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FAST MODAL EXTRACTION IN NASTRAN VIA THE FEER COMPUTER PROGRAM

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SUMMARY

A new eigensolution routine, FEER (Fast Eigensolution Extraction Routine), used in conjunction with NASTRAN at Israel Aircraft Industries is described. The FEER program is based on an automatic matrix reduction scheme whereby the lower modes of structures with many degrees of freedom can be accurately extracted from a tridiagonal eigenvalue problem whose size is of the same order of magnitude as the number of required modes. The process is effected without arbitrary lumping of masses at selected node points or selection of nodes to be retained in the analysis set.

The results of computational efficiency studies are presented, showing major arithmetic operation counts and actual computer run times of FEER as compared to other methods of eigenvalue extraction, including those available in the NASTRAN READ module. It is concluded that the tridiagonal reduction method used in FEER would serve as a valuable addition to NASTRAN for highly increased efficiency in obtaining structural vibration modes.

INTRODUCTION

One of the most burdensome computational tasks in discretized structural systems centers around the extraction of mode shapes and frequencies when the orders of the matrices are large. The difficulties are compounded as the number of required eigensolutions increases and multiple or near-multiple eigenvalues are encountered.

Currently, NASTRAN provides three methods for modal extraction (refs. 1 and 2): the Tridiagonal or Givens method, the Inverse Power Method with Shifts, and the Determinant method. In each method the problem size encountered is equal to the number of degrees-of-freedom in the analysis set which, given typical, present-day problem applications, may run into the thousands. One means of reducing the size of the analysis set is via the Guyan reduction (ref. 3), which has been incorporated into NASTRAN. This technique, which is similar in concept to the Kaufman - Hall reduction (ref. 4), requires a "judicious" elimination of selected mass degrees-of-freedom and an attempt is made to account for the influence of the eliminated nodes through equivalent energy criteria. As demonstrated by Levy (ref. 5), such an intuitive approach involves a great deal of guesswork and can lead to grossly inaccurate results, particularly in systems with relatively non-uniform mass distributions.

What is required to circumvent these difficulties is a more automated eigenreduction scheme which yields accurate lower modes of the structural system. In essence, the problem may be posed as follows:

Given the nth order eigenvalue problem

$$[K]\{\phi\} = \omega^2[M]\{\phi\} \tag{1}$$

where [K] and [M] are symmetric and non-negative definite, we wish to approximate the modal vectors by

$$\{\phi\} \approx [T]\{\delta\} \tag{2}$$

where [T] is a suitably constructed transformation matrix of size nxm (m<<n) and $\{\delta\}$ is an m-component vector of generalized coordinates. Using a Rayleigh-Ritz procedure the resulting reduced, mth order eigenproblem is of the form

$$[\bar{K}]\{\delta\} = \bar{\omega}^2[\bar{M}]\{\delta\} \tag{3}$$

where

$$[\bar{K}] = [T]^T [K] [T] \tag{4a}$$

$$[\bar{M}] = [T]^T [M] [T] \tag{4b}$$

and $\bar{\omega}$ is an approximate modal frequency. If a specified number of lower modes are to be accurately obtained, then the individual nth order vectors comprising the transformation matrix must be sufficiently rich in the corresponding modal vectors. Thus, the practical value of

the reduction scheme hinges on its ability to generate such a transformation matrix with a minimum of computational effort.

A number of closely related methods involving eigenreduction concepts have been proposed previously. In the work of Hestenes and Karush (ref. 6), eigensolutions were obtained via a block power method (iterating with several vectors simultaneously as opposed to a single vector) and a reduced eigenvalue problem was employed to orthogonalize and improve successive blocks of vectors between iteration steps. More recently, Jennings and Orr (ref. 7), Dong, Wolf and Peterson (ref. 8), and Bathe and Wilson (ref. 9) proposed similar techniques using the Inverse Power Method in conjunction with simultaneous sets of vectors (alternately called Simultaneous Iteration, Subspace Iteration and Block-Stodola methods). In each of these approaches, however, the functional role of the reduced eigenproblem is to improve a subspace of approximate modal vectors with central emphasis being placed on a block-type Inverse Power method.

In this report, a new eigenreduction routine, FEER (Fast Eigensolution Extraction Routine) is described, wherein a single reduced eigenproblem is generated for the accurate extraction of any specified number of lower modes. Further, the transformation matrix is generated vector-by-vector in such a way that the reduced eigenproblem is tridiagonal in form. The FEER program is now being used in conjunction with NASTRAN at Israel Aircraft Industries to obtain much more economical eigensolutions than currently possible with the NASTRAN READ module.

The tridiagonal reduction method employed in FEER was first suggested by Crandall (ref. 10) as a truncated version of the Lanczos algorithm (ref. 11). However, it was soon discovered that the original scheme possessed numerical instabilities (refs. 12 and 13). The necessary improvements to correct these weaknesses were made by Ojalvo and Newman (ref. 14), who were the first to develop a successful tridiagonal reduction program for large scale structural dynamics problems. The FEER computer program contains further refinements later introduced by Newman and Pipano (ref. 15), including:

1. Highly efficient numerical computation schemes, using packing techniques which take advantage of matrix sparsity.

2. Calculation of accurate upper and lower bounds on the extracted eigenvalues.

3. Accommodation of singular mass matrices and stiffness matrix singularities associated with rigid body modes.

TRIDIAGONAL REDUCTION METHOD

Preliminary Operations

Employing the NASTRAN notation, the structural eigenvalue problem is of the form

$$[K_{aa}]\{\phi_a\} = \omega^2[M_{aa}]\{\phi_a\} \tag{5}$$

Both $[K_{aa}]$ and $[M_{aa}]$ are nth order symmetric, non-negative and semi-definite matrices corresponding to the analysis set. Hence, they may both be singular, but all the eigenvalues are zero or positive.

