A B-B-G-K-Y Framework

For Fluid Turbulence

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ABSTRACT

A kinetic theory for fluid turbulence is developed from the Liouville equation and the associated BBGKY hierarchy. Real and imaginary parts of Fourier coefficients of fluid variables play the roles of "particles." Closure is achieved by the assumption of negligible five-coefficient correlation functions. Probability distributions of Fourier coefficients, rather than moments, are the basic variables of the theory. Neglect of the correlation functions does not necessarily imply a cumulant discard hypothesis or a quasi-normal assumption. However, a later additional approximation leads to a closed moment description similar to the so-called eddy-damped Markovian approximation. A kinetic equation is derived for which conservation laws and an H-theorem can be rigorously established. In the absence of viscosity and external driving forces, the H-theorem implies relaxation to the absolute equilibrium of Kraichnan. The equation can be cast in the Fokker-Planck form, and relaxation times estimated from its friction and diffusion coefficients. An undetermined parameter in the theory, about which significant assumptions have to be made, is the free decay time for triplet correlations. Some attention is given to the inclusion of viscous damping and external driving forces.

I. INTRODUCTION

Two major and distinct areas of investigation in non-equilibrium statistical mechanics over the last thirty years have been the BBGKY kinetic theory of gases[1-7] and the statistical theory of fluid turbulence[8-11]. It seems worthwhile to attempt to unify the two by considering the latter from the point of view of the former.

The purpose of this article is to present a framework for a systematic kinetic theory of turbulence originating from the Liouville equation[12] for the Fourier coefficients of the fluid variables. The real and imaginary parts of these Fourier coefficients play the role, in a somewhat abstract way, that particle coordinates (positions and momenta) play in the BBGKY theory. Early formulations in this direction are due to Edwards[13] and Herring[14,15], following general indications of Hopf.[16]

A related approach of some importance is due to Lundgren[17,18] (see also Monin[19] and Chung[20]). Lundgren's basic variables are configuration-space probabilities instead of Fourier-space distributions. Some apparent success using Lundgren's formulation is due to Fox,[21,22,23] who has reproduced the decay of an energy spectrum for grid turbulence.[24] The essential approximation in Fox's theory has been the neglect of the three point spatial correlation function, by analogy with the kinetic theory of plasmas. However, in the fluid context, the physical content of this approximation is quite obscure. The approximation is justified for the plasma case[7] by arguments based on thermal equilibrium theory which are lacking in the case of fluids.

One argument for the use of Fourier coefficients as dynamical variables is that, at least in the absence of viscosity, there is an absolute equilibrium distribution both in two and three dimensions (see Kraichnan[25,26] and Lee[12]). It can be said that relaxation to this absolute equilibrium can be considered to have been definitively established in two dimensions, by the numerical investigations of Seyler et al,[27,28] as the state to which the truncated inviscid Navier-Stokes system relaxes for arbitrary initial conditions. This equilibrium theory can be used as a guide for suggesting approximations for the non-equilibrium theory, as in the case of the BBGKY kinetic theory. No such equilibrium theory is available in the configuration-space representation.

It is our goal to introduce only quantities that can be given a sharp definition in terms of Liouville's equation and probability distributions (ensembles) which obey it. It is clear that at one level, everything about the statistics of the system must be obtainable from Liouville's equation. It is also our intent only to make approximations which appear to be in some well-defined sense experimentally testable; basically two will be required.

The approach is illustrated by considering what is algebraically the simplest case: two-dimensional, inviscid, incompressible Navier-Stokes flow. The methods are not restricted to this case and can also be applied to three-dimensional flow, to magnetohydrodynamic turbulence, and to cases when dissipation and external driving forces are present. Our intent is to introduce the <u>method</u> as simply as possible; detailed comparisons with data will have to await numerical solution of the kinetic equation to be derived, and additional refinements and revisions by other workers are to be expected.

