \tag{1.1}$$

where

$$\left. \begin{array}{l} B = M^{-\frac{1}{2}} D M^{-\frac{1}{2}} = B^T \\[2mm] C = M^{-\frac{1}{2}} K M^{-\frac{1}{2}} = C^T \\[2mm] \underline{q} = M^{-\frac{1}{2}} \underline{f}(t) \\[2mm] \underline{y}(0) = M^{\frac{1}{2}} \underline{a} \quad , \quad \dot{y}(0) = M^{\frac{1}{2}} \underline{b} \end{array} \right\} \tag{1.2}$$

B. Classical Normal Modes

The system (1.0) is said to possess classical normal modes if and only if (iff) it can be reduced to a set of N uncoupled second order systems. A necessary and sufficient condition for the existence of classical normal

modes is that the matrices B and C commute (Caughey, 1960; Caughey and O'Kelly, 1964). Thus

$$BC = CB \quad \text{or} \quad M^{-1}DM^{-1}K = M^{-1}KM^{-1}D \tag{1.3}$$

In this case there exists ($\exists$) an orthogonal matrix T, such that ($\ni$)

$$\left. \begin{aligned} T^T BT &= \Theta - \text{a diagonal matrix} - \Theta_{ij} = \delta_{ij}\theta_i \\ T^T CT &= \Lambda - \text{a diagonal matrix} - \Lambda_{ij} = \delta_{ij}\lambda_i \\ \text{and} \qquad T &= [\underline{t}^1, \underline{t}^2 \cdots \underline{t}^N] \end{aligned} \right\} \tag{1.4}$$

where $\underline{t}^i$, $i = 1, 2 \cdots N$ are the eigenvectors of C. Equation (1.0) reduces to

$$\left. \begin{aligned} I\underline{\ddot{z}} + \Theta\underline{\dot{z}} + \Lambda\underline{z} &= \underline{r}(t) \\ \underline{z} &= T^T\underline{y} \\ \underline{z}(0) &= T^T\underline{y}(0) \quad, \quad \underline{\dot{z}}(0) = T^T\underline{\dot{y}}(0) \\ \underline{r}(t) &= T^T\underline{q}(t) \end{aligned} \right\} \tag{1.5}$$

C. Application of Classical Normal Modes

In many practical problems, the damping matrix D, or its canonical equivalent B, is unknown. From extensive testing, however, it is known that the modal damping is in the range of one to two percent of critical. In this case one frequently assumes that the damping is classical and writes

$$\theta_i = 2\lambda_i^{\frac{1}{2}} \zeta\left(\lambda_i^{\frac{1}{2}}\right) \tag{1.6}$$

Now $B = T\Theta T^T$, using (1.6) and the Cayley-Hamilton theorem we may write

$$B = \zeta(C) \tag{1.7}$$

That is, the matrix B is a matrix function of matrix C. Or expressed in terms of M and K

$$D = M^{\frac{1}{2}} \zeta \left(M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \right) M^{\frac{1}{2}} \qquad (1.8)$$

If it is assumed that (1.8) is a basic property of the materials used in the structure, then (1.6), (1.7) and (1.8) remain unchanged in form as the mass and stiffness of the elements of the structure are changed. This assumption results in considerable simplification in the problem. Using (1.6) and recalling that Θ and Λ are diagonal, (1.5) may be written

$$\left.\begin{aligned}
&\ddot{z}_i + 2\lambda_i^{\frac{1}{2}} \zeta_i \dot{z}_i + \lambda_i z_i = r_i(t) \\
&z_i(0) = z_i^0 \quad, \quad \dot{z}_i(0) = \dot{z}_i^0 \\
&i = 1, 2 \cdots N
\end{aligned}\right\} \qquad (1.9)$$

Thus

$$z_i(t) = u_i(t) z_i^0 + v_i(t) \dot{z}_i^0 + \int_0^t v_i(t-\tau) r_i(\tau)\, dt \qquad i = 1, 2 \cdots N \qquad (1.10)$$

where

$$\left.\begin{aligned}
&u_i(t) = e^{-\lambda_i^{\frac{1}{2}} \zeta_i t} \left[\cos \overline{\lambda}_i^{\frac{1}{2}} t + \frac{\lambda_i^{\frac{1}{2}} \zeta_i}{\overline{\lambda}_i^{\frac{1}{2}}} \sin \overline{\lambda}_i^{\frac{1}{2}} t \right] \\
&v_i(t) = \frac{e^{-\lambda_i^{\frac{1}{2}} \zeta_i t}}{\overline{\lambda}_i^{\frac{1}{2}}} \sin \overline{\lambda}_i^{\frac{1}{2}} t \\
&\overline{\lambda}_i = \lambda_i (1 - \zeta_i^2)
\end{aligned}\right\} \qquad (1.11)$$

Equations (1.10) can be expressed compactly in matrix notation as follows

$$\left.\begin{aligned}
&\underline{z}(t) = u(t)\underline{z}(0) + v(t)\underline{\dot{z}}(0) + \int_0^t v(t-\tau)\, \underline{r}(\tau)\, d\tau \\
\text{where} \quad & \\
&u(t)_{ij} = \delta_{ij} u_i(t) \quad ; \quad v(t)_{ij} = \delta_{ij} v_i(t)
\end{aligned}\right\} \qquad (1.12)$$

Using Equations (1. 4) and (1. 5)

$$\underline{y}(t) = T\underline{z}(t) = Tu(t)T^T\underline{y}(0) + Tv(t)T^T\underline{\dot{y}}(0) + \int_0^t Tv(t-\tau)T^T q(\tau)\,d\tau \tag{1.13}$$

Using Equation (1. 2), the solution to Equations (1. 0) can be expressed as

$$\underline{x}(t) = Qu(t)Q^T M\underline{a} + Qv(t)Q^T M\underline{b} + \int_0^t Qv(t-\tau)Q^T \underline{f}(\tau)\,d\tau \tag{1.14}$$

where Q is the congruence transformation

$$Q = M^{-\frac{1}{2}}T \tag{1.15}$$

which has the properties that

$$\left. \begin{array}{l} 1)\ \ Q^T MQ = I \\[2mm] 2)\ \ Q^T DQ = \Theta \\[2mm] 3)\ \ Q^T KQ = \Lambda \end{array} \right\} \tag{1.16}$$

D. Nonclassical Normal Modes

If in Equation (1. 1) the matrices B and C do not commute, then in general it is impossible to reduce (1. 1) to a set of uncoupled second order equations. In this case we rewrite Equation (1. 1) in the form

$$\left. \begin{array}{l} \dfrac{d\underline{w}}{dt} = A\underline{w} + \underline{p}(t) \\[4mm] w(0) = \begin{bmatrix} M^{\frac{1}{2}}\underline{a} \\[2mm] M^{\frac{1}{2}}\underline{b} \end{bmatrix} \end{array} \right\} \tag{1.17}$$

where

$$\left. \begin{array}{l} \underline{w}(t) = \begin{bmatrix} \underline{y}(t) \\[2mm] \underline{\dot{y}}(t) \end{bmatrix} \\[6mm] A = \begin{bmatrix} 0 & I \\[1mm] -C & -B \end{bmatrix} \end{array} \right\} \tag{1.18}$$

$$p(t) = \begin{bmatrix} 0 \\ q(t) \end{bmatrix} \Bigg\}$$

The solution of Equation (1.17) is

$$\underline{w}(t) = e^{At}\underline{w}(0) + \int_0^t e^{A(t-\tau)}\underline{p}(\tau)\,d\tau \tag{1.19}$$

Alternatively, there always exists a non-singular matrix S, such that

$$S^{-1}AS = J \tag{1.20}$$

If the matrix A has a full complement of ordinary eigenvectors, then J is a diagonal matrix whose elements are the eigenvalues of A. Furthermore, the matrix S has, as its columns, the eigenvectors of A. If A does not possess a full complement of ordinary eigenvectors, then J is a Jordan matrix whose diagonal elements are the eigenvalues of A. In this case the matrix S has, as its columns, the ordinary and generalized eigenvectors of A.

In the case where A has a full complement of ordinary eigenvectors Equation (1.19) can be written in a more convenient form for computation. Since $S^{-1}AS = J = \Lambda$ a diagonal matrix $A = S\Lambda S^{-1}$

$$\therefore \quad e^{At} = S \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_2 t} \end{bmatrix} S^{-1} \tag{1.21}$$

$$\therefore \quad \underline{w}(t) = S \begin{bmatrix} \ddots & & \\ & e^{\lambda_i t} & \\ & & \ddots \end{bmatrix} S^{-1}\underline{w}(0) + \int_0^t S \begin{bmatrix} \ddots & & \\ & e^{\lambda_i (t-\tau)} & \\ & & \ddots \end{bmatrix} S^{-1}\underline{p}(\tau)\,d\tau \tag{1.22}$$

III. MATRIX PERTURBATION THEORY

A. Linear Eigenvalue Problems

Case 1 [the eigenvalues of C^0 distinct]

Consider the eigenvalue problem

$$\mu_i I \underline{\psi}^i + [C^0 + \epsilon C^1 + \epsilon^2 C^2 \cdots] \underline{\psi}^i = 0 \tag{2.0}$$

where C^0, C^1, C^2, etc., are symmetric N×N matrices and C^0 is positive definite with distinct roots, and ϵ is a small parameter. Let

$$\underline{\psi}^i = T \underline{\varphi}^i \tag{2.1}$$

where T is the orthogonal matrix which diagonalizes C^0. Thus

$$[\mu_i I + \Lambda_0 + \epsilon c^1 + \epsilon^2 c + \cdots] \underline{\varphi}^i = 0 \tag{2.2}$$

where $c^j = T^T C^j T = c^{jT}$, Λ_0 a diagonal matrix with elements λ_{i0}. Since

$$C = \Lambda_0 + \sum_{j=1}^{p} \epsilon^j c^j$$

is a symmetric matrix, we know that there exists an orthogonal matrix Z which will diagonalize C. In addition, we know that the columns of Z are the eigenvectors $\underline{\varphi}^i$ of C, and both they and the eigenvalues μ_i, $i=1,2\cdots$, are analytic functions of ϵ. Knowing this we expand μ_i and $\underline{\varphi}^i$ in a power series in ϵ. Thus

$$\left. \begin{array}{l} \underline{\varphi}^i = \underline{\varphi}_0^i + \epsilon \underline{\varphi}_1^i + \epsilon^2 \underline{\varphi}_2^i \cdots \\[2mm] \mu_i = \mu_{i0} + \epsilon \mu_{i1} + \epsilon^2 \mu_{i2} \cdots \end{array} \right\} \tag{2.3}$$

