APPLICATION OF ANTIRESONANCE THEORY TO HELICOPTERS

Felton D. Bartlett, Jr.
Research Engineer

William G. Flannelly
Senior Staff Engineer

Kaman Aerospace Corporation
Bloomfield, Connecticut

Abstract

Antiresonance theory is the principle underlying nonresonant nodes in a structure and covers both nonresonant nodes occurring naturally and those introduced by devices such as dynamic absorbers and antiresonant isolators. The Dynamic Antiresonant Vibration Isolator (DAVI) developed by Kaman Aerospace Corporation and the Nodal Module developed by the Bell Helicopter Company are specific examples of the applications of transfer antiresonances. A new and convenient technique is presented to numerically calculate antiresonant frequencies. It is shown that antiresonances are eigenvalues and that they can be determined by matrix iteration.

Novel applications of antiresonance theory to helicopter engineering problems, using the antiresonant eigenvalue equation introduced in this paper, are suggested.

Notation

f force vector

K stiffness matrix

M mass matrix

y response vector

Z impedance matrix

θ antiresonant eigenvector

ω forcing frequency

ω_r antiresonant frequency

Presented at the AHS/NASA-Ames Specialists' Meeting on Rotorcraft Dynamics, February 13-15, 1974.

In forced vibrations an antiresonance or "off-resonance node", is that frequency for which a system has zero motion at one or more points. A nodal point in a normal mode is a special case of an antiresonance. Driving point antiresonances have a readily grasped physical interpretation since they are the resonances of the system when it is restrained at the driving point. However, transfer antiresonances are not all real and, in general, have not been susceptible to analysis except in special cases. The eigenvalue equation for antiresonances used in this paper renders them as amenable to analysis as are resonances. The mathematics for analyzing resonances are conventional and well-known[1].

Although general analytical methods for transfer antiresonances were not heretofore commonly used, the existence of both driving point and transfer antiresonances in the forced vibration of a string were described by Lord Rayleigh[2]. The invention of the dynamic vibration absorber in 1909 gave antiresonances some practical engineering importance[3]. The absorber is an appendant dynamic system which has a driving point antiresonance at its fixed base natural frequency and it therefore reacts the forces at its base in the direction in which it acts. Isolating devices based on transfer antiresonances were not invented until this decade[4]. Sometimes natural fuselage transfer antiresonances for major hub excitations occurred near a critical point and at the proper frequency (e.g., the pilot's seat at blade passage frequency) by fortuity of helicopter design. Occasionally, engineers have manipulated transfer antiresonance frequencies and positions in design through lengthy trial-and-error response analyses. However, the industry has not used a direct analytical method for calculating the positions and frequencies of natural antiresonances.

Structures have antiresonances as an intrinsic "natural" property much as they have "natural" resonant frequencies. Natural transfer, or "off-diagonal", antiresonances are as important to structural dynamics engineering as are resonances. Unfortunately, many of the theorems which underly conventional analyses do not apply to transfer antiresonances. The antiresonant dynamical matrix is in general nonsymmetrical and therefore not positive definite. This results in both left-handed and right-handed eigenvectors which are unequal and require a new orthogonality condition for the calculation of successive eigenvectors. The antiresonant frequencies of the transfer antiresonance determinants are not necessarily real and the imaginary roots do not have a simple physical interpretation. These matters, along with the lack of an engineering eigenvalue formulation for antiresonances, may, in part, account for the relatively little attention given to natural antiresonances over the years.

Theory

The steady-state equations of motion for an undamped spring-mass system vibrating in the vicinity of equilibrium are:

$$(K - \omega^2 M)y = f \qquad (1)$$

where the impedance matrix is defined as

$$Z = [\partial f_i/\partial y_j] = (K - \omega^2 M) \qquad (2)$$

Let all the forces be zero except the force acting at the k-th generalized coordinate and further impose the restraint of zero motion for the j-th generalized coordinate. The resulting eigenvalues are jk antiresonances of Equation (1). Since Z is real and symmetric the antiresonance eigenvectors are real and the jk and kj antiresonance eigenvalues are real (positive or negative) and equal.