The basic relations of the problem are the incompressible Navier-Stokes equations in two dimensions with zero viscosity:

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\nabla(p/\rho_M) ,$$

$$\nabla \cdot \underset{\sim}{u} = 0,$$

where $\underset{\sim}{u}$ is the fluid velocity, p is the pressure, and ρ_M is the (uniform) density. $\underset{\sim}{u}$ lies in a plane normal to the z-axis, and is a function of x and y only. It is convenient to work in the vorticity representation, for which

$$\underset{\sim}{\rho} = \rho \hat{b} = \nabla \times \underset{\sim}{u} .$$

$\hat{b}$ denotes a unit vector in the z-direction. All variables are represented as Fourier series in a large but finite box. Both $\underset{\sim}{u}$ and p are eliminated in favor of an equation involving ρ alone in the usual way (see Eq. (1) below). A well-known alternative interpretation of the equations, that they describe the electrostatic guiding-center plasma, is possible,[29] in which ρ is the electrostatic charge density and $\underset{\sim}{u}$ is the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocity of a fluid element.

In Sec. II, a notation is introduced which expresses neatly the dynamics of the real and imaginary parts of the Fourier coefficients in terms of a simple set of nonlinear coupled ordinary differential equations. A Liouville equation and BBGKY hierarchy can be derived in the phase space defined by these coefficients (Sec. III). Though the moments of

the probability distribution functions of the Fourier coefficients have been more popular as a set of dynamical variables for statistical theories of turbulence, our concern throughout will be with the distribution functions themselves.

Closure of the hierarchy is achieved in Sec. IV by the hypothesis that the five-coefficient correlation function is negligible. This closure is explicitly shown to include all cumulants in the moment hierarchy, and not to be a variant of the cumulant-discard, or quasi-normal, hypothesis.[30] The theory at this point owes a debt to the direct-interaction approximation of Kraichnan,[31-34] where the assumption is made that the three-coefficient correlations are more basic to the dynamics than any higher grouping. But we have not introduced the unit infinitesimal response matrix of Kraichnan, and have limited our apparatus to quantities which can be defined in terms of the single-time Liouville distribution. The kinetic equation derived in Sec. V shares, however, the non-Markovian property of the direct-interaction approximation: The time derivatives of the distribution functions depend not only upon their present values, but on their recent history over a time of the order of a decay time for triplet correlations. A second hypothesis (for which experimental evidence for or against appears to be lacking) will result in a Markovian kinetic equation: That the decay of three-coefficient correlations in the absence of their source terms is rapid compared to the evolution of the one-coefficient distributions. Numerous features of this Markovian kinetic equation (Sec. VI) can be rigorously established, including that of a monotonic approach to the absolute equilibrium distribution of Kraichnan (H-theorem), a result believed to be the first of its kind. With no further approximation,

the kinetic equation can be shown to be of a standard (nonlinear) Fokker-Planck type. The Fokker-Planck equation is of the kind derived from the Langevin equation for Brownian motion,[35] with the difference that the friction and diffusion coefficients are time-dependent integrals of the distributions of the other Fourier coefficients. At this level, a closed description in terms of low-order moments is possible, but the Fokker-Planck equation provides information on moments of arbitrarily high order. A relaxation time can be estimated from the Fokker-Planck equation, to be compared with the decay times for triplet correlations, when the latter become available.

Only two significant hypotheses or approximations (beyond the standard but significant one of the representation of the fluid variables as truncated Fourier series) are employed, and both involve only well-defined quantities for the system:

(1) Neglect of the quintuplet correlation and a subsequent assumption concerning the mode of relaxation of the triplet correlation; and (2) the assumption that the decay of the triplet correlation is rapid compared to the evolution of the single-coefficient distributions. A significant fraction of the theory (specifically, everything up to Eq. (32)) can be based on the first hypothesis alone, and we re-emphasize that this is not quite a cumulant discard hypothesis. The second assumption is analogous to the Bogolyubov[5] "functional assumption." Both are subject to some degree of experimental verification or disproof.

Section VII discusses the problems associated with the inclusion of viscosity and external driving forces. Section VIII summarizes the results and anticipates future inquiries.