Substituting (2.3) into (2.2) and equating coefficients of like powers of ϵ to zero, we have

$$\epsilon^0 \qquad [\mu_{i0}I + \Lambda_0]\underline{\varphi}_0^i = 0 \tag{2.4}$$

$$\epsilon^1 \qquad [\mu_{i0}I + \Lambda_0]\underline{\varphi}_1^i = -c^1\underline{\varphi}_0^i - \mu_{i1}\underline{\varphi}_0^i \tag{2.5}$$

$$\epsilon^2 \qquad [\mu_{i0}I + \Lambda_0]\underline{\varphi}_2^i = -c^1\underline{\varphi}_1^i - c^2\underline{\varphi}_0^i - \mu_{i1}\underline{\varphi}_1^i - \mu_{i2}\underline{\varphi}_0^i \tag{2.6}$$

etc.

From (2.4)

$$\left. \begin{aligned} &\mu_{i0} = -\lambda_{i0} \quad i = 1, 2 \ldots N \\[1em] &\underline{\varphi}_0^i = \underline{e}_i \quad - \text{ a column vector whose elements are} \\ &\qquad\qquad \text{all zero except the } i^{th} \text{ element} \\[1em] &\left(\underline{\varphi}_0^i, \underline{\varphi}_0^j\right) = \delta_{ij} \\[1em] &\left(\underline{\varphi}_0^i, \Lambda\underline{\varphi}_0^j\right) = -\lambda_{i0}\delta_{ij} \end{aligned} \right\} \tag{2.7}$$

The $\underline{\varphi}_0^i$'s form a complete orthonormal set. From (2.5)

$$[\mu_{i0}I + \Lambda_0]\underline{\varphi}_1^i = -c^1\underline{\varphi}_0^i - \mu_{i1}\underline{\varphi}_0^i \tag{2.8}$$

Since the $\underline{\varphi}_0^i$'s form a complete orthornormal set they span the N space, thus any vector in that N space may be represented as a linear combination of the $\underline{\varphi}_0^i$'s. Let

$$\underline{\varphi}_1^i = \sum_{j=1}^{N} \alpha_{ji}\underline{\varphi}_0^j \tag{2.9}$$

Substituting (2.9) into (2.8) we have

$$\sum_{j=1}^{N} \alpha_{ji}[\mu_{i0}I + \Lambda_0]\underline{\varphi}_0^j = -c^1\underline{\varphi}_0^i - \mu_{i1}\underline{\varphi}_0^i \tag{2.10}$$

If the inner product of (2.10) is taken with $\underline{\varphi}_0^k$ and use is made of (2.7), then

$$\alpha_{ki}[\lambda_{i0} - \lambda_{k0}] = -\left(\underline{\varphi}_0^k, C^1 \underline{\varphi}_0^i\right) - \mu_{i1} \delta_{ik} \qquad (2.11)$$

Since $\lambda_{i0} \neq \lambda_{k0}$, $i \neq k$, $\underline{\varphi}_0^j = \underline{e}_j$, then

$$\alpha_{ki} = -\frac{C_{ik}^1}{(\lambda_{i0} - \lambda_{k0})} \qquad i \neq k \qquad (2.12)$$

We note in passing that since C^1 is a symmetric matrix $C_{ij}^1 = C_{ji}^1$. Hence

$$\alpha_{ik} = -\frac{C_{ki}^1}{\lambda_{k0} - \lambda_{i0}} = -\alpha_{ki} \qquad (2.13)$$

If $i=k$, then we have

$$\mu_{i1} = -\left(\underline{\varphi}_0^i, C^1 \underline{\varphi}_0^i\right) = -C_{ii}^1 \qquad (2.14)$$

To determine α_{ii} we make use of the normalization properties of the $\underline{\varphi}^i$'s

$$(\underline{\varphi}^i, \underline{\varphi}^i) = 1 = \left(\underline{\varphi}_0^i, \underline{\varphi}_0^i\right) + 2\epsilon\left(\underline{\varphi}_0^i, \underline{\varphi}_1^i\right) + O(\epsilon^2) \qquad (2.15)$$

Equating coefficients of like powers of ϵ, we have

$$\left(\underline{\varphi}_0^i, \underline{\varphi}_1^i\right) = \alpha_{ii} = 0 \qquad (2.16)$$

Thus

$$\mu_i = \lambda_{i0} - \epsilon C_{ii}^1 + O(\epsilon^2) \qquad (2.17)$$

$$\underline{\varphi}^i = \underline{\varphi}_0^i - \epsilon \sum_{j=1}^{N} \frac{C_{ij}^1 \underline{\varphi}_0^j}{\lambda_{i0} - \lambda_{j0}} + O(\epsilon^2) \qquad (2.18)$$

Using (2.18) we may define a matrix Z, where

$$Z = I + \epsilon S \qquad (2.19)$$

where

$$S = -S^T = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \cdots \alpha_{1N} \\ \alpha_{21} & 0 & \alpha_{23} \cdots \alpha_{2N} \\ \vdots & & \\ \alpha_{N1} & \alpha_{N2} \cdots \cdots \cdots 0 \end{bmatrix} \tag{2.20}$$

The matrix Z has the following properties

i) $\quad Z^T Z = I + O(\epsilon^2)$ $\qquad$ (2.21)

ii) $\quad Z^T \left[\Lambda_0 + \epsilon C + \epsilon^2 C^2 + \cdots \right] Z = \Lambda_0 + \epsilon \Lambda_1 + O(\epsilon^2)$ $\qquad$ (2.22)

where

$$\Lambda_0 = \begin{bmatrix} \lambda_{10} & & & \\ & \lambda_{20} & & \\ & & \ddots & \\ & & & \lambda_{N0} \end{bmatrix} \tag{2.23}$$

$$\Lambda_1 = \begin{bmatrix} \mu_{11} & & & \\ & \mu_{21} & & \\ & & \ddots & \\ & & & \mu_{N1} \end{bmatrix} \tag{2.24}$$

where λ_{i0} are the eigenvalues of C^0 and μ_{i1}, $i=1,2,\cdots N$ are given by (2.14).

Case 2 [some eigenvalues of C^0 repeated]

Suppose that the first r eigenvalues of C^0 are repeated. We know that the corresponding eigenvectors are not unique and hence the orthogonal transformation which diagonalizes C^0 is not unique. Let

$$T^* = T \left[\begin{array}{c|c} T_r & 0 \\ \hline 0 & I \end{array} \right] \tag{2.25}$$

where T is any orthogonal matrix which diagonalizes C^0 and T_r is any $r \times r$ orthogonal matrix. Then

$$T^{*T} T^* = I \tag{2.26}$$

and

$$T^{*T}C^0T^* = \left[\begin{array}{c|c} \lambda_{10}I_r & \\ \hline & \Lambda_0^{N-r} \end{array}\right] = \Lambda_0 \qquad (2.27)$$

where λ_{10} is repeated r times.

Properties (2.26) and (2.27) give us the necessary freedom to solve the perturbations in the case of repeated roots. The first step is to apply the transformation

$$\underline{y} = T\underline{z} \qquad (2.28)$$

where T is any orthogonal matrix which diagonalizes C^0. Thus

$$\left[\mu_i I + \Lambda_0 + \epsilon C^1 + \epsilon^2 C^2\right]\underline{\varphi}^i = 0 \qquad (2.29)$$

Direct application of the matrix perturbation technique, applied to Case 1, fails unless $C^1_{ij} = 0$, $i \neq j$, $i, j \in (1, r)$. In general these conditions are not satisfied, however we can always construct a new transformation T^* such that this condition does hold.

Let $B^r = [C^1_{ij}]$ $i, j \in (1, r)$, and let T_r be the $r \times r$ matrix whose columns are the eigenvectors of

$$[\mu I + B^r]\underline{t}_r = 0 \qquad (2.30)$$

The new orthogonal transformation (2.25) using this particular matrix T_r has the property that

$$T^{*T}C^1 T^* = \widetilde{C}^1 \qquad (2.31)$$

where $\widetilde{C}^1$ is a symmetric matrix with the property that

$$\widetilde{C}^1_{ij} = \widetilde{C}^1_{ii}\delta_{ij} \qquad i, j \in (1, r) \qquad (2.32)$$

Thus if T^* is used in place of T, $T^{*T}C^1T^* = \widetilde{C}^1$ has the necessary properties so that the matrix perturbation technique can be used.