Partition Equation (1) so that the kj-th element of the impedance matrix appears in the lower right-hand corner.

$$\left[\begin{array}{c|c} Z_{mn} \quad \begin{matrix} m \neq k \\ n \neq j \end{matrix} & Z_{mj} \\ \hline Z_{kn} \quad n \neq j & Z_{kj} \end{array}\right] \left\{\begin{array}{c} y_\ell \\ \ell \neq j \\ \hline 0 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline f_k \end{array}\right\} \qquad (3)$$

where

$$\left[\begin{array}{c|c} Z_{mn} \quad \begin{matrix} m \neq k \\ n \neq j \end{matrix} & \begin{matrix} Z_{mj} \\ m \neq k \end{matrix} \\ \hline Z_{kn} \quad n \neq j & Z_{kj} \end{array}\right] \equiv \left[\begin{array}{c|c} Z_A & Z_C \\ \hline Z_R & Z_{kj} \end{array}\right] \qquad (4)$$

If the impedance matrix is similarly partitioned so that the upper left-hand matrix does not contain the j-th row or the k-th column, then

$$Z = \left[\begin{array}{c|c} \tilde{Z}_A & \tilde{Z}_C \\ \hline \tilde{Z}_R & Z_{jk} \end{array}\right] \qquad (5)$$

It follows from Equations (4) and (5) that

$$Z_A = \tilde{Z}_A^T \qquad (6)$$

From Equation (3) we obtain

$$Z_A y = 0 \qquad (7)$$

A kj antiresonance is defined such that for a force at k alone, the response at j is zero. Normalizing y and substituting for Z_A in Equation (7) results in the antiresonance eigenvalue equation.

$$M_A \theta_r = \frac{1}{\omega_r^2} K_A \theta_r \qquad (8)$$

A jk antiresonance eigenvalue equation is similarly defined by considering Equation (5) and making use of Equation (6).

$$\tilde{\theta}_s^T M_A = \frac{1}{\omega_s^2} \tilde{\theta}_s^T K_A \qquad (9)$$

Equations (8) and (9) constitute a set of right-handed and left-handed eigenvectors. Since Z_A is not symmetrical, the jk eigenvectors are not orthogonal but instead are biorthogonal with the kj eigenvectors[1]. Premultiply Equation (8) by $\tilde{\theta}_s^T$, postmultiply Equation (9) by θ_r, and subtract to obtain

$$\left(\frac{1}{\omega_r^2} - \frac{1}{\omega_s^2}\right) \tilde{\theta}_s^T K_A \theta_r = 0 \qquad (10)$$

when $s \neq r$ we have

$$\tilde{\theta}_s^T K_A \theta_r = 0 \qquad (11)$$

Thus, the kj antiresonance eigenvector is biorthogonal to the jk antiresonance eigenvector.

When s = r the corresponding generalized mass and stiffness are defined as

$$\tilde{\theta}_r^T M_A \theta_r = M_r \quad (12)$$

$$\tilde{\theta}_r^T K_A \theta_r = K_r \quad (13)$$

Successive antiresonance eigenvectors are found by applying the biorthogonality condition and using classical matrix iteration techniques. The (n + 1)st jk antiresonant eigenvector is obtained from Equation (14),

$$\left(K_A^{-1} - \sum_{i=1}^{n} \frac{\theta_i \tilde{\theta}_i^T}{K_i}\right) M_A \theta_{n+1} = \frac{1}{\omega_{n+1}^2} \theta_{n+1} \quad (14)$$

which establishes the method of sweeping.[5]

Discussion of Theory

Each antiresonant eigenvector consists of a pair which is biorthogonal with respect to both mass and stiffness. For driving point antiresonances (j = k), the two eigenvectors are, obviously, the same. An N-degree-of-freedom system has N^2 possible antiresonant eigenvectors corresponding to all possible forcing and response coordinates.