II. BASIC DYNAMICAL DESCRIPTION

For our purposes, it is useful to write the Fourier-transformed, two-dimensional, inviscid, Navier-Stokes equation in the vorticity representation:

$$\frac{\partial \rho\ (\underset{\sim}{k},t)}{\partial t} = \sum_{\underset{\sim}{p}+\underset{\sim}{r}=\underset{\sim}{k}} M(\underset{\sim}{r},\underset{\sim}{p})\ \rho\ (\underset{\sim}{r},t)\ \rho\ (\underset{\sim}{p},t)\ , \tag{1}$$

where the coupling coefficients between the various Fourier modes are given by

$$M(\underset{\sim}{r},\underset{\sim}{p}) = M(\underset{\sim}{p},\underset{\sim}{r}) = \frac{\hat{b} \cdot (\underset{\sim}{p} \times \underset{\sim}{r})}{2} \left\{ \frac{1}{r^2} - \frac{1}{p^2} \right\} \tag{2}$$

If the fluid velocity is $\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{x},t)$, the vorticity vector is $\underset{\sim}{\rho}(\underset{\sim}{x},t) = \nabla \times \underset{\sim}{u} = \hat{b}\rho(\underset{\sim}{x},t)$, where $\hat{b}$ is a unit vector lying in the z-direction, say. $\rho(\underset{\sim}{x},t)$ is written as a Fourier series

$$\rho(\underset{\sim}{x},t) = \Sigma_{\underset{\sim}{k}}\ \rho(\underset{\sim}{k},t)\ \exp\ (i\underset{\sim}{k} \cdot \underset{\sim}{x})$$

and $\rho(\underset{\sim}{k},t) = L^{-2} \int d\underset{\sim}{x}\ \rho(\underset{\sim}{x},t)\ \exp(-i\underset{\sim}{k} \cdot \underset{\sim}{x})$, and the integral runs over a large square box of edge L. The vectors $\underset{\sim}{u}$, $\underset{\sim}{x}$, $\underset{\sim}{k}$, $\underset{\sim}{r}$, $\underset{\sim}{p}$ all lie in the xy plane, and are normal to $\hat{b}$. The allowed values of the wave numbers $\underset{\sim}{k}$, $\underset{\sim}{p}$, $\underset{\sim}{r}$ are $2\pi\underset{\sim}{n}/L$, where $\underset{\sim}{n}$ is a non-null vector (n_x,n_y) with integer components. The wave numbers' magnitudes are bounded from below by virtue of the finite size of the box and from above by a large but finite maximum, which remains somewhat arbitrary. Extensive numerical studies of the relaxation of Eqs. (1)

were carried out by Seyler et al,[27] starting from highly non-thermal initial conditions.

It is the statistical theory of Eqs. (1) which is of interest, but it is convenient to introduce a more compact notation (see, e.g., Kraichnan,[31] Betchov,[36] or Herring[14]). We order the wave numbers by associating them in any convenient way with the integers and represent them by Roman subscripts i, j, k, l, ... There are to be two integers per wave number. By the variables X_i, X_j, X_k, ..., we shall mean the real and imaginary parts of the various $\rho(\underset{\sim}{k},t)$ associated with the ordered wave numbers.

The system of equations represented by Eq. (1) can then be written in the simple-looking form

$$\frac{d X_i}{dt} = \sum_{jk} C_{ijk} X_j X_k , \tag{3}$$

where the C_{ijk} are a set of real constant coefficients which can be inferred in terms of the $M(\underset{\sim}{r},\underset{\sim}{p})$ from Eq. (1). All terms in Eq. (3) are now real. The subscripts range from 1 to N, say, where N>>1. N is the number of allowed wave numbers. (There are 2N total real and imaginary parts of Fourier coefficients, but only half the variables are independent, since $\rho(\underset{\sim}{k},t) = \rho^*(-\underset{\sim}{k},t)$.) Only variables which can be prescribed independently initially are included in Eq. (3).

For detailed numerical applications, the exact values of the C_{ijk} coefficients can be read off, but for the formal theory to be presented here, only five properties of the C_{ijk} are needed:

(1) All $C_{ijk} = 0$ unless i, j, k correspond to different wave numbers (i.e., unless i, j, k are all different);

(2) $C_{ijk} = C_{ikj}$;

(3) i, j, and k all run from 1 to N;

(4) $C_{ijk} + C_{jki} + C_{kij} = 0$;

(5) $\frac{C_{ijk}}{|i|^2} + \frac{C_{jki}}{|j|^2} + \frac{C_{kij}}{|k|^2} = 0.$

Property (4) expresses conservation of enstrophy (or mean-square charge density, for the plasma case). Property (5) expresses conservation of energy. In property (5), $|i|^2$ is a symbolic notation which means the square of the wave number associated with the subscript i. (The constants of the motion guaranteed by properties (4) and (5) survive an essentially arbitrary Fourier truncation.)