If in Equation (2.1) T^* is used in place of T, Equation (2.2) becomes

$$\left[\mu_i I + \Lambda_0 + \epsilon \widetilde{C}^1 + \epsilon^2 \widetilde{C}^2 \cdots\right] \underline{\varphi}^i = 0 \qquad (2.33)$$

where $\widetilde{C}^j = T^{*T} C^j T^*$ $j = 1, 2 \cdots$. As before, let

$$\left.\begin{array}{l} \underline{\varphi}^i = \underline{\varphi}_0^i + \epsilon \underline{\varphi}_1^i + \epsilon^2 \underline{\varphi}_2^i + \cdots \\[2mm] \mu_i = \mu_{i0} + \epsilon \mu_{i1} + \epsilon^2 \mu_{i2} + \cdots \end{array}\right\} \qquad (2.34)$$

Substituting (2.34) into (2.33) and equating coefficients of like powers of ϵ to zero, we have

$$\epsilon^0 \qquad [\mu_{i0} I + \Lambda_0] \underline{\varphi}_0^i = 0 \qquad (2.35)$$

$$\epsilon^1 \qquad [\mu_{i0} I + \Lambda_0] \underline{\varphi}_1^i = -\widetilde{C}^1 \underline{\varphi}_0^i - \mu_{i1} \underline{\varphi}_0^i \qquad (2.36)$$

$$\epsilon^2 \qquad [\mu_{i0} I + \Lambda_0] \underline{\varphi}_2^i = -\widetilde{C}^1 \underline{\varphi}_1^i - \widetilde{C}^2 \underline{\varphi}_0^i - \mu_{i1} \underline{\varphi}_1^i - \mu_{i2} \underline{\varphi}_0^i \qquad (2.37)$$

etc.

From (2.35), we have

$$\left.\begin{array}{l} \mu_{i0} = -\lambda_{i0} \qquad i = 1, 2 \cdots N \\[2mm] \underline{\varphi}_0^i = \underline{e}_i \\[2mm] \left(\underline{\varphi}_0^i, \underline{\varphi}_0^j\right) = \delta_{ij} \\[2mm] \left(\underline{\varphi}_0^i, \Lambda_0 \underline{\varphi}_0^j\right) = -\lambda_{i0} \delta_{ij} \end{array}\right\} \qquad (2.38)$$

The $\underline{\varphi}_0^i$'s form a complete orthonormal set. From (2.36)

$$[\mu_{i0} I + \Lambda_0] \underline{\varphi}_1^i = -\widetilde{C}^1 \underline{\varphi}_0^i - \mu_{i0} \underline{\varphi}_0^i \qquad (2.39)$$

As before, let $\underline{\varphi}_1^i$ be represented as a linear combination of the $\underline{\varphi}_0^j$'s. Thus

$$\underline{\varphi}_1^i = \sum_{j=1}^{N} \alpha_{ji}\underline{\varphi}_0^j \tag{2.40}$$

Substituting (2.40) into (2.39) and taking the inner product with $\underline{\varphi}_0^k$, we have

$$\alpha_{ki}\left(\lambda_{i0}-\lambda_{k0}\right) = -\left(\underline{\varphi}_0^k, \widetilde{c}^1\underline{\varphi}_0^i\right) - \mu_{i0}\underline{\varphi}_0^i \tag{2.41}$$

Thus $i \neq k$ and either i or $k \notin (1,r)$, we have as before

$$\alpha_{ki} = -\frac{\widetilde{c}^1_{ik}}{\lambda_{i0}-\lambda_{k0}} \tag{2.42}$$

If $i = k$, then

$$\mu_{i1} = -\left(\underline{\varphi}_0^i, \widetilde{c}^1\underline{\varphi}_0^i\right) = -\widetilde{c}_{ii} \tag{2.43}$$

If $i \neq k$, but i and $k \in (1,r)$, we cannot determine α_{ik}. As before, normalization gives us $\alpha_{ii} \equiv 0, \forall i$.

To determine α_{ik}, $i \neq k$, i and $k \in (1,r)$ we proceed to the $O(\epsilon^2)$. From (2.37), we have

$$[\mu_{i0}I + \Lambda_0]\underline{\varphi}_2^i = -\widetilde{c}^1\underline{\varphi}_1^i - \widetilde{c}^2\underline{\varphi}_0^i - \mu_{i1}\underline{\varphi}_1^i - \mu_{i2}\varphi_0^i \tag{2.44}$$

Since the $\underline{\varphi}_0^j$'s span the N space, let $\underline{\varphi}_2^i$ be represented by a linear combination of the φ_0^j's. Thus

$$\underline{\varphi}_2^i = \sum_{j=1}^{N} \beta_{ji}\underline{\varphi}_0^j \tag{2.45}$$

Substituting (2.45) into (2.44) and taking the inner product with $\underline{\varphi}_0^k$, we have

$$\left[\lambda_{i0}-\lambda_{k0}\right]\beta_{ki} = -\left(\underline{\varphi}_0^k, \widetilde{c}^1\underline{\varphi}_1^i\right) - \left(\varphi_0^k, \widetilde{c}^2\underline{\varphi}_0^i\right) - \mu_{i1}\alpha_{ki} - \mu_{i2}\delta_{ik} \tag{2.46}$$

Making use of (2.40), we have

$$[\lambda_{i0} - \lambda_{k0}]\beta_{ki} = -\sum_{j=1}^{N} \alpha_{ji}\widetilde{C}_{jk}^{1} - \widetilde{C}_{ik}^{2} - \mu_{i1}\alpha_{ki} - \mu_{i2}\delta_{ik} \qquad (2.47)$$

if $i \ne k$, but i and $k \in (1, r)$. Then, since $\widetilde{C}_{jk}^{1} = \widetilde{C}_{jj}^{1}\delta_{jk}$ for j and $k \in (1, r)$, (2.47) gives us

$$0 = (\mu_{k1} - \mu_{i1})\alpha_{ki} - \sum_{j=1+r}^{N} \alpha_{ji}\widetilde{C}_{jk}^{1} - \widetilde{C}_{ik}^{2} \qquad (2.48)$$

Therefore

$$\left.\begin{array}{l} \alpha_{ki} = -\dfrac{\displaystyle\sum_{j=1+r}^{N} \alpha_{ji}\widetilde{C}_{jk}^{1} + \widetilde{C}_{ik}^{2}}{\mu_{i1} - \mu_{k1}} \\[4mm] \qquad = -\alpha_{ik} \end{array}\right\} \qquad (2.49)$$

Hence

$$\mu_i = \lambda_{i0} - \epsilon\widetilde{C}_{ii}^{1} + O(\epsilon^2) \qquad (2.50)$$

$$\underline{\varphi}^{i} = \underline{\varphi}_{0}^{i} + \epsilon\sum_{j=1}^{N} \alpha_{ji}\underline{\varphi}_{0}^{j} + O(\epsilon^2) \qquad (2.51)$$

where the α_{ji} are given by (2.42) and (2.49).

Using (2.51) we may define a matrix Z, where

$$Z = I + \epsilon S \qquad (2.52)$$

where

$$S = -S^{T} = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \cdots \alpha_{1N} \\ \alpha_{21} & 0 & \alpha_{23} \cdots \alpha_{2N} \\ \cdots \cdots & & \\ \alpha_{N1} & \alpha_{N2} & \cdots\cdots\cdots 0 \end{bmatrix} \qquad (2.53)$$

The matrix Z has the following properties, as before

$$\text{i)} \quad Z^T Z = I + O(\epsilon^2) \tag{2.54}$$

$$\text{ii)} \quad Z^T \left[\Lambda_0 + \epsilon \tilde{c}^1 + \epsilon^2 \tilde{c}^2 + \cdots \right] Z = \Lambda_0 + \epsilon \Lambda_1 + O(\epsilon^2) \tag{2.55}$$

where

$$\Lambda_0 = \begin{bmatrix} \lambda_{10} & & & \\ & \lambda_{20} & & \\ & & \ddots & \\ & & & \lambda_{N0} \end{bmatrix} \tag{2.56}$$

$$\lambda_{i0} = \lambda_{10} \qquad i = 1, 2, \cdots r$$

$$\Lambda_1 = \begin{bmatrix} \mu_{11} & & & \\ & \mu_{21} & & \\ & & \ddots & \\ & & & \mu_{N1} \end{bmatrix} \tag{2.57}$$

where λ_{i0}, $i = 1, 2 \cdots N$ are the eigenvalues of C^0 and the μ_{ii} are given by (2.43).

Case 3 [some eigenvalues of C^0 close to one another]

Suppose that the first r roots of Λ_0 are close such that

$$|\lambda_{i0} - \lambda_{j0}| \le k\epsilon \qquad (i,j) \in (1,r) \tag{2.58}$$

In this case we observe that $\epsilon \underline{\varphi}_1^i$ is no longer guaranteed to be of $O(\epsilon)$. This difficulty is easily remedied by writing our eigenequation in the form

$$\left[\mu_i I + \tilde{\Lambda}_0 + \epsilon \tilde{c}^1 + \epsilon^2 c^2 \cdots \right] \underline{\varphi}^i = 0 \tag{2.59}$$

where

$$\tilde{\Lambda}_0 = \left[\begin{array}{c|c} \lambda_1 I^r & \\ \hline & \Lambda_0^{N-r} \end{array} \right] \Bigg\} \tag{2.60}$$

$$\tilde{C}^1 = C^1 + \frac{1}{\epsilon}\left[\begin{array}{ccc|c} 0 & & & \\ & \lambda_2 - \lambda_1 & & 0 \\ & & \ddots & \\ & & \lambda_r - \lambda_1 & \\ \hline & 0 & & 0 \end{array}\right] \Bigg\}$$

<div align="right">(2.60)
cont.</div>

Thus the problem has been reduced to one in which the first r eigenvalues of C^0 are repeated. In this case the methods used in Case 2 are applicable.