In order to obtain a decomposable matrix, a small, positive shift parameter, α , is chosen such that

$$\omega^2 = \omega_o^2 - \alpha \tag{6}$$

Then

$$[K_{aa} + \alpha M_{aa}]\{\phi_a\} = \omega_o^2[M_{aa}]\{\phi_a\}. \tag{7}$$

It can be easily shown that the shifted stiffness matrix is non-singular and positive-definite provided that the system masses generate kinetic energy due to any kinematically admissible rigid body motions of the structure. This requirement is always satisfied by the mass matrix in a physically well-posed problem.

In order to maintain the elements of the subsequent trial vectors on the order of unity, a positive mass-scaling parameter, S, is also employed, such that

$$[M_{aa}] = \frac{1}{S}[\bar{M}] \tag{8}$$

If a Cholesky symmetric decomposition of the shifted stiffness matrix is performed:

$$[K_{aa} + \alpha M_{aa}] = [L]\,[L]^T \tag{9}$$

it follows that the eigenvalue problem equation (7), can be converted

to the form

$$[B]\ \{x\} = \lambda\{x\} \tag{10}$$

where

$$[B] = [L]^{-1}[\bar{M}]\ [L^{-1}]^T \tag{11a}$$

$$\{x\} = [L]^T\ \{\phi_a\} \tag{11b}$$

and

$$\lambda = \frac{S}{\omega^2+\alpha} \tag{11c}$$

The above triangular matrix inverses are treated as purely operational symbols, since in actual numerical operations forward and backward passes on vectors are employed.

Generation of the Reduced Eigenproblem

A reduction of the nth order eigenvalue problem, equation (10), is effected through the transformation

$$\underset{nx1}{\{\bar{x}\}} = \underset{nxm}{[V]}\ \underset{mx1}{\{y\}} \tag{12}$$

where $\{\bar{x}\}$ is an approximation of $\{x\}$ and $m < n$. The transformation matrix is taken to be unitary, so that

$$[V]^T[V] = [I]. \tag{13}$$

The reduced mth order eigenproblem is then

$$[A]\ \{y\} = \bar{\lambda}\{y\} \tag{14}$$

where

$$[A] = [V]^T[B][V] \tag{15}$$

and $\bar{\lambda}$ is an approximation of the eigenvalue λ.

The essence of the reduction scheme lies in the choice of the

transformation matrix [V]. In the tridiagonal reduction method, the Lanczos algorithm (refs. 11 and 13) is used to build the [V] matrix, vector by vector, i.e.,

$$[V] = [\{v_1\}\ \{v_2\}\text{-------}\ \{v_m\}] \tag{16}$$

such that the reduced mxm matrix [A], is tridiagonal and its eigenvalues approximate the higher end of the eigenspectrum of [B] (or, equivalently, the lower natural frequencies of the structure).

The algorithm yields

$$[A] = \begin{bmatrix} a_{11} & d_2 & & & \\ d_2 & a_{22} & d_3 & & \\ & d_3 & a_{33} & \ddots & \\ & & \ddots & \ddots & d_m \\ & & & d_m & a_{mm} \end{bmatrix} \tag{17}$$

where the matrix coefficients are <u>theoretically</u> given by the recurrence formulas

$$\begin{aligned} a_{ii} &= \{v_i\}^T[B]\{v_i\} \\ d_i &= \{v_{i-1}\}^T[B]\{v_i\} \qquad ;\ i = 1,m \\ \{\bar{v}_{i+1}\} &= [B]\{v_i\} - a_{ii}\{v_i\} - d_i\{v_{i-1}\} \\ \{v_{i+1}\} &= \{\bar{v}_{i+1}\}/[\{\bar{v}_{i+1}\}^T\{\bar{v}_{i+1}\}]^{1/2} \end{aligned} \tag{18}$$

The process is initialized by choosing an initial trial vector, $\{v_1\}$ and setting $\{v_{-1}\} = \{0\}$; $d_1 = 0$.

The initial trial vector should contain all components of the system eigenvectors and must be constrained to eliminate spurious eigensolutions ($\omega^2 \to \infty$) due to mass matrix singularities. These requirements are

satisfied by setting

$$\{v_1\} = [B]\{w\}/[(Bw)^T(Bw)]^{1/2}, \tag{19}$$

where {w} is an n-element vector obtained from a random number generator routine.

Reorthogonalization of the Trial Vectors

Although the trial vectors $\{v_i\}$ generated in equations (18) form a theoretically orthogonal set, it has been shown (ref. 16) that they rapidly degrade as the computations proceed, such that the later vectors are far removed from orthogonality to the earlier ones. This is caused by unavoidable computational round-off, which, because of repeated multiplications by the unreduced eigenmatrix, [B], tends to amplify the contributions of the lower frequency eigenvector components. To correct this problem, Gregory (ref. 12) experimented with the use of higher precision arithmetic, but found only marginal improvements in the final results. Later, Lanczos suggested a single reorthogonalization of the trial vectors. While this improves matters substantially, it still does not eliminate the precision problem adequately. However, Ojalvo and Newman (ref. 14) found that the introduction of an iterative reorthogonalization loop can make the trial vectors as orthogonal as necessary for extremely large systems. The procedure is as follows:

Denote each new vector obtained from equations (18) as $\{v^o_{i+1}\}$ and iterate,

$$\begin{aligned}\{v^1_{i+1}\} &= \{v^o_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T\{v^o_{i+1}\}]\{v_j\} \\ \{v^2_{i+1}\} &= \{v^1_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T\{v^1_{i+1}\}]\{v_j\}\end{aligned} \tag{19}$$

until an acceptable vector

$$\{v^{s+1}_{i+1}\} = \{v^s_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T\{v^s_{i+1}\}]\{v_j\}$$

is found which satisfies the orthogonality criterion

$$\max_{1\le j\le i} \left|\{v_j\}^T\{v^s_{i+1}\}\right| \le 10^{2-t} \tag{20}$$

where t is the total number of decimal digits carried by the computer.