Equations (3) may also be used to discuss the three-dimensional case, and much of the theory to be developed applies to the three-dimensional case as well. The major difference is that in three dimensions, the analogue of property (4) does not apply. The consequences that result if property (4) does not hold will be noted later.

The initial-value problem involves specifying the initial values of the X_j and following their subsequent evolution according to Eqs. (3). We turn now to the introduction of probabilistically distributed initial conditions.

III. LIOUVILLE'S EQUATION AND THE BBGKY HIERARCHY

The various X_i in Eq. (3) can be thought of a coordinates in a many-dimensional phase space. Eq. (3) apparently does not define a Hamiltonian system. It is, however, a conservative system and possesses two constants[25] of the motion as a consequence of properties (4) and (5) of Section II: The energy $\mathcal{E}$ and enstrophy Ω; these are

$$\mathcal{E} = \Sigma_i X_i^2/|i|^2 \tag{4}$$

$$\Omega = \Sigma_i X_i^2 . \tag{5}$$

Moreover, a Liouville equation, or probability conservation law, exists and governs the evolution of any probability distribution law $D = D(X_1, X_2 \ldots X_n, t)$ defined over the phase space:

$$\frac{\partial D}{\partial t} + \sum_i \frac{\partial}{\partial X_i}\left(\frac{dX_i}{dt} D\right) = 0 . \tag{6}$$

In Eq. (6), dX_i/dt is given by Eq. (3). Using property (1),

$$\frac{\partial D}{\partial t} + \sum_{ijk} C_{ijk} X_j X_k \frac{\partial}{\partial X_i} D = 0 . \tag{7}$$

Conservation of the positive semi-definite character of D and conservation of the normalization integral

$$\int dX_1 \ldots dX_N D = 1$$

follow immediately from Eq. (7).

A family of reduced probability distributions $f_{ijk...r} = f_{ijk...r}(X_i, X_j, X_k, \ldots X_r, t)$ can be defined by integrating D over all the phase space coordinates which are not of interest:

$$f_{ijk...r} \equiv \int D \left(dX_1 \, dX_2 \ldots dX_N \right)_{ijk...r} \tag{8}$$

where the notation in the integral means to integrate D over all the phase space variables except those associated with the indices ijk...r. The reduced distribution functions defined by Eq. (8) differ from those used in the conventional BBGKY theory[1-7] mainly in that they are not symmetric under interchanges of the different phase space coordinates. Thus $f_i(X_i,t)$ is not the same function of X_i that $f_j(X_j,t)$ is of X_j, if i and j are different. Most of the added complexity of the theory stems from this lack of symmetry. Note that the subscripts on the $f_{ijk...r}$ are by definition always different. There are never any repeated subscripts in the distributions.

The basic variables in most statistical theories of turbulence are moments, such as

$$\langle X_i^2 \rangle \equiv \int X_i^2 \, f_i \, dX_i$$

or

$$\langle X_i \, X_j \rangle \equiv \int X_i \, X_j \, f_{ij} \, dX_i \, dX_j \, .$$

However, in these present calculations, the variables to be treated as basic are the distributions f_i, f_{ij}, f_{ijk}, etc.

By integrating Eq. (7), discarding surface integrals at infinity in phase space, and using Eq. (8), a BBGKY hierarchy of equations for the reduced distributions can be readily generated. For example, the first two members are

$$\frac{\partial f_i}{\partial t} = -\frac{\partial}{\partial X_i} \sum_{jk} C_{ijk} \int dX_j \, dX_k \, X_j \, X_k \, f_{ijk} \tag{9}$$

and

$$\frac{\partial f_{ijk}}{\partial t} + \left(C_{ijk} \, X_j \, X_k \frac{\partial}{\partial X_i} + \text{c. p.} \right) f_{ijk}$$

$$= -\frac{\partial}{\partial X_i} \sum_{rs} C_{irs} \int dX_j \, dX_k \, f_{ijkrs} + \text{c. p.} \tag{10}$$