B. "Nonlinear" Eigenvalue Problems

In the case of nonclassical normal modes we have to deal with an eigenvalue problem in which the eigenvalue occurs quadratically, viz.

$$\left[\gamma_i^2 I + \epsilon\gamma_i\left(B^1 + \epsilon B^2 + \cdots\right) + C\right]\underline{\psi}^i = 0 \tag{2.61}$$

where B^i, $i=1,2\cdots$ and C are symmetric matrices which do not commute. Let

$$\underline{\psi}^i = T\underline{\varphi}^i \tag{2.62}$$

where T is the orthogonal matrix which diagonalizes C. Substituting (2.62) in (2.61)

$$\left[\gamma_i^2 I + \gamma_i\left(\epsilon\mathcal{B}^1 + \epsilon^2\mathcal{B}^2\cdots\right) + \Lambda_0\right]\underline{\varphi}^i = 0 \tag{2.63}$$

where

$$\Lambda_0 = T^T C T = \left[\begin{array}{ccccc} w_1^2 & & & & \\ & w_2^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & w_n^2 \end{array}\right]\Bigg\}$$

$$\mathcal{B}^1 = T^T B^i T \quad i=1,2\cdots$$

<div align="right">(2.64)</div>

Equation (2.63) may also be written in the form

$$[\gamma_i I + A] \underline{\Phi}^i = 0 \tag{2.65}$$

where

$$
A = \begin{bmatrix} 0 & -I \\ \hline \Lambda_0 & \in \mathbb{B}^1 + \epsilon^2 \mathbb{B}^2 \end{bmatrix}
$$

and

$$
\underline{\Phi}^i = \begin{bmatrix} \underline{\varphi}^i \\ \gamma_i \underline{\varphi}^i \end{bmatrix}
\tag{2.66}
$$

Let

$$
\underline{\varphi}^i = \underline{\varphi}_0^i + \in \underline{\varphi}_1^i + \epsilon^2 \underline{\varphi}_2^i \cdots
$$

$$
\gamma_i = \gamma_{i0} + \in \gamma_{i0} + \epsilon^2 \gamma_{i2} \cdots
\tag{2.67}
$$

Substituting (2.67) into (2.63) and equating coefficients of like powers of $\in$ to zero, we have

$$\epsilon^0 \quad \left[\gamma_{i0}^2 I + \Lambda_0\right]\underline{\varphi}_0^i = 0 \tag{2.68}$$

$$\epsilon^1 \quad \left[\gamma_{i0}^2 I + \Lambda_0\right]\underline{\varphi}_1^i = -\left[2\gamma_{i0}\gamma_{i1}\underline{\varphi}_0^i + \gamma_{i0}\mathbb{B}^1\underline{\varphi}_0^i\right] \tag{2.69}$$

From (2.68) we have

$$
\gamma_{i0}^+ = jw_i
$$

$$
\gamma_{i0} = -jw_i = \overline{\gamma_{i0}^+}
$$

$$
\underline{\varphi}_0^i = \underline{e}_i \quad \text{(real)}
$$

$$
\left(\underline{\varphi}_0^i, \underline{\varphi}_0^j\right) = \delta_{ij}
\tag{2.70}
$$

$$\left(\underline{\varphi}_0^i, \Lambda_0 \underline{\varphi}_0^j\right) = w_i^2 \delta_{ij} \Big\}$$

(2.70)
cont.

The $\underline{\varphi}_0^i$'s form a complete orthonormal set.

For each $\underline{\varphi}_0^i$ there are two Φ^i's, one corresponding to γ_{i0}^+, the other to γ_{i0}^-. Thus

$$\underline{\Phi}_0^{i+} = \begin{bmatrix} \underline{\varphi}_0^i \\ \gamma_{i0}^+ \, \underline{\varphi}_0^i \end{bmatrix} \quad , \quad \underline{\Phi}_0^{i-} = \begin{bmatrix} \underline{\varphi}_0^i \\ \gamma_{i0}^- \, \underline{\varphi}_0^i \end{bmatrix} \tag{2.71}$$

We see that

$$\underline{\Phi}_0^{i-} = \bar{\underline{\Phi}}_0^{i+} \tag{2.72}$$

Since the $\underline{\varphi}_0^i$'s form an orthonormal set, let us express $\underline{\varphi}_1^i$ as a linear combination of the $\underline{\varphi}_0^j$'s, $j=1,2\cdots N$. Thus

$$\underline{\varphi}_1^i = \sum_{j=1}^{N} \alpha_{ji} \underline{\varphi}_0^j \tag{2.73}$$

Substituting (2.73) into (2.69) and taking the inner product with $\underline{\varphi}_0^k$ we have

$$\left[\gamma_{i0}^{\pm 2} - \gamma_{k0}^{\pm 2}\right] \alpha_{ki}^\pm = -2\gamma_{i0}^\pm \gamma_{i1}^\pm \delta_{ik} + \gamma_{i0}^\pm \beta_{ik}^1 \tag{2.74}$$

We shall consider only the case where the γ_{i0}'s are distinct; the case of repeated and close roots can be handled in the same way as in the linear eigenvalue problem. Thus if $i \neq k$

$$\alpha_{ki}^\pm = -\frac{\gamma_{i0}^\pm \beta_{ik}^1}{\gamma_{i0}^{\pm 2} - \gamma_{k0}^{\pm 2}} \tag{2.75a}$$

$$\therefore \quad \alpha_{k1}^\pm = \frac{\pm j w_i \beta_{ik}^1}{w_i^2 - w_k^2} \tag{2.75b}$$

If $i \equiv k$ then

$$\gamma_{i1}^{\pm} = -\frac{1}{2}\beta_{ii}^{1} \qquad (2.76)$$

The α_{ii}'s can be shown to be zero by using the normalization condition on the $\underline{\varphi}^{i}$'s. Thus

$$\gamma_i^{\pm} = \pm jw_i - \frac{\epsilon}{2}\beta_{ii}^1 + O(\epsilon^2) \qquad (2.77)$$

$$\underline{\varphi}^{i\pm} = \underline{\varphi}_0^i \pm \epsilon \sum_{k=1}^{N} j\frac{w_i \beta_{ik}^1}{w_i^2 - w_k^2}\underline{\varphi}_0^k + O(\epsilon^2) \qquad (2.78)$$

Using (2.78) we can construct the two $N \times N$ matrices

$$Z_1 = \begin{bmatrix} 1 & \epsilon\alpha_{12}^+ & \epsilon\alpha_{13}^+ \cdots \epsilon\alpha_{1N}^+ \\ \epsilon\alpha_{21}^+ & 1 & \epsilon\alpha_{23}^+ & \epsilon\alpha_{2N}^+ \\ \cdots\cdots \\ \epsilon\alpha_{N1}^+ & \epsilon\alpha_{N2}^+ & \cdots\cdots & 1 \end{bmatrix} \qquad (2.79)$$

and

$$Z_2 = \overline{Z}_1 \qquad (2.80)$$

Let us define the matrices Ω_0, S, Λ_1,

$$\Omega_0 = \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_N \end{bmatrix} \qquad (2.81)$$

$$S = \begin{bmatrix} 0 & \dfrac{\beta_{12}^1}{w_1^2 - w_2^2} & \cdots\cdots & \dfrac{\beta_{1N}^1}{w_1^2 - w_N^2} \\[3ex] \dfrac{\beta_{21}^1}{w_2^2 - w_1^2} & 0 & \cdots\cdots & \dfrac{\beta_{2N}^1}{w_2^2 - w_N^2} \\[3ex] \dfrac{\beta_{N1}^1}{w_N^2 - w_1^2} & \cdots\cdots\cdots\cdots & 0 \end{bmatrix} \qquad (2.82)$$

The matrix S is skew symmetric, I. e.,

$$S^T = -S \qquad (2.83)$$

$$\Lambda_1 = \begin{bmatrix} \frac{1}{2} \beta_{11}^1 & & & \\ & \frac{1}{2} \beta_{22}^1 & & \\ & & \ddots & \\ & & & \frac{1}{2} \beta_{NN}^1 \end{bmatrix} \qquad (2.84)$$

Then the matrix Z_1 can be expressed in the form

$$Z_1 = [I + j \in S \Omega_0] \qquad (2.85)$$

The nonsingular $2N \times 2N$ matrix, T, whose columns are the eigenvectors of matrix A is given by

$$T = \left[\begin{array}{c|c} Z_1 & \overline{Z}_1 \\ \hline Z_1 \Omega & \overline{Z}_1 \overline{\Omega} \end{array} \right] \qquad (2.86)$$

where

$$\Lambda = j \Omega_0 - \in \Lambda_1 \qquad (2.87)$$

Matrix T can be expressed in terms of S, Ω_0 and Λ_1. Thus

$$T = \left[\begin{array}{c|c} I + j \in S \Omega_0 & I - j \in S \Omega_0 \\ \hline j \Omega_0 - \in \left(\Lambda_1 + S \Omega_0^2 \right) & -j \Omega_0 - \in \left(\Lambda_1 + S \Omega_0^2 \right) \end{array} \right] + O(\in^2) \qquad (2.88)$$

It may be shown, after some tedious algebra, that

$$T^{-1} = \frac{1}{2} \left[\begin{array}{c|c} I + j \in \Omega_0^{-1} \left[\Lambda_1 + S \Omega_0^2 \right] & -j \Omega_0^{-1} - \in S \\ \hline I - j \in \Omega_0^{-1} \left[\Lambda_1 + S \Omega_0^2 \right] & j \Omega_0^{-1} - \in S \end{array} \right] + O(\in^2) \qquad (2.89)$$

The matrix T has the property that

$$T^{-1}AT = \left[\begin{array}{c|c} -j\Omega_0 + \epsilon\Lambda_1 & 0 \\ \hline 0 & j\Omega_0 + \epsilon\Lambda_1 \end{array}\right] + O(\epsilon^2) \qquad (2.90)$$

Thus T is the required similarity transformation matrix which will diagonalize A to $O(\epsilon^2)$.

IV. APPLICATIONS OF MATRIX PERTURBATION THEORY

Two examples will now be given to illustrate the use of matrix perturbation theory.

A. The Mating of a Small Complex Structure to a Large Complex Structure

Consider the problem of mating a small, but complex, structure with M degrees of freedom to a much larger structure having N degrees of freedom. A good example of such a problem is the mating of a space craft to its booster. Suppose that the eigenvalues and eigenvectors are known for each structure separately. Can one compute the eigenvalues and eigenvectors for the composite structure using this information? As we shall now show this can be done without having to solve the eigenvalue problem for the (N+M) degree of freedom system.