A normalized form of the reorthogonalized trial vector is finally obtained through

$$\{v_{i+1}\} = \{v_{i+1}^{s+1}\}/[\{v_{i+1}^{s+1}\}^T\{v_{i+1}^{s+1}\}]^{1/2} \tag{21}$$

Experiences gained through application of the FEER program to a large variety of problem types and sizes have indicated that an average of only two reorthogonalizations are required per trial vector generation.

Size Criteria for the Reduced Eigenproblem

As a result of numerical experiments and applications (refs. 14, 15, 17-19), it has been found that in cases where m<<r (where r is the total number of structural modes, including rigid body modes, and m is the size of the reduced eigenvalue problem), a first grouping of more than m/2 lower frequencies of the reduced system are in accurate agreement with the corresponding number of exact frequencies, provided that $m \geq 7$, i.e., when at least seven trial vectors are chosen. The remaining reduced system frequencies are spread across the remaining exact spectrum, with the last one representing a lower bound on the highest exact frequency of the unreduced problem.

Thus, if the user requests q lower frequencies of the structure, the order of the reduced eigenvalue problem is

$$m = \begin{cases} \min [2q+1, r]; & q > 3 \\ \min [7, r]; & q \leq 3 \end{cases} \tag{22}$$

It should be noted that in all cases $m \leq r$, and whenever m is set equal to r, <u>all</u> the structural modes of the unreduced problem are generated.

Error Bounds on the Computed Eigenvalues

One of the inherently striking features of the tridiagonal reduction method is that the solution of the reduced, tridiagonal eigenproblem

$$[A]\{y\} = \bar{\lambda}\{y\} \tag{23}$$

and the off-diagonal elements of [A] automatically provide accurate error-bound parameters for the extracted eigenvalues. In particular, it can be shown (ref.20) that absolute error bounds for each approximate root, $\bar{\omega}_i^2$, are found from the inequality

$$\left| \frac{\bar{\omega}_i^2 + \alpha}{\omega_i^2 + \alpha} - 1 \right| \leq \frac{\left| d_{m+1} \cdot (y_f)_i \right|}{\bar{\lambda}_i} \qquad (24)$$

where ω_i^2 is an exact system root, d_{m+1} is the (m+1)th off-diagonal element of an [A] matrix of order m+1, and y_{fi} is the last element of the eigenvector corresponding to $\bar{\lambda}_i$.

Program FEER Flow Diagrams and Sample Output

The overall flow diagram for implementation of the tridiagonal reduction method in FEER is shown in figure 1. The reduced system eigenvalue problem is solved in block 7 by means of a Q-R algorithm which takes advantage of the symmetrical, tridiagonal form of the eigenmatrix and the physical modal vectors and frequencies are finally computed in block 9. The details of block 6, "Execute Tridiagonal Reduction Algorithm", are given in figure2. Block 6.4 and the associated peripheral test conditions are used to generate re-start vectors whenever premature vanishing of a trial vector occurs. This is usually due to the existence of multiple or near-multiple eigenvalues, as described in reference 13. Figure 3 shows a representative eigenvalue table produced by FEER. In this example, the order of the stiffness matrix was 3,072, while the size of the reduced problem was 41. As shown by the error bound listing, FEER generated 21 lower frequencies to within an accuracy of .01%, using only 362 seconds of CPU time on a CDC-6600 computer.

COMPUTATIONAL EFFICIENCY STUDIES

A count of the major arithmetic operations expended in FEER is summarized in Table 1, where n denotes the size of the stiffness matrix in the analysis set, b and $\bar{b}$ are average semi-band widths of the stiffness and mass matrices, respectively, and q is the number of accurate modes requested by the user. Each operation is assumed to consist of a

multiplication followed by an addition.

It should be noted that the average bandwidth parameters are used primarily to provide a measure of the number of non-zero matrix elements. In actuality, FEER employs efficient packing routines which do not require a uniform band structure for efficient computational operations. It can be seen that the major computational effort involves decomposition of the modified stiffness matrix (step 3) and provides the leading term of $1/2\ nb^2$ in the total operation count. One of the positive features of the tridiagonal reduction method is that only one such decomposition is performed regardless of the number of roots required.

Operation count and storage requirements for several alternate eigensolution methods are compared with FEER in Table 2. The purpose of this comparison is to provide an indication of the potential efficiency of each method, assuming that an equally adept and knowledgable programmer has had a chance to employ the same time-saving tricks in each case. For this reason, several excellent solution techniques which achieve high efficiency through special data handling and storage methods (see for example, refs. 21 and 22), but nevertheless show a high minimum operation count, have not been included in the comparison. As in Table 1, the counts are presented in terms of average bandwidths which are again to be interpreted as a measure of non-zero matrix entries rather than in terms of a specific band structure.

It can be seen that in the Givens method the operation count $(\frac{2}{3}n^3)$ and the storage requirements as well $\left(O(n^2) \right)$ become prohibitively large when the size of analysis set grows beyond more than a few hundred degrees-of-freedom.

The leading term in the Inverse Power Method (NASTRAN) is $qnb^2/2$ as compared to $nb^2/2$ for FEER, since at least one shift per extracted root and a subsequent triangular decomposition is typically required in the former method. Based on this assumption and the additional supposition that an average of seven iterations per eigenvector are required in the Inverse Power method, theoretical operation-count ratios (Inverse Power Method/FEER) are presented as a function of semi-band width and the number of required roots in figures 4 and 5 for the cases of diagonal and consistent mass matrices. These curves provide only an approximate estimate of the relative time savings actually accrued for several reasons. First, the structure of the stiffness matrix influences the decomposition strategy employed in NASTRAN via the active column approach. In addition, there is no a-priori knowledge of the actual number of shifts and iterations which will be required in the Inverse Power method for any given problem application. In general, both the number of shifts and iterations tend to increase with the number of roots extracted, so that the curves indicating improved efficiency of the Inverse Power method for a very large number of extracted roots

and small bandwidths are unrealistic.