The notation "c. p." will always mean, add to the immediately preceding term the same term with the unsummed indices cyclically permuted, and continue doing this until the original ordering is reached. Thus

$$C_{ijk} \, X_j \, X_k \frac{\partial}{\partial X_i} + \text{c. p.} =$$

$$C_{ijk} \, X_j \, X_k \frac{\partial}{\partial X_i} + C_{jki} \, X_k \, X_i \frac{\partial}{\partial X_j} + C_{kij} \, X_i \, X_j \frac{\partial}{\partial X_k} . \tag{11}$$

As in the case of the BBGKY hierarchy for interacting point particles, the higher members of the hierarchy become complicated. So far, there has been no incentive to define a notation complicated enough to write down the general term. But its structure is obvious. For s phase space

coordinates, it will involve the Liouville equation for s coordinates on the left and a large number of similar integro-differential linear operators on the right, which act on distributions of s+2 coordinates. One interesting feature is that reduced distribution functions containing only odd numbers of the X_i's directly enter the hierarchy. Though the distributions of even numbers of coordinates can be obtained from the odd distributions by integration, they play no direct dynamical role. This contrasts with the all-important role that the pair correlation function plays in particle kinetic theory, and with Lundgren's configuration-space description.[17]

A cluster expansion can be used to replace the multibody distributions f_{ijk}, f_{ijkrs}, ..., by appropriately expressed correlation functions T_{ijk}, Q_{ijkrs}, ..., as follows:

$$f_{ijk} = f_i f_j f_k + T_{ijk} \tag{12}$$

$$\begin{aligned} f_{ijkrs} = {} & f_i f_j f_k f_r f_s + f_i f_j T_{krs} + f_i f_k T_{jrs} \\ & + f_i f_r T_{jks} + f_i f_s T_{jkr} + f_j f_k T_{irs} \\ & + f_j f_r T_{iks} + f_j f_s T_{ikr} + f_k f_r T_{ijs} \\ & + f_k f_s T_{ijr} + f_r f_s T_{ijk} + Q_{ijkrs} . \end{aligned} \tag{13}$$

The "quintuplet" correlation Q_{ijkrs} is the completely nonfactorable part of the f_{ijkrs}, and so forth.

We shall restrict attention to the case of vanishing first moments

$$\langle X_i \rangle \equiv \int X_i f_i \, dX_i = 0, \text{ all } i. \tag{14}$$

This corresponds to the conditions $\langle \rho \rangle = 0$, $\langle u \rangle = 0$, etc. in configuration space.[8] However, we do not impose all the conditions that have been sometimes associated with more restrictive definitions of "homogeneous turbulence." Also, it is not assumed that $\langle X_i^2 X_j^2 \rangle = \langle X_i^2 \rangle \langle X_j^2 \rangle$ when i and j are identified with different wave numbers, and we do not assume that the initial values of the univariate distributions are Gaussian.

From Eq. (14), it follows that an alternative way of writing Eq. (9) is

$$\frac{\partial f_i}{\partial t} = -\frac{\partial}{\partial X_i} \sum_{jk} C_{ijk} \int dX_j \, dX_k \, X_j \, X_k \, T_{ijk} \, . \tag{15}$$

From Eq. (9) or Eq. (15), it is clear that what is required for a kinetic description, in the kinetic theory sense[5,6,7] of the term, is an approximate expression for the f_{ijk} or T_{ijk} in terms of the f_i's. It is clear that some additional complexity (over and above that encountered in hierarchy derivations of Boltzmann's equation or the Fokker-Planck equation) is to be expected, since the different f_i's are distinct functions.

So far, everything that has been said is exact. Further progress depends upon developing satisfactory approximations with which to close or truncate the hierarchy.

IV. CLOSURE OF THE HIERARCHY

A hierarchy generated by taking moments of Eq. (1) or Eq. (3) has in the past been used as a starting point for statistical theories of turbulence.[9,10] Closure schemes for this moment hierarchy, based on discard of higher cumulants,[30] have largely fallen into disrepute.[9,10,37] This is in large part due to the negative spectral densities ($\langle X_1^2 \rangle$, in our notation) which result. It is not true, as we shall later prove explicitly, that closure schemes for the just-derived BBGKY hierarchy based on the neglect of one of the higher-order correlation functions (such as Q_{ijkrs}) are equivalent to the neglect of higher cumulants in the moment hierarchy. It will also be seen that neglect of the higher-order correlation functions need not, in general, lead to negative values of any intrinsically non-negative quantities such as $\langle X_1^2 \rangle$.