Consider the following problem

$$\left. \begin{array}{c} M\ddot{\underline{x}} + D\dot{\underline{x}} + K\underline{x} = \underline{f}(t) \\ \underline{x}(0) = \underline{a} \qquad \dot{\underline{x}}(0) = \underline{b} \end{array} \right\} \qquad (3.0)$$

where

$$M = \left[\begin{array}{c|c} M_1 & \\ \hline & \epsilon^2 M_2 \end{array}\right] \qquad (3.1)$$

is the inertia matrix for the composite structure and consists of the two submatrices M_1 and $\epsilon^2 M_2$. M_1 is an $N \times N$ symmetric positive definite matrix and represents the inertia matrix for the large structure. $\epsilon^2 M_2$ is an $M \times M$ symmetric positive definite matrix and represents the inertia matrix of the small structure. It is purposely written in the form $\epsilon^2 M_2$ to emphasize that the inertial elements of the small structures are very much smaller than those for the large structure. Where

$$K = \left[\begin{array}{c|c} K_1 & 0 \\ \hline 0 & 0 \end{array} \right] + \epsilon^2 \left[\begin{array}{c|c} K_{21} & K_{22} \\ \hline K_{23} & K_{24} \end{array} \right] \tag{3.2}$$

is the stiffness matrix for the composite structure and consists of the five submatrices K_1, $\epsilon^2 K_{2i}$ i=1,2,3,4.

K_1 is an $N \times N$ symmetric nonnegative definite matrix and represents the stiffness matrix of the large structure and $\epsilon^2 K_{2i}$, i=1,2,3,4 are coupling and stiffness matrices for the small structure. $\epsilon^2 K_{21}$ is an $N \times N$ symmetric nonnegative definite matrix. $\epsilon^2 K_{22}$ is an $N \times M$ symmetric nonnegative definite matrix. $\epsilon^2 K_{23}$ is an $M \times N$ symmetric nonnegative definite matrix. $\epsilon^2 K_{24}$ is an $M \times M$ symmetric nonnegative definite matrix. These matrices are written in the form $\epsilon^2 K_{2i}$, i=1,2,3,4 to emphasize that the elements of the stiffness and coupling matrices for the small structure are very much smaller than those for the large structure. It should be noted that the eigenvalues for the two systems separately can be of the same order of magnitude since they are governed by the eigenequations

$$\left. \begin{array}{c} [\lambda_1 M_1 + K_1] \underline{x}_1 = 0 \\[2mm] \epsilon^2 [\lambda_2 M_2 + K_2] \underline{x}_2 = 0 \end{array} \right\} \tag{3.3}$$

and are independent of ϵ.

In Equation (3. 0) D is assumed to be an $(M+N) \times (M+N)$ nonnegative matrix which satisfies the conditions for classical normal modes, i. e.

$$M^{-1}KM^{-1}D = M^{-1}DM^{-1}K \tag{3.4}$$

In Equation (3. 0) the force vector $\underline{f}(t)$ is given by

$$\underline{f}(t) = \left\{ \begin{array}{c} \underline{p(t)} \\ \hline 0 \end{array} \right\} \begin{array}{c} N \\ M \end{array} \tag{3.5}$$

where $\underline{p(t)}$ is an N vector and acts only on the large structure.

Consider the eigenvalue problem associated with (3. 0)

$$[\mu_i M + K] \underline{x_i} = 0 \tag{3.6}$$

[Note: Since by assumption D is such that (3. 0) has classical normal modes, the eigenequation (3. 6) does not include damping.] The transformation

$$\underline{x_i} = M^{-1/2} \underline{\psi}^i \tag{3.7}$$

reduces (3. 6) to canonical form.

$$\left[\mu_i I + C^0 + \epsilon C^1 + \epsilon^2 C^2 \right] \underline{\psi}^i = 0 \tag{3.8}$$

where

$$C^0 = \left[\begin{array}{c|c} M_1^{-1/2} K_1 M_1^{-1/2} & 0 \\ \hline 0 & M_2^{-1/2} K_{24} M_2^{-1/2} \end{array} \right] \tag{3.9}$$

$$C^1 = \left[\begin{array}{c|c} 0 & M_1^{-1/2} K_{22} M_2^{-1/2} \\ \hline M_2^{-1/2} K_{23} M_1^{-1/2} & 0 \end{array} \right] \tag{3.10}$$

$$C^2 = \left[\begin{array}{c|c} M_1^{-1/2} K_{21} M_1^{-1/2} & 0 \\ \hline 0 & 0 \end{array} \right] \tag{3.11}$$

If T_1 is the orthogonal transformation which diagonalizes $M_1^{-1/2} K_1 M_1^{-1/2}$ and T_2 is the orthogonal transformation which diagonalizes $M_2^{-1/2} K_{24} M_2^{-1/2}$, let

$$T = \begin{bmatrix} T_1 & 0 \\ \hline 0 & T_2 \end{bmatrix} \tag{3.12}$$

The transformation

$$\underline{\psi}^i = T\underline{\varphi}^i \tag{3.13}$$

reduces (3.8) to

$$\left[\mu_i I + \Lambda_0 + \epsilon C^1 + \epsilon^2 C^2 \right] \underline{\varphi}^i \tag{3.14}$$

where

$$\Lambda_0 = \begin{bmatrix} \Lambda_{01} & 0 \\ \hline 0 & \Lambda_{02} \end{bmatrix} \tag{3.15}$$

where

$$\Lambda_{01} = \begin{bmatrix} \lambda_{10}^N & & \\ & \ddots & \\ & & \lambda_{N0}^N \end{bmatrix} \tag{3.16}$$

λ_{10}^N, are the eigenvalues of the large structure

and

$$\Lambda_{02} = \begin{bmatrix} \lambda_{10}^M & & \\ & \ddots & \\ & & \lambda_{M0}^M \end{bmatrix} \tag{3.17}$$

λ_{10}^M, are the eigenvalues of the small structure

$$C^1 = T^T C^1 T \tag{3.18}$$

and

$$C^2 = T^T C^2 T \tag{3.19}$$

The eigenequation (3.6) has now been reduced to the standard form where the techniques of Section 2 may be applied. The exact details will be reflected in two numerical examples.

By perturbation analysis we obtain

$$Z = I + \epsilon S \tag{3.20}$$

and

$$\Lambda = \Lambda_0 + \epsilon \Lambda_1 \tag{3.21}$$

where S is given by (2.20) and the elements of Λ by (2.18). The orthogonal transformation which diagonalizes (3.8) is given by

$$T_2 = TZ = T + \epsilon TS + O(\epsilon^2) \tag{3.22}$$

The congruence transformation which diagonalizes (3.0) is given by

$$Q_2 = M^{-1/2} T_2 \tag{3.23}$$

$$Q_2 = \left[\begin{array}{c|c} M_1^{-1/2} T_1 (I + \epsilon S_1) & \epsilon M_1^{-1/2} T_1 S_2 \\ \hline M_2^{-1/2} T_2 S_3 & \frac{1}{\epsilon} M_2^{-1/2} T_2 (I + \epsilon S_4) \end{array} \right] \tag{3.24}$$

$$\underbrace{\hspace{2.5cm}}_{\text{N modes}} \underbrace{\hspace{2.5cm}}_{\text{M modes}}$$

where S_1, S_2, S_3, S_4 are the partitions of S

$$S = \begin{array}{cc} & \begin{array}{cc} N & M \end{array} \\ \left[\begin{array}{c|c} S_1 & S_2 \\ \hline S_3 & S_4 \end{array} \right] & \begin{array}{c} N \\ M \end{array} \end{array} \tag{3.25}$$

It will be seen from (3.24) that the first N modes are ordinary or global modes. By that we mean that they are simply extensions of the modes of the main structure into the secondary structure. The next M modes are however "local modes". By this we mean that the response of the secondary structure is $O(\frac{1}{\epsilon})$, or $O(\frac{1}{\epsilon^2})$ times as large as that of the main structure. That is the motion is highly localized in the secondary structure.

Consider now the forced vibration problem (3.0) with $\underline{a} = \underline{b} = 0$. Using Equation (1.14), the solution of (3.0) is given by

$$\underline{x}(t) = Q \int_0^t v(t-\tau) Q^T \underline{f}(\tau) \, d\tau \tag{3.26}$$

where $v(t)$ is a diagonal matrix whose elements are given by (1.11).

Defining the partitions Q_i, $i=1,2,3$ of Q by

$$Q = \begin{matrix} & N & M \\ & \begin{bmatrix} Q_1 & \epsilon Q_2 \\ \hline Q_3 & \frac{1}{\epsilon} Q_4 \end{bmatrix} & \begin{matrix} N \\ M \end{matrix} \end{matrix} \tag{3.27}$$

where Q_i, $i=1,2,3,4$ are $O(1)$.

Defining the partitions $\underline{x}_i$, $i=1,2$ of $\underline{x}$ by

$$\underline{x}(t) = \begin{Bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{Bmatrix} \begin{matrix} N \\ M \end{matrix} \tag{3.28}$$

Defining the partitions v_i, $i=1,2$ of v by

$$v(t) = \begin{matrix} & N & M \\ & \begin{bmatrix} v_1 & 0 \\ \hline 0 & v_2 \end{bmatrix} & \begin{matrix} N \\ M \end{matrix} \end{matrix} \tag{3.29}$$

Introducing the partitions (3.27), (3.28) and (3.29) into (3.26) we have

$$\underline{x}(t) = \begin{Bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{Bmatrix} = \int_0^t \begin{bmatrix} Q_1 v_1 (t-\tau) Q_1^T \underline{p}(\tau) + O(\epsilon^2) \\ \hline \left[Q_4 v_2 (t-\tau) Q_2^T + Q_3 v_1 (t-\tau) Q_1^T \right] \underline{p}(\tau) \end{bmatrix} d\tau \tag{3.30}$$

Thus, we see that the response of the main structure and the secondary structure depend only on the frequencies in the input and the distribution of the eigenvalues. In particular, the response of the secondary system is of $O(1)$ with respect to ϵ and not $O\left(\frac{1}{\epsilon^2}\right)$ as one might have expected from the eigenvectors of the local modes. In order to make the exact details of the procedure clear we shall now consider two numerical examples.