Since the mass and stiffness matrices are nonsymmetric in the antiresonance eigenvalue problem and consequently not positive definite when $j \neq k$, the antiresonant generalized masses and stiffnesses may be either positive or negative. In other words, the antiresonance frequencies are not necessarily real. When j = k the antiresonant mass and stiffness matrices are symmetrical and positive definite, resulting in at least N-1 positive real antiresonances. As shown in Reference 6 the driving point antiresonances lie between the natural resonant frequencies.

Applications of Antiresonance Theory

To illustrate the practical potential of antiresonance theory, consider a ten-degree-of-freedom beam specimen with springs to ground at stations 3 and 9 and mass and stiffness parameters simulating a 9000 pound helicopter. Antiresonances are continuous functions of frequency and position and Figure 1 presents a typical position spectrum plot of the specimen forcing at station 3 alone. The dashed vertical lines are the natural resonant frequencies determined conventionally. When an antiresonance line crosses a natural frequency line there is a nodal point in the "natural mode".

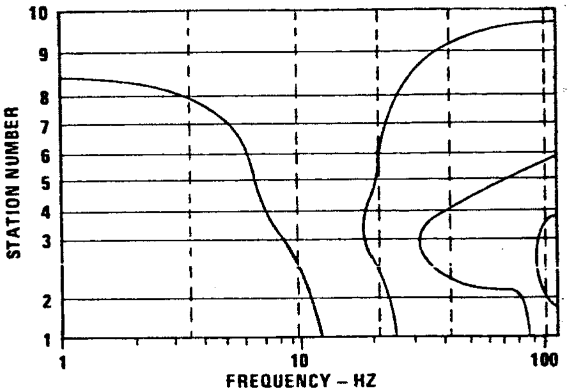


Figure 1. Antiresonance Lines Forcing at Station 3

With the same techniques of altering masses and stiffnesses to avoid undesirable natural resonances, the engineer can manipulate natural antiresonances. The stiffness between stations 2 and 3 was increased by 11.8% in the K_{23} term of the stiffness matrix and Figure 2 illustrates this effect in the natural frequencies and antiresonance lines. Similar changes in the mass of the structure have a similar effect. This possibility for response control indicates a profitable area for further exploration.

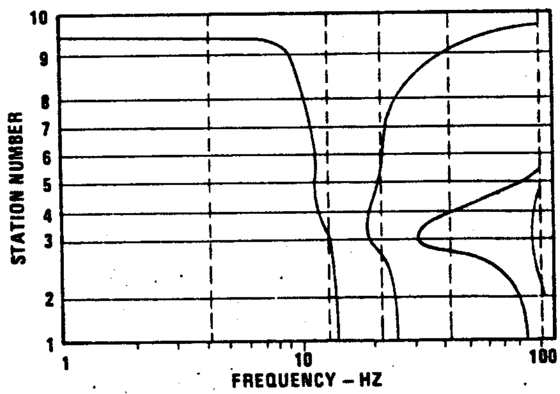
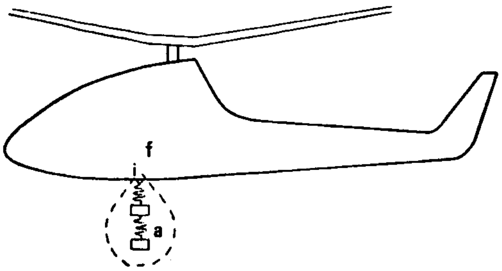


Figure 2. Antiresonance Lines with Stiffness Change Forcing at Station 3

Conventional Use of the Dynamic Absorber

A dynamic absorber is an appendant dynamic system attached to a helicopter, usually at a point, as shown in Figure 3. When we eliminate the i-th row and column, corresponding to the attachment point (see Figure 3) we obtain two uncoupled systems.