Table 2 also shows approximate operation counts and storage requirements for Gupta's Sturm Sequence method (ref. 23) and a current version of the subspace or Block-Stodola method (ref. 9). The storage requirements for each of these methods, as well as the Inverse Power and FEER methods, are all on the same order of magnitude. In Gupta's method the count of $25nb^2q$ is based on his assumption that approximately $2nb^2$ operations are involved in examining the Sturm sequence for one trial root value, and that about twelve such values must be examined for each accurately predicted root (ref. 9). With regard to the Subspace Iteration method, the leading term in the count, nb^2, is twice as large as in FEER and all other terms involving the same functional forms of the parameters n, b, $\bar{b}$, q are also much larger. In addition, the reduced eigenproblem which is solved for improvement of the subspace is not tridiagonal so that the count for this operation is on the order of q^3 as compared to q^2 for the tridiagonal reduction method. Finally, the assumption of eight subspace iterations may not be very reliable, since this depends on the choice of the starting subspace, which is somewhat arbitrary.

Table 3 presents a set of actual computer runs comparing the CPU execution times of FEER vs. the Inverse Power and Givens methods in the NASTRAN READ module. The results indicate that the more efficient decomposition operations and shift strategy incorporated into Level 15 have yielded significant improvements in the Inverse Power method as compared to the Level 12 version (see also ref.24).

However, the run times for comparable or identical problems are generally 5 to 20 times faster with FEER than with the Level 15 Inverse Power method when between 5 and 20 accurate modes are requested. This result is in rough agreement with the operation count ratios shown in figures 4 and 5. In problem No.2, which is relatively small and could therefore be treated with the Givens method, the execution time via FEER was approximately 3 times as fast, since only 35 modes were requested, while in the Givens method the user has no choice and must pay the penalty of having all the eigenvalues calculated (in this particular case, 105).

CONCLUDING REMARKS

Significant computational efficiencies are achieved in the FEER program primarily due to the tridiagonal reduction method of modal extraction. Basically, the subspace of trial vectors generated via this method are sufficiently rich in the lower modes to provide a

single, reduced, tridiagonal eigenproblem whose solution provides these modes with a high degree of accuracy. This feature distinguishes it from the usual subspace or block iteration methods, where the trial vector subspace is established somewhat arbitrarily and subsequently improved through repeated solutions of reduced eigenproblems. The tridiagonal reduction method employs only a single, intitial shift of eigenvalues and hence requires only one matrix decomposition. It is consequently much more efficient than the Inverse Power Method with shifts when more than one or two lower modes are required. FEER is also extremely efficient for out-of-core operations and requires only (15,000 + 7.n) central memory words, where n is the order of the analysis set. Another feature of the method is that the reduced problem is generated automatically, starting with a random trial vector, and this avoids one of the basic weaknesses of techniques requiring either a judicious selection of starting vectors or retained nodes.

It is concluded that the tridiagonal reduction method used in FEER would serve as a valuable addition to NASTRAN for increased efficiency in obtaining structural vibration modes.

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TABLE 1

SUMMARY OF MAJOR OPERATIONS AND STORAGE REQUIREMENTS

- TRIDIAGONAL REDUCTION METHOD (PROGRAM FEER)

Operation	Major Calculations	Approximate Operation Count	Approximate Storage Requirements*
1. Stiffness matrix shift	$[K] = [K_{aa} + sM_{aa}]$	nb	
2. Mass matrix scaling	$[M] = s[M_{aa}]$	nb	
3. Cholesky decomposition of $[K]$	$[K] = [L]\,[L]^T$	$\frac{1}{2}nb^2$	
4. Tridiagonal Reduction Algorithm and reorthogonalization of trial vectors; $i = 1,2,\ldots,2q$	$[B] = [L]^{-1}[M][L^{-1}]^T$		
	$a_{i,i} = \{v_i\}^T[B]\{v_i\}$		
	$d_i = \{v_{i-1}\}^T[B]\{v_i\}$		
	$\{\bar{v}_{i+1}\} = [B]\{v_i\} - a_{ii}\{v_i\} - d_i\{v_{i-1}\}$	$4qn(b+5+2q+1)$	$n(b+2q)$
	$d_{i+1} = [\bar{v}_{i+1}^T\,\bar{v}_{i+1}]^{\frac{1}{2}}$		
	$\{v_{i+1}^1\} = \{\bar{v}_{i+1}\}/d_{i+1}$		
	$\{v_{i+1}^{s+1}\} = \{v_{i+1}^s\} - \sum_{j=1}^{i} [v_j^T\,v_{i+1}^s]\{v_j\}$; $s=1,2,\ldots,t$		
	$d_{i+1}^t = [\{v_{i+1}^t\}^T\{v_{i+1}^t\}]^{\frac{1}{2}}$		
	$\{v_{i+1}\} = \{v_{i+1}^t\}/d_{i+1}^t$		
5. Q-R iterations to obtain 2q modes of the tridiagonal [A]matrix; $i=1,2,\ldots,2q$	$[T_1] = [A]$		
	$[T_{k+1}] = [Q_k]^T[T_k][Q_k]$; $k = 1,2,\ldots,s$		
	$[\lambda_i] = [T_s]$	$80q^2$	
	$([A]-\lambda_i[I])\{y_i^{k+1}\} = \{y_i\}^k$; $k = 1,2,\ldots,t$		
	$\{y_i\} = \{y_i^t\}$		
6. Recovery of physical eigenvalues and eigenvectors; $i=1,2,\ldots 2q$	$\omega_i^2 = \frac{s}{\lambda_i}-\alpha$		
	$\{\phi_i\} = [L^{-1}]^T[V]\{y_i\}$	$2nq^2$	
Total operations for q lowest eigenvalues and associated eigenvectors, assuming two reorthogonalizations per trial vector and two Q-R iterations of the reduced, tridiagonal matrix		$\approx n[\frac{1}{2}b^2 + 4q(b+5+1) + 10q^2+2t] + 80q^2$	