Example A1 (Distinct Eigenvalues)

In Equation (3.0) let the matrices M and K be given by

$$M = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \epsilon^2 \end{bmatrix} \tag{3.31}$$

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\epsilon^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{3.32}$$

$$M^{-1/2} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{\epsilon} \end{bmatrix} \tag{3.33}$$

Using the transformation $\underline{x} = M^{-1/2}\underline{\psi}$ reduces (3.6) to the canonical form

$$\left[\mu_i I + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} \right] \underline{\psi}^i = 0 \tag{3.34}$$

Now, the matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is diagonalized by the orthogonal matrix

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{bmatrix}$$

and the matrix [2] is diagonalized by the matrix [1]. Thus

$$T = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.35}$$

The transformation $\underline{\psi}^i = T\underline{\varphi}^i$ reduces (3.34) to the standard form

$$\left[\mu_i I + \begin{bmatrix} 1 & & \\ & 3 & \\ & & 2 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{bmatrix} \right] \underline{\varphi}^i = 0 \tag{3.36}$$

This is a case of distinct eigenvalues (1, 2, 3) and hence the standard matrix perturbation method can be used. Thus

$$Z = I + \epsilon S = \begin{bmatrix} 1 & 0 & -\epsilon\sqrt{2} \\ 0 & 1 & -\epsilon\sqrt{2} \\ \epsilon\sqrt{2} & \epsilon\sqrt{2} & 1 \end{bmatrix} \quad (3.37)$$

$$T_2 = TZ = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & -2\epsilon \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\ \epsilon\sqrt{2} & \epsilon\sqrt{2} & 1 \end{bmatrix} \quad (3.38)$$

$$Q_2 = M^{-1/2} T_2 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & -2\epsilon \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\ \sqrt{2} & \sqrt{2} & \epsilon^{-1} \end{bmatrix} \quad (3.39)$$

Q_2 has the following properties

i) $\quad Q_2^T M Q_2 = I + O(t^2) \quad (3.40)$

ii) $\quad Q_2^T K Q_2 = \Lambda_0 + \epsilon \Lambda_1 + O(t^2) = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 2 \end{bmatrix} + O(\epsilon^2) \quad (3.41)$

Thus we see that the first body, or global mode, has eigenvalue 1 and eigenvector $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \sqrt{2}\right)$ and is simply an extension of the first mode of the main structure into the secondary structure. The second body, or global mode, has eigenvalue 3 and eigenvector $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, \sqrt{2}\right)$ and is simply an extension of the second mode of the main structure into the secondary structure.

The third mode is a local mode and has the eigenvector $[-2\epsilon, 0, \epsilon^{-1}]$. For example, if $\epsilon = 10^{-1}$, then the motion of the secondary structure is fifty times as large as that of the main structure.

If, for example, we take the matrix D to be

$$D = 2\zeta \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2\epsilon^2 \\ 0 & -2\epsilon^2 & 2\epsilon^2 \end{bmatrix} \tag{3.42}$$

Then

$$Q_2^T D Q_2 = 2\zeta \begin{bmatrix} 1 & & \\ & 3 & \\ & & 2 \end{bmatrix} \tag{3.43}$$

Using the partition (3.27) we see that

$$Q_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{3.44}$$

$$Q_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \tag{3.45}$$

$$Q_3 = [\sqrt{2} \quad \sqrt{2}] \tag{3.46}$$

$$Q_4 = [1] \tag{3.47}$$

Thus, using (3.30)

$$\underline{x}_1(t) = \frac{1}{2} \int_0^t \begin{bmatrix} \dfrac{e^{-\zeta(t-\tau)}}{\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,(t-\tau) & \dfrac{e^{-3\zeta(t-\tau)}}{\sqrt{3-9\zeta^2}} \sin\sqrt{3-9\zeta^2}\,(t-\tau) \\ \dfrac{e^{-\zeta(t-\tau)}}{\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,(t-\tau) & -\dfrac{e^{-3\zeta(t-\tau)}}{\sqrt{3-9\zeta^2}} \sin\sqrt{3-9\zeta^2}\,(t-\tau) \end{bmatrix} \begin{Bmatrix} (p_1(\tau)+p_2(\tau)) \\ \\ (p_1(\tau)-p_2(\tau)) \end{Bmatrix} dt$$
$$+ O(\epsilon^2) \tag{3.48}$$

$$\underline{x}_2(t) = x_3(t) = \int_0^t \frac{e^{-\zeta(t-\tau)}}{\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,(t-\tau)(p_1(\tau)+p_2(\tau))$$

$$+ \frac{e^{-3\zeta(t-\tau)}}{\sqrt{3-9\zeta^2}} \sin\sqrt{3-9\zeta^2}\,(t-\tau)(p_1(\tau)-p_2(\tau))$$

$$- \frac{2e^{-2\zeta(t-\tau)}}{\sqrt{2-4\zeta^2}} \sin\sqrt{2-4\zeta^2}\,(t-\tau)p_1(\tau)\,dt + O(\epsilon^2) \tag{3.49}$$

As pointed out previously in connection with (3.30), the response of the secondary system does not blow up as ϵ tends to zero as might be suggested by the local mode, whose eigenvector is $[-2\epsilon, 0, \epsilon^{-1}]$.

Example A2 [Repeated Eigenvalues]

As a second example consider Equation (3.0) with the matrices M, D and K given by

$$M = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \epsilon^2 \end{bmatrix} \tag{3.50}$$

$$D = 2\zeta \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -\epsilon^2 \\ 0 & -\epsilon^2 & \epsilon^2 \end{bmatrix} \tag{3.51}$$

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon^2 \\ 0 & -\epsilon^2 & \epsilon^2 \end{bmatrix} \tag{3.52}$$

Using the transformation $\underline{x} = M^{-1/2} \psi$ reduces (3.6) to the canonical form

$$\left[\mu_i I + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \right] \psi^i = 0 \tag{3.53}$$

Now, the matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is diagonalized by the orthogonal matrix

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{bmatrix}$$

and the matrix $[1]$ is diagonalized by the matrix $[1]$. Thus

$$T = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\[2mm] \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\[2mm] 0 & 0 & 1 \end{bmatrix} \qquad (3.54)$$

The transformation $\underline{\psi}^i = T\underline{\varphi}^i$ reduces (3.53) to the standard form

$$\left[I\mu_i + \begin{bmatrix} 1 \\ & 3 \\ & & 1 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & -\dfrac{1}{\sqrt{2}} \\[2mm] 0 & 0 & \dfrac{1}{\sqrt{2}} \\[2mm] -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \end{bmatrix} \right] \underline{\varphi}^i = 0 \qquad (3.55)$$

We note that in this case the first and third elements of Λ_0 are equal, this is then a case of repeated roots and we must use the techniques of Section 2 applicable to the problem of repeated roots.

First we form the auxiliary eigenequation

$$[\lambda I + B^r]\underline{t}_n = 0 \qquad (3.56)$$

[Note: In the formulation of Section 2 the first $\underline{r}$ roots were repeated; here the first and third roots are equal. This poses no difficulty since it can be put into standard form by interchanging the second and third columns of T in (3.54). This is really unnecessary provided we use the elements of C^1 associated with the repeated roots.]

In the case of the present problem

$$B^r = \begin{bmatrix} 0 & -\dfrac{1}{\sqrt{2}} \\[2mm] -\dfrac{1}{\sqrt{2}} & 0 \end{bmatrix} \qquad (3.57)$$

Solving the eigenvalue problem (3.56), the associated orthogonal matrix T_r is

$$T_r = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\[2ex] \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{3.58}$$

The required orthogonal matrix T^* is given by

$$T^* = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\[2ex] \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\[2ex] 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\[2ex] 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{3.59}$$

[Note: The reordering of T_r to accommodate the fact that the first and third roots are repeated.] Thus

$$T^* = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{2} \\[2ex] \dfrac{1}{2} & -\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} \\[2ex] \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{3.60}$$

Using the transformation $\underline{\psi}^i = T^* \underline{\varphi}^i$ reduces (3.53) to the canonical form for repeated roots

$$\left[\mu_i I + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \epsilon \begin{bmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} & 0 \\[2ex] \dfrac{1}{2} & 0 & -\dfrac{1}{2} \\[2ex] 0 & -\dfrac{1}{2} & +\dfrac{1}{\sqrt{2}} \end{bmatrix} \right] \underline{\psi}^i = 0 \tag{3.61}$$

Application of the perturbation technique applicable to repeated roots gives

$$Z = \begin{bmatrix} 1 & \dfrac{\epsilon}{4} & 0 \\[2ex] -\dfrac{\epsilon}{4} & 1 & \dfrac{\epsilon}{4} \\[2ex] 0 & -\dfrac{\epsilon}{4} & 1 \end{bmatrix} \tag{3.62}$$

$$T_2 = T^* Z \tag{3.63}$$

$$Q_2 = M^{-1/2} T^* Z$$

$$= \begin{bmatrix} \dfrac{1}{2} - \dfrac{\epsilon}{4\sqrt{2}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{2} + \dfrac{\epsilon}{4\sqrt{2}} \\[2ex] \dfrac{1}{2} + \dfrac{\epsilon}{4\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} - \dfrac{\epsilon}{4\sqrt{2}} \\[2ex] \epsilon^{-1} \dfrac{1}{\sqrt{2}} & \dfrac{1}{2\sqrt{2}} & -\epsilon^{-1} \dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{3.64}$$

The congruence matrix Q_2 has the following properties

i) $\quad Q_2^T M Q_2 = I + O(\epsilon^2) \tag{3.65}$

ii) $\quad Q_2^T K Q_2 = \begin{bmatrix} 1 - \dfrac{\epsilon}{\sqrt{2}} & 0 & 0 \\[1.5ex] 0 & 3 & 0 \\[1.5ex] 0 & 0 & 1 + \dfrac{\epsilon}{\sqrt{2}} \end{bmatrix} + O(\epsilon^2) \tag{3.66}$

In this particular example we see that the first and third modes of vibration are both local modes and are associated with the reduction of the degenerate eigenvalue 1. It should be noted that in this case the relative motion of the secondary system is of order $1/\epsilon$ compared to that of the main system, whereas in the previous case the relative motion was of order $1/\epsilon^2$. The second mode of vibration is a body, or global, mode which has eigenvalue 3 and eigenvector $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{2\sqrt{2}} \right)$ and is simply an extension of the second mode of the main structure into the secondary structure.