$$\begin{bmatrix} Z_{ff} & 0 \\ 0 & Z_{aa} \end{bmatrix} \begin{Bmatrix} Y_f \\ Y_a \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \qquad (15)$$



$$\begin{bmatrix} Z_{ff} & Z_{fi} & 0 \\ Z_{if} & Z_{ii} & Z_{ia} \\ 0 & Z_{ai} & Z_{aa} \end{bmatrix} \begin{Bmatrix} Y_f \\ Y_i \\ Y_a \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \\ 0 \end{Bmatrix}$$

Figure 3. Conventional Absorber

The antiresonant eigenvalue equation is obtained from Equation (15) as

$$[K_{aa}^{-1} M_{aa}]\theta_r = \frac{1}{\omega_r^2}\theta_r \qquad (16)$$

which is of the form of Equation (7).

If the absorber system were attached at I points, instead of one, we would eliminate the I rows and columns corresponding to the attachments and find the simultaneous antiresonant frequencies of all I points.

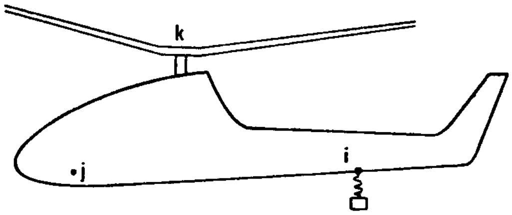
Unconventional Use of the Dynamic Absorber

In some instances there may be only one significant unreacted force on the helicopter as, for example, when an in-plane isolation system or in-plane hub or flapping absorbers leave small hub moments but a relatively large vertical oscillatory force. We can use a dynamic absorber in the fuselage at some point i as a "resonator" to shift antiresonance lines so that there exists an antiresonance at another point j (e.g., the pilot's seat) for the one remaining large force or moment along the k-th generalized coordinate. This is creating a jk antiresonance by manipulation of a "resonator" at point i. The jk antiresonant frequency is not at the tuned frequency of the "resonator" and does not necessarily produce an antiresonance at j for excitations along generalized coordinates other than k.

The aforementioned system and equations of motion are shown in Figure 4. To obtain an antiresonance at j for a force at k we eliminate the k-th row and j-th column from the equations of motions. This results in the antiresonant eigenvalue equation,

$$\begin{bmatrix} Z_{ff'} \;\; {\scriptstyle f'\neq j,\; f\neq k} & Z_{fi} \;\; {\scriptstyle f\neq k} & 0 \\ Z_{if'} \;\; {\scriptstyle f'\neq j} & Z_{ii} & Z_{ia} \\ 0 & Z_{ai} & Z_{aa} \end{bmatrix} \begin{Bmatrix} \theta \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \qquad (17)$$

which is of the form of Equation (7).



$$\begin{bmatrix} Z_{ff^1} \;\; {\scriptstyle f\neq k,\; f^1\neq j} & Z_{fj} \;\; {\scriptstyle f\neq k} & Z_{fi} \;\; {\scriptstyle f\neq k} & 0 \\ Z_{kf^1} \;\; {\scriptstyle f^1\neq j} & Z_{kj} & Z_{ki} & 0 \\ Z_{if^1} \;\; {\scriptstyle f^1\neq j} & Z_{ij} & Z_{ii} & Z_{ia} \\ 0 & 0 & Z_{ai} & Z_{aa} \end{bmatrix} \begin{Bmatrix} Y_{f^1} \\ Y_j \\ Y_i \\ Y_a \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_k \\ 0 \\ 0 \end{Bmatrix}$$

Figure 4. Antiresonance at Station j from a Resonator at Station i

This technique of using a remote dynamic absorber as a "resonator" allows the engineer to obtain an antiresonance, to a given excitation, at points where structural limitations prevent installation of an absorber. When the new resonant frequency introduced by the "resonator" cuts across a natural antiresonance line, the shifts are dramatic as shown in Figure 5. Figure 5 illustrates the antiresonance lines in the specimen, forcing at station 3, when an absorber of 77.2 pounds tuned to 7.7 Hz is added to station 2. The natural frequency introduced by the absorber intersects the antiresonance line of Figure 1 and produces new antiresonances at all stations, forcing at station 3.