*Minimum core storage = (15,000 + 7n) central memory words

TABLE 2

OPERATION COUNT AND STORAGE COMPARISONS FOR ALTERNATE EIGENSOLUTION METHODS

Method	Approximate Operation Count (Significant Terms)	Approximate Storage Requirements
Givens' Method - NASTRAN, plus Q-R iterations (all eigenvalues and q eigenvectors obtained)	$\frac{2}{3}n^3 + n^2(q + 10) + n\left(q(9 + \bar{b}) + \frac{1}{2}\bar{b}^2\right) + 4b\bar{b}$	$O(n^2)$
Inverse Power Method - NASTRAN (assuming one shift per root and seven iterations per eigenvector)	$nq\left(\frac{1}{2}b^2 + 15(b + \bar{b}) + 7q - 5\right)$	$O(nb)$
Gupta's Sturm Sequence method (assuming twelve iterations per root)	$25nb^2q$	
Subspace Iteration Method (assuming eight iterations and one Q-R eigensolution per iteration)	$n\left(b^2 + 32q(b + \bar{b}) + 64q^2 + 8b + \bar{b}\right) + 40q^2(q + 16)$	
Tridiagonal Reduction Method - Program FEER (assuming two reorthogonalizations per trial vector and two Q-R iterations of the reduced, tridiagonal matrix)	$n\left(\frac{1}{2}b^2 + 4q(b + \bar{b}) + 10q^2 + 2\bar{b}\right) + 80q^2$	

TABLE 3

EXECUTION TIME (CPU SECONDS): FEER vs. NASTRAN READ MODULE

Problem No.	Description	Unrestrained Degrees of Fredom	Average Semi-Band Width	No. of Accurate Modes Obtained	CPU Execution Time, Sec.*					
					CDC 6500 - 145 K_8			CDC 6600 - 145 K_8		
					FEER	NASTRAN Level 12.1	NASTRAN Level 15.1	FEER	NASTRAN Level 12.1	NASTRAN Level 15.1
1	Simply Supported Plate	105	30	16	11	790/INV				
2	Simply Supported Plate	105	30	35	17	58/GIV				
3	Free-Free Plate	130	30	10	12			7		91/INV
4	Engine Mount	195	20	7	12		95/INV			
5	Wing/Fuselage	195	25	4			63/INV			
6	Clamped Plate	225	51	10	26					
7	Control Surface	322	37	12	34		250/INV**			
8	Commodore Jet-Total Aircraft	456	30	14	42	1,044/INV				
9	Delta Wing	485	90	30	213					
10	Wing/Fuselage	594	55	1			455/INV			
11***	Ships Deck	1,028		25					4,752/ GUYAN+GIV	
12		1,056	102	10	235					
13	Clamped Plate	1,056	102	30	616					
14		3,072	102	20				362		
15		4,128	102	7				244		

* INV - Inverse Power Method with Shifts
GIV - Givens' Method
GUYAN + GIV - Guyan Reduction followed by Givens' Method.