Using the values of the matrix D in (3.42), then

$$Q_2^T D Q_2 = 2\zeta \begin{bmatrix} 1 - \dfrac{\epsilon}{\sqrt{2}} & & \\ & 3 & \\ & & 1 + \dfrac{\epsilon}{\sqrt{2}} \end{bmatrix} \tag{3.67}$$

Using Equation (3.30) the solution of (3.0) is

$$x_1(t) = \int_0^t \frac{1}{4}\nu_1(t-\tau)[p_1(\tau)+p_2(\tau)] + \frac{1}{2}\nu_2(t-\tau)[p_1(\tau)-p_2(\tau)]$$

$$+ \frac{1}{4}\nu_3(t-\tau)[p_1(\tau)+p_2(\tau)]\,d\tau + O(\epsilon) \qquad (3.68)$$

$$x_2(t) = \int_0^t \frac{1}{4}\nu_1(t-\tau)[p_1(\tau)+p_2(\tau)] - \frac{1}{2}\nu_2(t-\tau)[p_1(\tau)-p_2(\tau)]$$

$$+ \frac{1}{4}\nu_3(t-\tau)[p_1(\tau)+p_2(\tau)]\,d\tau + O(\epsilon) \qquad (3.69)$$

$$x_3(t) = \int_0^t \frac{\epsilon^{-1}}{2\sqrt{2}}[\nu_1(t-\tau)-\nu_3(t-\tau)][p_1(\tau)+p_2(\tau)]$$

$$+ \frac{1}{4}\nu_2(t-\tau)[p_1(\tau)-p_2(\tau)]$$

$$- \frac{1}{8}[\nu_1(t-\tau)+\nu_3(t-\tau)][p_1(\tau)-p_2(\tau)]\,d\tau + O(\epsilon) \qquad (3.70)$$

where

$$\nu_1(t) = e^{-\zeta\left(1-\frac{\epsilon}{\sqrt{2}}\right)t} \sin\sqrt{1-\frac{\epsilon}{\sqrt{2}}-\zeta^2\left(1-\frac{\epsilon}{\sqrt{2}}\right)^2}\,t \Big/ \sqrt{1-\frac{\epsilon}{\sqrt{2}}-\zeta^2\left(1-\frac{\epsilon}{\sqrt{2}}\right)^2}$$

$$\nu_2(t) = e^{-3\zeta t}\sin\sqrt{3-9\zeta^2}\,t\Big/\sqrt{3-9\zeta^2} \qquad (3.71)$$

$$\nu_3(t) = e^{-\zeta\left(1+\frac{\epsilon}{\sqrt{2}}\right)t} \sin\sqrt{1+\frac{\epsilon}{\sqrt{2}}-\zeta^2\left(1+\frac{\epsilon}{\sqrt{2}}\right)^2}\,t \Big/ \sqrt{1+\frac{\epsilon}{\sqrt{2}}-\zeta^2\left(1+\frac{\epsilon}{\sqrt{2}}\right)^2}$$

It will be observed that the term $x_3(t)$ contains a term of order $1/\epsilon$. One might think that this term will become singular as $\epsilon \to 0$; this is not the case and it is easily shown that the limit of $x_3(t)$ as $\epsilon \to 0$ is finite.

$$\text{Limit } x_3(t) =$$
$$\underset{\epsilon \to 0}{}$$

$$-\frac{1}{\sqrt{2}} \int_0^t \left[(t-\tau)e^{-\zeta(t-\tau)} \left(\frac{\frac{1}{2\sqrt{2}}(1-2\zeta^2)}{1-\zeta^2} \cos\sqrt{1-\zeta^2}\,(t-\tau) - \frac{\zeta}{\sqrt{2}\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,(t-\tau) \right) \right.$$

$$\left. - \frac{\frac{1}{2\sqrt{2}}(1-2\zeta^2)}{\left(1-\zeta^2\right)^{3/2}} e^{-\zeta(t-\tau)} \sin\sqrt{1-\zeta^2}\,(t-\tau) \left(p_1(\tau)+p_2(\tau)\right) d\tau \right]$$

$$+ \int_0^t \frac{1}{4} \left[\nu_2(t-\tau) - \nu_1(t-\tau) \right] \left[p_1(\tau) - p_2(\tau) \right] d\tau \qquad (3.72)$$

Thus we see that $x_3(t)$ is finite if ζ is greater than zero and p_1 and p_2 are finite.

Example B

Consider now the problem

$$\left. \begin{array}{c} M\ddot{\underline{x}} + \epsilon D\dot{\underline{x}} + K\underline{x} = \underline{f}(t) \\ \underline{x}(0) = \underline{a} \quad , \quad \dot{\underline{x}}(0) = \underline{b} \end{array} \right\} \qquad (3.73)$$

where M, D and K are symmetric, M is positive definite and D and K are non-negative definite. Suppose that $M^{-1}D$ does not commute with $M^{-1}K$. Equation (3.73) does not possess classical normal modes. Let us first reduce (3.73) to canonical form. Let

$$\underline{y} = M^{1/2}\underline{x} \quad , \quad \underline{q}(t) = M^{-1/2}\underline{f}(t)$$

Then

$$\left. \begin{array}{c} I\ddot{\underline{y}} + \epsilon B\dot{\underline{y}} + C\underline{y} = \underline{q}(t) \\ \underline{y}(0) = M^{1/2}\underline{a} \quad , \quad \dot{\underline{y}}(0) = M^{1/2}\underline{b} \end{array} \right\} \qquad (3.74)$$

Now let $\underline{y} = T\underline{z}$, where T is the orthogonal matrix which diagonalizes C. Thus

$$\left. \begin{array}{c} I\ddot{\underline{z}} + \epsilon B\dot{\underline{z}} + \Lambda_0\underline{z} = \underline{r}(t) \end{array} \right\} \qquad (3.75)$$

$$\underline{z}(0) = T^T M^{1/2} \underline{a} \;, \quad \underline{\dot{z}}(0) = T^T M^{1/2} \underline{b} \Bigg\} \qquad \text{(3.75)}$$
<div align="right">cont</div>

Equation (3.75) may be written as a system of first order equations

$$\left. \begin{array}{l} \dfrac{d\underline{w}}{dt} = -A\underline{w} + \underline{s}(t) \\[2mm] \underline{w}(0) = \underline{d} \end{array} \right\} \qquad \text{(3.76)}$$

where

$$A = \begin{bmatrix} 0 & -I \\ \Lambda_0 & \epsilon B \end{bmatrix} \qquad \text{(3.77)}$$

$$\underline{w} = \begin{bmatrix} \underline{z} \\ \underline{\dot{z}} \end{bmatrix} \qquad \text{(3.78)}$$

$$\underline{s} = \begin{bmatrix} 0 \\ \underline{r}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ T^T \underline{q}(t) \end{bmatrix} = \begin{bmatrix} \bar{0} \\ Q^T \underline{f}(t) \end{bmatrix} \qquad \text{(3.79)}$$

$$\underline{d} = \begin{bmatrix} T^T M^{1/2} \underline{a} \\ T^T M^{1/2} \underline{b} \end{bmatrix} = \begin{bmatrix} Q^T M \underline{a} \\ Q^T M \underline{b} \end{bmatrix} \qquad \text{(3.80)}$$

where $Q = M^{-1/2} T$ is the congruence transformation which simultaneously diagonalizes M and K.