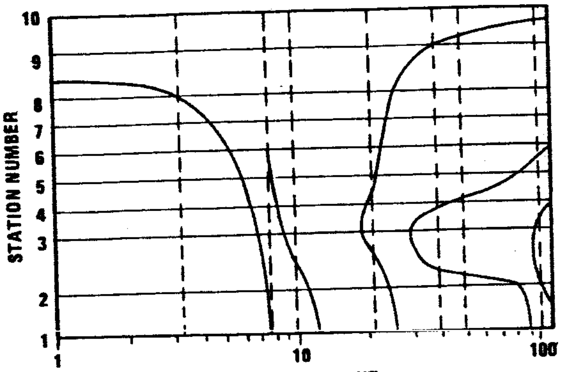


Figure 5. Antiresonance Lines with Dynamic Absorber at Station 2, Forcing at Station 3

The effect at station 5 of the 77.2 pound absorber located at station 2 and tuned to 7.7 Hz, in terms of both antiresonant frequency and bandwidth, is the same as the effect produced by a 193 pound absorber located at station 5 itself and tuned to 8.0 Hz. Bandwidth is here defined as the difference between the antiresonance frequency and the nearest natural frequency. This comparison is presented in Figure 6. The approximately two to one reduction in absorber weight does not imply that such savings are always obtainable.

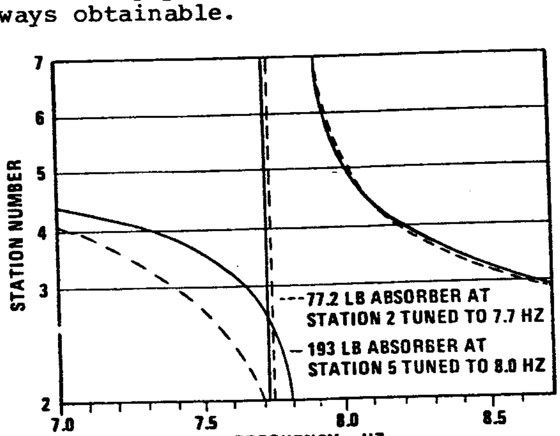
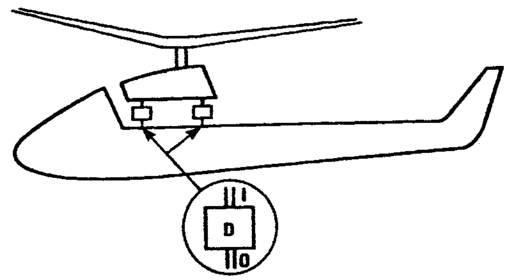


Figure 6. Comparison of Antiresonance Lines for Two Absorbers

Antiresonant Isolators

Passive antiresonant isolation devices have received considerable attention from the industry in recent years. Notable among these are Bell Helicopter's Nodal Module, Kaman's DAVI series, and the Kaman COZID.

Figure 7 illustrates the antiresonant isolation system and corresponding equations of motion. The excited structure



$$\begin{bmatrix} Z_{II} & Z_{ID} & Z_{IO} \\ Z_{DI} & Z_{DD} & Z_{DO} \\ Z_{OI} & Z_{OD} & Z_{OO} \end{bmatrix} \begin{Bmatrix} Y_I \\ Y_D \\ Y_O \end{Bmatrix} = \begin{Bmatrix} f_I \\ 0 \end{Bmatrix}$$