** Only one mode extracted by NASTRAN at this time limit.

*** Reference 25.

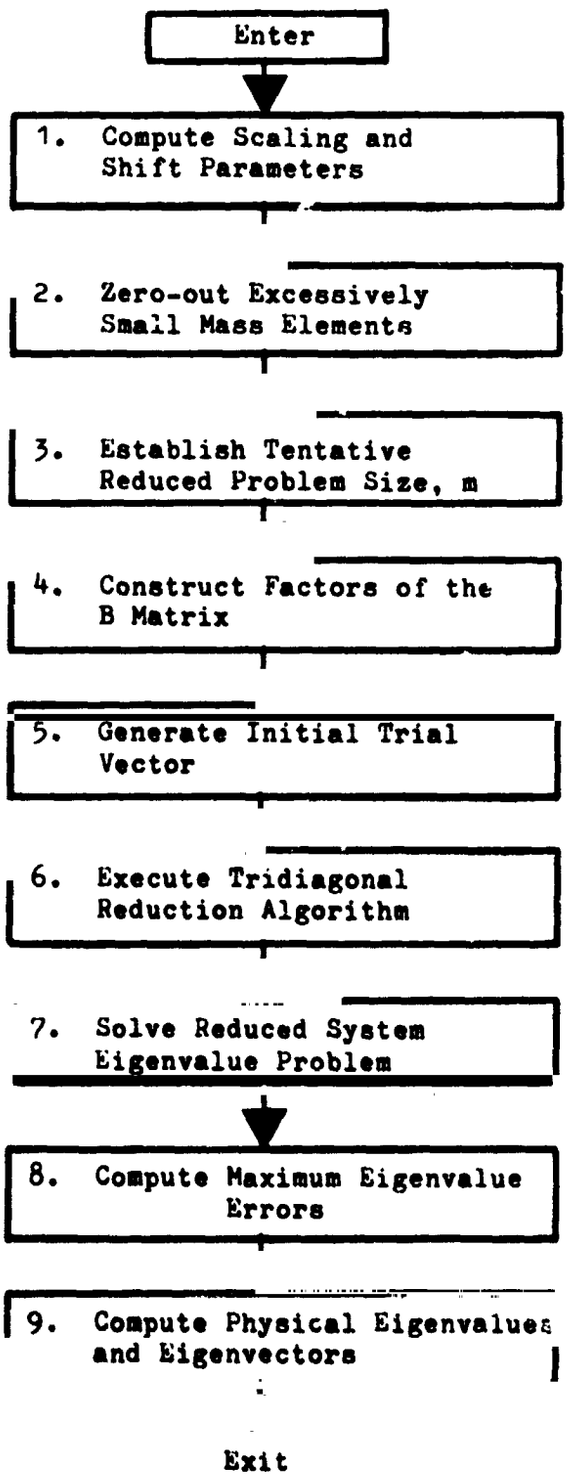

Figure 1. - Overall flow diagram for tridiagonal reduction method

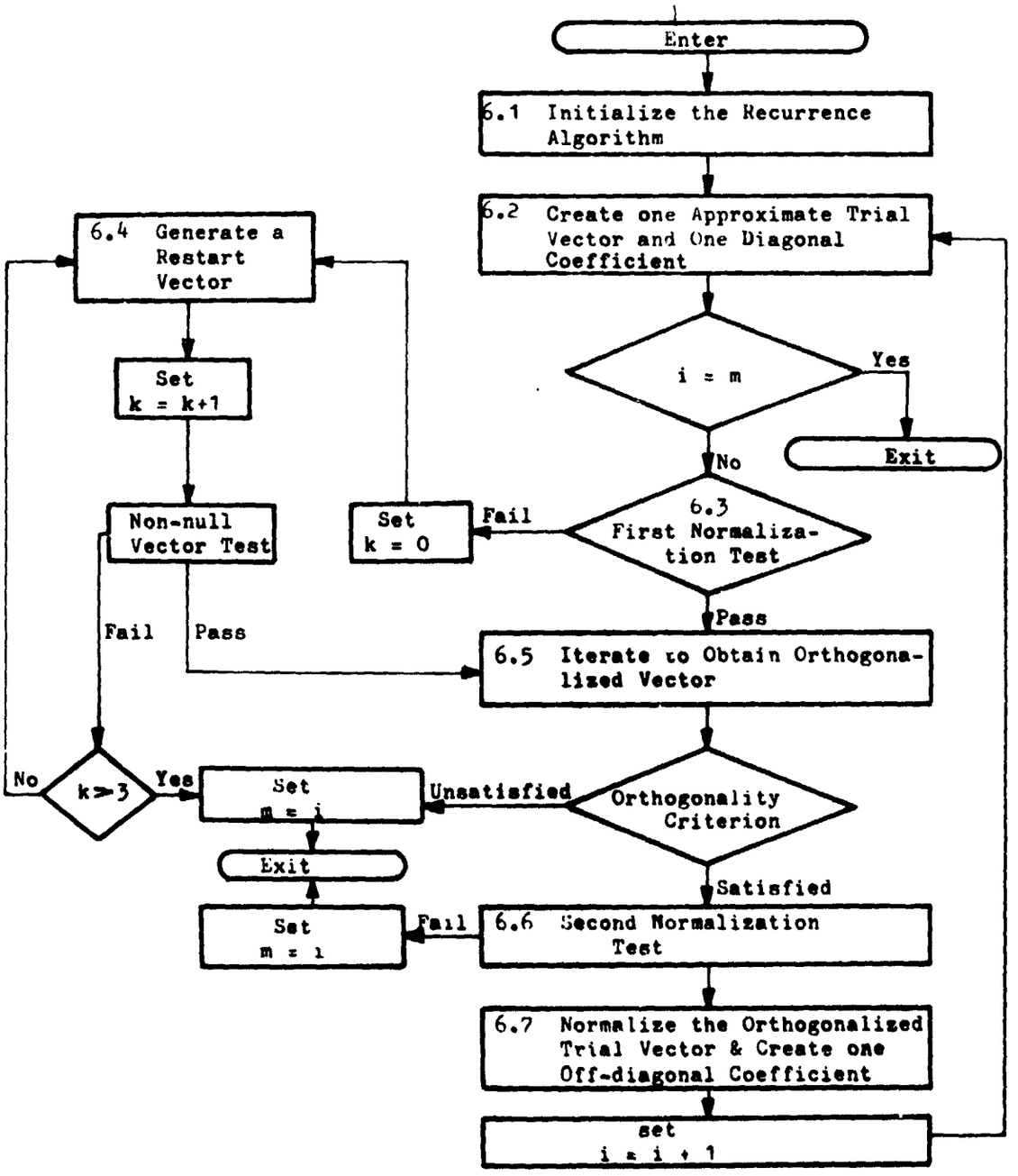

Figure 2. - Flow diagram for block 6, execute tridiagonal reduction

IN THE FOLLOWING TABLE 21 LOWEST EIGENVALUES ARE ACCURATE TO WITHIN A MAXIMUM ERROR OF .01008 PERCENT

MODE NO.	EIGENVALUE	RADIANS	CYCLES	GENERALIZED MASS	GENERALIZED STIFFNESS	PERCENT ERROR IN EIGENVALUE
1	.1516298E+07	.1231381E+04	.1959863E+03	.5957395E-03	.9033186E+03	.3605223E-26
2	.6303399E+07	.2510657E+04	.3995835E+03	.6269459E-03	.3951890E+04	.8112854E-25
3	.6303399E+07	.2510657E+04	.3995835E+03	0.	0.	.5102502E-28
4	.1365802E+08	.3695675E+04	.5881858E+03	0.	0.	.9802889E-25
5	.2025778E+08	.4500864E+04	.7163348E+03	0.	0.	.1685580E-26
6	.2045452E+08	.4522668E+04	.7198049E+03	0.	0.	.5055913E-24
7	.3169457E+08	.5630148E+04	.8960659E+03	0.	0.	.4462429E-20
8	.3169457E+08	.5630148E+04	.8960659E+03	0.	0.	.1171522E-20
9	.5186382E+08	.7201654E+04	.1146179E+04	0.	0.	.4522379E-11
10	.5186382E+08	.7201654E+04	.1146179E+04	0.	0.	.1861807E-11
11	.5613077E+08	.7492647E+04	.1192396E+04	0.	0.	.6991707E-20
12	.6821361E+08	.8259153E+04	.1314485E+04	0.	0.	.1963082E-15
13	.6879773E+08	.8294438E+04	.1320101E+04	0.	0.	.4804483E-16
14	.1015335E+09	.1007638E+05	.1603706E+04	0.	0.	.3552703E-08
15	.1015556E+09	.1007748E+05	.1603881E+04	0.	0.	.1346240E+01
16	.1116596E+09	.1056691E+05	.1681776E+04	0.	0.	.6173331E-07
17	.1118599E+09	.1057629E+05	.1683269E+04	0.	0.	.1709451E-07
18	.1349183E+09	.1161543E+05	.1848652E+04	0.	0.	.1786617E-07