The problem has now been reduced to the form where the matrix perturbation theory of Section 2B can be applied. In Section 2B it was shown that the similarity matrix T which will diagonalize A is given by

$$T = \left[\begin{array}{c|c} I + j\epsilon S\Omega_0 & I - j\epsilon S\Omega_0 \\ \hline j\Omega_0 - \epsilon\left(\Lambda_1 + S\Omega_0^2\right) & -j\Omega_0 - \epsilon\left(\Lambda_1 + S\Omega_0^2\right) \end{array} \right] \qquad \text{(3.81)}$$

$$T^{-1} = \frac{1}{2} \left[\begin{array}{c|c} I + j\epsilon\Omega_0^{-1}\,\Lambda_1 + S\Omega_0^2 & -j\Omega_0^{-1} - \epsilon S \\ \hline I - j\epsilon\Omega_0^{-1}\,\Lambda_1 + S\Omega_0^2 & j\Omega_0^{-1} - \epsilon S \end{array} \right] \qquad \text{(3.82)}$$

where

$$\Omega_0 = \Lambda_0^{1/2} = \begin{bmatrix} w_1 & & & \\ & w_2 & \cdot & \\ & & \cdot & \\ & & & \cdot w_N \end{bmatrix} \qquad (3.83)$$

where w_i, $i=1,2\ldots N$ are the radial frequencies of the undamped system.

$$\Lambda_1 = \frac{1}{2} \begin{bmatrix} \beta_{11} & & & \\ & \beta_{22} & \cdot & \\ & & \cdot & \\ & & & \cdot \beta_{NN} \end{bmatrix} \qquad (3.84)$$

$$S = \begin{bmatrix} 0 & \dfrac{\beta_{12}}{w_1^2 - w_2^2} & \cdot\cdot & \dfrac{\beta_{1N}}{w_1^2 - w_N^2} \\[3mm] \dfrac{\beta_{21}}{w_2^2 - w_1^2} & 0 & \cdot\cdot\cdot\cdot\cdot & \dfrac{\beta_{2N}}{w_2^2 - w_N^2} \\[3mm] \dfrac{\beta_{N1}}{w_N^2 - w_1^2} & \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & 0 \end{bmatrix} \qquad (3.85)$$

Making the transformation

$$\underline{w} = T\underline{p} \qquad (3.86)$$

Equation (3.76) becomes

$$\left. \begin{aligned} \frac{d\underline{p}}{dt} &= \left[\begin{array}{c|c} j\Omega_0 - \epsilon \Lambda_1 & 0 \\ \hline 0 & -j\Omega_0 - \epsilon \Lambda_1 \end{array} \right] \underline{p} + T^{-1} \underline{s}(t) \\[3mm] \underline{p}(0) &= T^{-1} \underline{d} \end{aligned} \right\} \qquad (3.87)$$

Thus

$$\underline{p}(t) = \begin{bmatrix} e^{(j\Omega_0 - \epsilon \Lambda_1)t} & 0 \\ 0 & e^{(-j\Omega_0 - \epsilon \Lambda_1)t} \end{bmatrix} T^{-1} \underline{w}(0) + \int_0^t \begin{bmatrix} e^{(j\Omega_0 - \epsilon \Lambda_1)(t-\tau)} & 0 \\ 0 & e^{(-j\Omega_0 - \epsilon \Lambda_1)(t-\tau)} \end{bmatrix} T^{-1} \underline{s}(\tau) \, d\tau \qquad (3.88)$$

Therefore

$$\underline{w}(t) = T \begin{bmatrix} e^{(j\Omega_0 - \epsilon\Lambda_1)t} & 0 \\ 0 & e^{(-j\Omega_0 - \epsilon\Lambda_1)t} \end{bmatrix} T^{-1} \underline{w}(0) + \int_0^t T \begin{bmatrix} e^{(j\Omega_0 - \epsilon\Lambda_1)t} & 0 \\ 0 & e^{(-j\Omega_0 - \epsilon\Lambda_1)t} \end{bmatrix} T^{-1} \begin{bmatrix} 0 \\ \underline{q}(t) \end{bmatrix} d\tau \qquad (3.89)$$

Substituting (3.78), (3.81), (3.82), (3.83), (3.84) into (3.89) and taking the special case where $\underline{w}(0) \equiv \underline{0}$

$$\underline{z}(t) = \int_0^t \begin{bmatrix} e^{-\frac{\epsilon}{2} \beta_{ii}(t-\tau)} \dfrac{\sin w_i(t-\tau)}{w_i} \end{bmatrix} Q^T \underline{f}(\tau)\, d\tau$$

$$+ \epsilon \int_0^t \left[SR(t-\tau) - R(t-\tau)S \right] Q^T \underline{f}(\tau)\, d\tau + O(\epsilon^2) \qquad (3.90)$$

where

$$R(\tau) = \begin{bmatrix} e^{-\frac{\epsilon}{2}\beta_{ii}t} \cos w_i t \end{bmatrix} \qquad (3.91)$$

Now

$$\underline{x}(t) = M^{-1/2} T\underline{z}(t) = Q\underline{z}(t)$$

$$\therefore \quad \underline{x}(t) = \int_0^t Q \begin{bmatrix} e^{-\frac{\epsilon}{2}\beta_{ii}(t-\tau)} \dfrac{\sin w_i(t-\tau)}{w_i} \end{bmatrix} Q^T \underline{f}(\tau)\, d\tau$$

$$+ \epsilon \int_0^t Q \left[SR(t-\tau) - R(t-\tau)S \right] Q^T \underline{f}(\tau)\, d\tau + O(\epsilon^2) \qquad (3.92)$$

The first term in (3.92) is the solution which would have been obtained if the matrix $\beta = Q^T B Q$ had been diagonal. Thus the error in neglecting the off-diagonal terms is

$$\underline{x}(t) - \underline{x}_0(t) = \epsilon \int_0^t Q \left[SR(t-\tau) - R(t-\tau)S \right] Q^T \underline{f}(\tau)\, d\tau + O(\epsilon^2) \qquad (3.93)$$

To obtain some idea of the errors introduced by the off-diagonal terms in the matrix B, let us compare the two matrices $v(t)$ and $\epsilon[SR(t)-R(t)S]$. We have

$$v(t) = \begin{bmatrix} e^{-\frac{\epsilon}{2}\beta_{11}t}\dfrac{\sin w_1 t}{w_1} & & \\ & \ddots & \\ & & e^{-\frac{\epsilon}{2}\beta_{NN}t}\dfrac{\sin w_N t}{w_N} \end{bmatrix}$$

(3.94)

and

$\epsilon[SR(t)-R(t)S]$

$$= \begin{bmatrix} 0 & \dfrac{\epsilon\beta_{12}(R_1-R_2)}{w_1^2-w_2^2} & \dfrac{\epsilon\beta_{13}(R_1-R_3)}{w_1^2-w_2^2} & \cdots & \dfrac{\epsilon\beta_{1N}(R_1-R_N)}{w_1^2-w_N^2} \\ \dfrac{\epsilon\beta_{12}(R_1-R_2)}{w_1^2-w_2^2} & 0 & \dfrac{\epsilon\beta_{23}(R_2-R_3)}{w_2^2-w_3^2} & \cdots \cdots \\ \dfrac{\epsilon\beta_{1N}(R_1-R_N)}{w_1^2-w_N^2} & \cdots \cdots \cdots \cdots \cdots & 0 \end{bmatrix}$$

(3.95)

The contribution of $v(t)$ to the solution depends on

i) the diagonal terms β_{ii}

ii) the frequencies w_i

iii) the frequencies and amplitudes of the model excitation

The contribution of $\epsilon[SR(t)-R(t)S]$ to the solution depends on

i) the off-diagonal term $\epsilon\beta_{ij}$ $i\neq j$

ii) the frequency separation of the modes $w_i^2-w_j^2$ $i\neq j$

iii) the frequencies and amplitudes of the modal excitation

From the above one may conclude the following:

i) The effect of the off-diagonal terms will be smaller the smaller the magnitude of the off-diagonal terms.

ii) The effect of the off-diagonal terms will be smaller the wider the separation in the modal frequencies.

To illustrate some of the points in this case, consider the numerical Example B.

Example B

Consider the steady state solution of the problem

$$I\ddot{y} + \epsilon \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \dot{y} + \begin{bmatrix} 1.05 & -0.05 \\ -0.05 & 1.05 \end{bmatrix} y = \begin{pmatrix} \cos t \\ \cos t \end{pmatrix} \tag{3.96}$$

The transformation

$$y = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{bmatrix} z$$

will diagonalize the stiffness matrix, giving

$$I\ddot{z} + \epsilon \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix} z = \begin{pmatrix} \sqrt{2} \cos t \\ 0 \end{pmatrix} \tag{3.97}$$

The associated eigenvalue problem is

$$\left(\gamma_i^2 I + \epsilon \gamma_i \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix} \right) \underline{\varphi}^i = 0 \tag{3.98}$$

Applying the methods of Section 2B, we have

$$\Omega_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1.1} \end{bmatrix} \tag{3.99}$$

$$\Lambda_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.100}$$

$$S = \begin{bmatrix} 0 & \dfrac{1}{0.1} \\ \dfrac{1}{0.1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix} \tag{3.101}$$

$$Q = T = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{3.102}$$

Thus the steady state solution is given by (3.92) with the lower limit set to $-\infty$.

$$\underline{\zeta}(t) = \underline{x}(t) = \int_{-\infty}^{t} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{-\epsilon(t-\tau)} \sin(t-\tau) & 0 \\ 0 & e^{-\epsilon(t-\tau)} \frac{\sin\sqrt{1.1}(t-\tau)}{\sqrt{1.1}} \end{bmatrix} \begin{bmatrix} \sqrt{2}\cos t \\ 0 \end{bmatrix} d\tau$$

$$+ \epsilon \int_{-\infty}^{t} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 10[e^{-\epsilon(t-\tau)}][\cos(t-\tau)-\cos\sqrt{1.1}(t-\tau)] \\ 10[e^{-\epsilon(t-\tau)}][\cos(t-\tau)-\cos\sqrt{1.1}(t-\tau)] & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}\cos t \\ 0 \end{bmatrix} d\tau + O(\epsilon^2)$$

$$(3.103)$$

Therefore

$$\underline{y}(t) = \begin{bmatrix} \frac{\sin t}{2\epsilon} \\ \frac{\sin t}{2\epsilon} \end{bmatrix} + \begin{bmatrix} 5\cos t \\ -5\cos t \end{bmatrix} + O(\epsilon) \qquad (3.104)$$

The first term represents the response $\underline{y}_0(t)$ of the system neglecting the off-diagonal term, thus the error due to the off-diagonal terms in B is given by

$$\underline{\epsilon} = \underline{y}(t) - \underline{y}_0(t) = \begin{bmatrix} 5\cos t \\ -5\cos t \end{bmatrix} + O(\epsilon) \qquad (3.105)$$

A measure of the relative error is

$$\eta = \frac{\text{Sup}\,\|\underline{\epsilon}\|}{\text{Sup}\,\|\underline{y}_0(t)\|} = 10\epsilon \qquad (3.106)$$

Thus we see that as long as ϵ is small the relative error in neglecting the off-diagonal terms in the B matrix is small in this case.

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