Figure 7. Antiresonance Isolation

is coupled to another structure through, and only through, the antiresonant isolation system which has inertial and elastic elements. Any isolator with a single input and single output, or a symmetrical arrangement having the same effect, has antiresonant frequencies given by the eigenvalues of

$$\begin{bmatrix} K_{DI} & K_{DD} \\ K_{OI} & K_{OD} \end{bmatrix}^{-1} \begin{bmatrix} M_{DI} & M_{DD} \\ M_{OI} & M_{OD} \end{bmatrix} \begin{Bmatrix} \theta \end{Bmatrix} = \frac{1}{\omega_A^2} \begin{Bmatrix} \theta \end{Bmatrix} \quad (18)$$

where I, O, and D represent the input, output and internal isolator degrees-of-freedom, respectively. The two-dimensional and three-dimensional DAVIs have, respectively, each two and three uncoupled equations of the form of Equation (18). Two outputs displaced with dynamic symmetry from a given input, or the converse, are also described by Equation (18) because the roots are not changed by transposing a matrix.

It is possible to solve for simultaneous antiresonances on arbitrarily placed multiple outputs for an equal number of arbitrarily placed multiple inputs by letting I and O be greater than one in Equation (18). However, such simultaneous antiresonances will, in the general case, occur only for those distributions of input forces given by the product of the rectangular impedance matrix of rows corresponding to the forced degrees of freedom and the vector of displacements. This is the reason why multiple input-output antiresonant isolators are not used in engineering. It is observed that the impedance matrix of

Figure 7, is, in general, nonsymmetric while the impedance matrix of Figure 3 is necessarily symmetric. That is the mathematically distinguishing feature between absorbers and antiresonant isolators.

It is obvious from Equation (18) that an infinite number of mechanical systems exist which will produce antiresonant transmissibilities at more than one frequency. Such systems can be analytically synthesized using desired antiresonant frequencies, the biorthogonality condition, and the methods of Reference 7. However, not all such synthesized systems will be physically realizable and not all of the physically realizable synthesized systems will be practical from an engineering standpoint.

An immediately practical application of Equation (18) would be the investigation of physical multi-input antiresonant isolators with internal coupling using simpler engineering arrangements for multi-harmonic antiresonances than has yet been achieved.

Conclusion

This paper has presented a solution to the antiresonant eigenvalue problem. It has been shown that antiresonances can be determined by matrix iteration techniques. Antiresonant nodes introduced by dynamic absorbers and antiresonant isolators have been discussed to illustrate the novel application of the theory to helicopter engineering problems.

References

1. Meirovitch, L., ANALYTICAL METHODS IN VIBRATIONS, McGraw-Hill Book Co., New York, 1967.

2. Strutt, J.W., Baron, Rayleigh, THE THEORY OF SOUND, 2nd Edition, Volume 1, Sec. 142a, Dover Publications, New York, 1945.

3. Den Hartog, J.P., MECHANICAL VIBRATIONS, 4th Edition, McGraw-Hill Publishing Co., New York, 1956.

4. Kaman Aircraft Report RN 63-1, DYNAMIC ANTIRESONANT VIBRATION ISOLATOR (DAVI), Flannelly, W.G., Kaman Aircraft Corporation, Bloomfield, Connecticut, November 1963.

5. Rehfield, L.W., HIGHER VIBRATION MODES BY MATRIX ITERATION, Journal of Aircraft, Vol. 9, No. 7, July 1972, p. 505.

6. Biot, M.A., COUPLED OSCILLATIONS OF AIRCRAFT ENGINE-PROPELLER SYSTEMS, Journal of Aeronautical Society, Vol. 7, No. 9, July 1940, p. 376.

7. USAAMRDL Technical Report 72-63B, RESEARCH ON STRUCTURAL DYNAMIC TESTING BY IMPEDANCE METHODS, Giansante, N., Flannelly, W.G., Berman, A., U. S. Army Air Mobility Research and Development Laboratory, Fort Eustis, Virginia, November 1972.