19	.1586191E+09	.1259441E+05	.2004462E+04	0.	0.	.1326203E-05
20	.1780326E+09	.1334289E+05	.2123586E+04	0.	0.	.5758552E-03
21	.1791420E+09	.1338439E+05	.2130192E+04	0.	0.	.8491525E-02
22	.2137861E+09	.1461869E+05	.2326637E+04	0.	0.	.1601388E-02
23	.2444191E+09	.1563398E+05	.2488209E+04	0.	0.	.8689251E+00
24	.2476670E+09	.1573744E+05	.2504691E+04	0.	0.	.8118341E+01
25	.2582427E+09	.1591906E+05	.2517692E+04	0.	0.	.1519093E+01
26	.3006779E+09	.1734075E+05	.2759757E+04	0.	0.	.1230935E+00
27	.3637937E+09	.1907338E+05	.3035622E+04	0.	0.	.3486279E+01
28	.3812355E+09	.1952525E+05	.3107540E+04	0.	0.	.4125696E+01
29	.4131341E+09	.2032590E+05	.3234951E+04	0.	0.	.1852755E+01
30	.5212692E+09	.2283132E+05	.3633718E+04	0.	0.	.1102166E+02
31	.6037141E+09	.2457839E+05	.3911500E+04	0.	0.	.7405199E+01
32	.7102992E+09	.2665144E+05	.4241708E+04	0.	0.	.1292237E+02
33	.8542059E+09	.2922805E+05	.4651590E+04	0.	0.	.2118042E+02
34	.1046307E+10	.3234666E+05	.5148131E+04	0.	0.	.2159101E+02
35	.1373608E+10	.3706221E+05	.5898634E+04	0.	0.	.3831538E+02
36	.1813366E+10	.4258363E+05	.6777395E+04	0.	0.	.4064077E+02
37	.2567842E+10	.5067388E+05	.8064999E+04	0.	0.	.4085212E+02
38	.4030054E+10	.6348271E+05	.1010359E+05	0.	0.	.4964692E+02
39	.7178549E+10	.8472232E+05	.1348461E+05	0.	0.	.6007299E+02
40	.1564334E+11	.1250733E+06	.1990644E+05	0.	0.	.9484936E+02
41	.5015635E+11	.2239561E+06	.3564373E+05	0.	0.	.1162810E+03

Figure 3. - Extraction of 21 accurate roots via Program FEER; CDC - 6600 Computer

Order of stiffness matrix = 3,072
Average semi-band width = 102
No. of vectors generated = 2
CPU time = 362 sec.

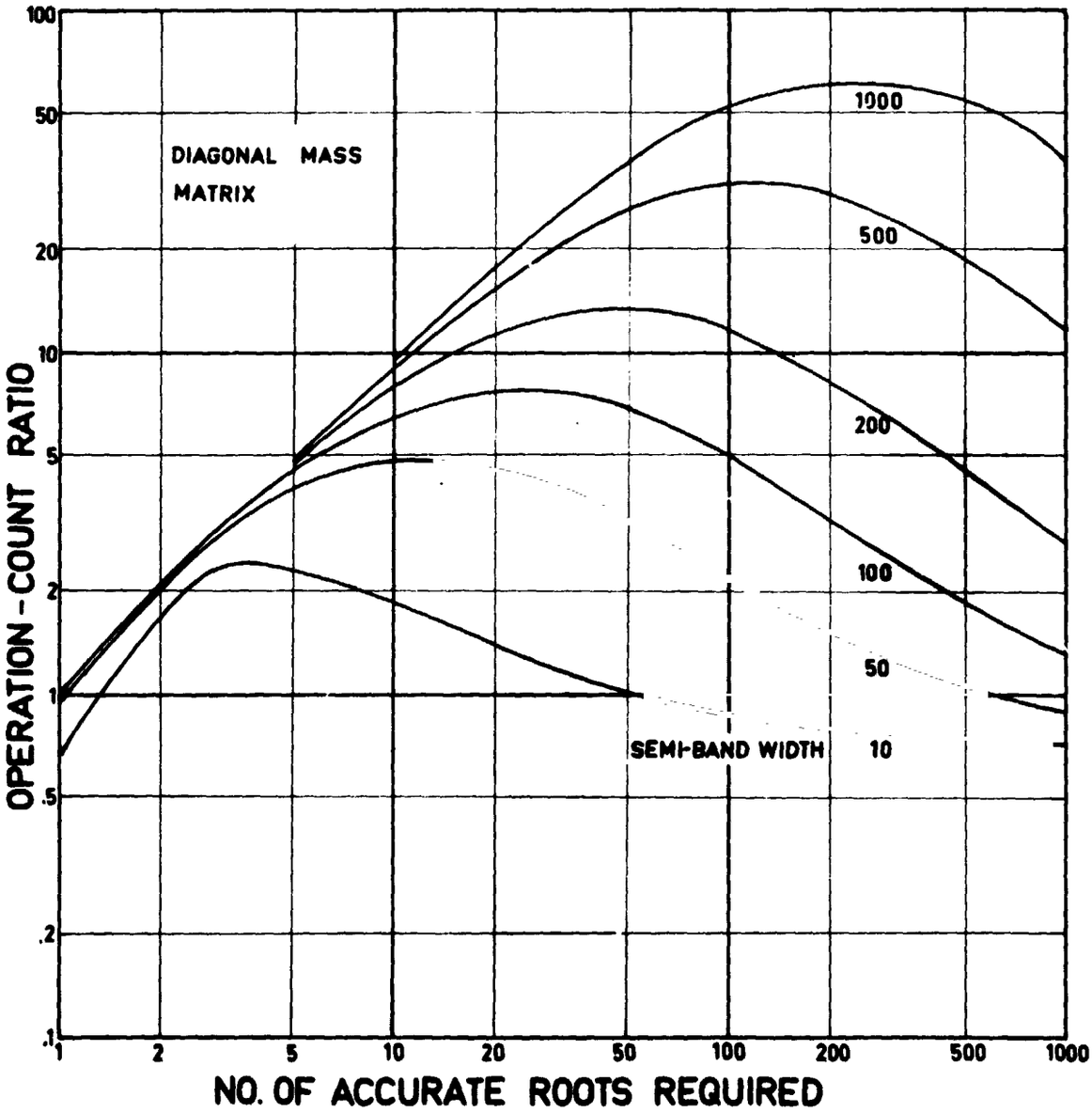

FIGURE 4.- THEORETICAL OPERATION-COUNT RATIO (INVERSE POWER METHOD/FEER)-DIAGONAL MASS MATRIX; INDEPENDENT OF PROBLEM SIZE FOR N≥200

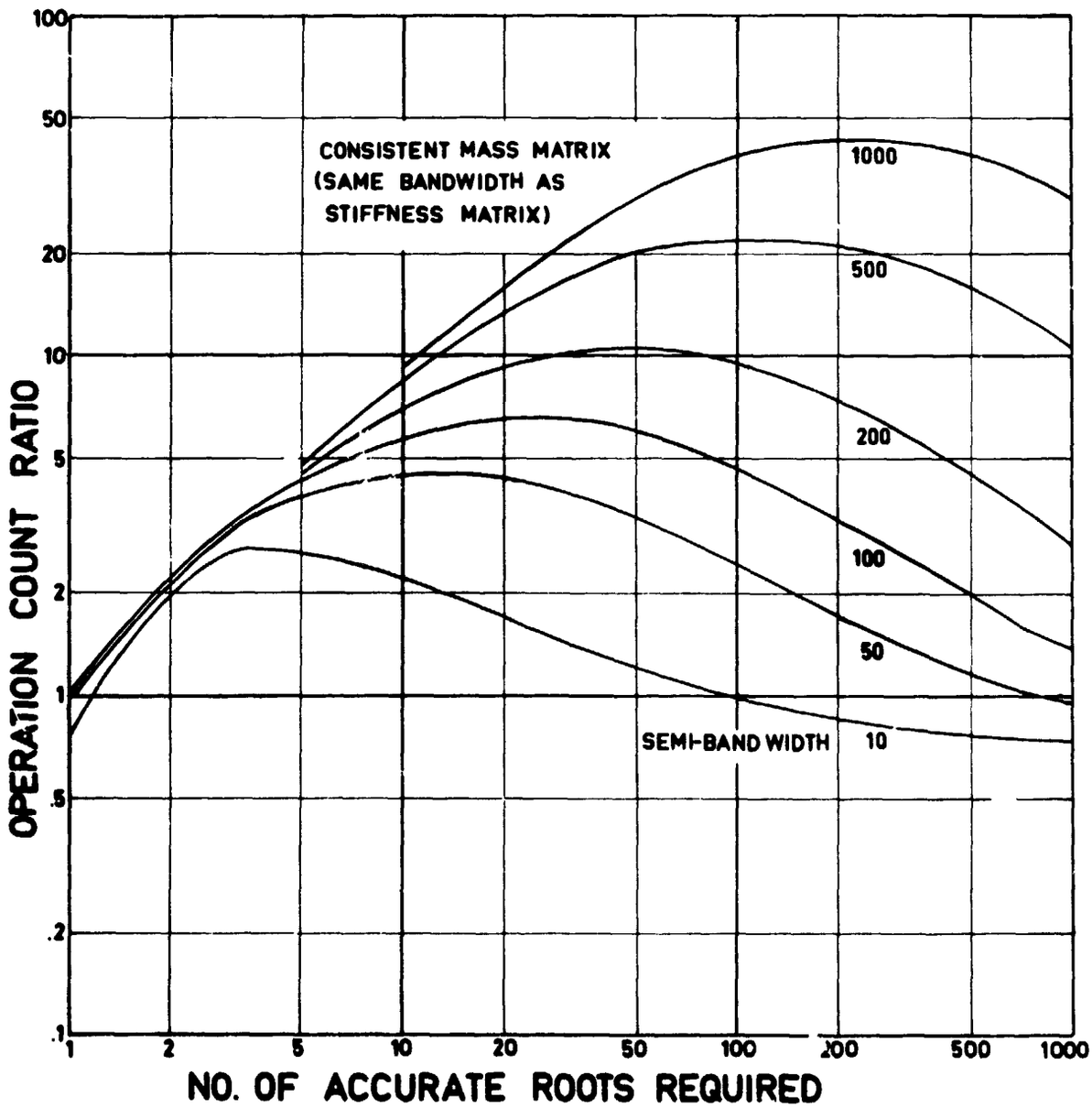

FIGURE 5 – THEORETICAL OPERATION-COUNT RATIO (INVERSE POWER METHOD/ FEER) – CONSISTENT MASS MATRIX; INDEPENDENT OF PROBLEM SIZE FOR N≥200