Most closure schemes for the interacting-particle BBGKY hierarchy rest on the identification of some small parameter which enters as a multiplicative factor in the hierarchy (e.g., the density for Boltzmann's equation or the coupling constant for the Fokker-Planck equation). This small parameter then serves as the basis for a perturbation expansion.[7] One of the more stubborn impediments in turbulence theory has been the lack of any such readily-identifiable small parameters. The only parameters which enter Eqs. (3), (7), (9), (10), or (15) are the coupling coefficients C_{ijk}, and they can be considered in no sense "small." In particular, the usual Fokker-Planck arguments (in which weak coupling is argued to imply weak correlations) would appear to be inappropriate.

Nevertheless, there is a sense in which the correlations may still be argued to be small. One exact solution to Eq. (7) is the absolute equilibrium[25,26] distribution of Kraichnan:

$$D_{eq.} = c \exp \{ -\alpha \mathcal{E} - \beta \Omega \} \tag{16}$$

where c is a normalizing constant, α and β are constants which play the role of inverse temperatures, and $\mathcal{E}$ and Ω are given by Eqs. (4) and (5). Equation (16) implies Maxwell distributions for the individual X_i's,

$$f_i^{eq.}(X_i) = c_i \exp\left\{ -\left(\frac{\alpha}{|i|^2} + \beta \right) X_i^2 \right\}, \tag{17}$$

and implies also that all correlations vanish identically. Extensive numerical evidence accumulated by Seyler et al[27] shows that in fact a quite definite relaxation of the properties of the system to those predicted by Eqs. (16) and (17) takes place. This relaxation to equilibrium seems to be beyond dispute.

If we are in some sense not too far from the absolute equilibrium, predicted by Eqs. (16) and (17), then it is clear that the correlation functions must be in some sense small, so that, for example, $f_i f_j f_k >> T_{ijk}$, $f_i f_j f_k f_r f_s >> Q_{ijkrs}$, and so forth. Dynamical arguments to support this conjecture are given later in this section.

The only assumption required for closure at a non-trivial level is that the quintuplet correlation Q_{ijkrs} is negligible compared to the product $f_i f_j T_{krs}$. This is somewhat in the spirit of the "direct interaction" approximation,[31-34] which assumes that correlations among three modes are more important than among more elaborate groupings.

Dropping terms containing Q_{ijkrs}, expressing f_{ijkrs} in terms of the f_{ijk} and the f_i, using Eq. (14), and performing some tedious but entirely straightforward algebra on Eq. (7) reduces it to the following form

$$\left\{\frac{\partial}{\partial t} + C_{ijk} X_j X_k \frac{\partial}{\partial X_i} + \text{c. p.}\right\} f_{ijk}$$

$$= - \sum_{rs \neq jk \text{ or } kj} C_{irs} \frac{\partial}{\partial X_i} \int dX_r \, dX_s \, X_r X_s \left[f_i f_j f_{krs} + f_i f_k f_{jrs} + f_j f_k f_{irs} \right] + \text{c. p.}$$

$$= (Vf)_{ijk} \qquad (18)$$

V denotes a linear differentio-integral operator containing the f_i whose action on the f_{krs} is defined by the right-hand side of Eq. (18).

Equations (9) or (15) and (18) constitute the basic closed description that results from the neglect of quintuplet correlations. It is not possible to proceed farther analytically without more insight into the properties of the linear operator V than is now available, unless we make plausible simplifying assumptions about the effect of the linear operator V. This is deferred to Section V. Numerical solution to the coupled pair of equations (9) and (18) may not be significantly more difficult than the solution of the Eulerian direct interaction equations, but this has not yet been attempted.

To close this section, two other remarks are in order concerning Eqs. (9) and (18).

First, it can be seen easily that the neglect of the quintuplet correlation is in no sense a "cumulant discard"[30] approximation. For, applying $\int dX_i\, X_i^2$ to Eq. (9) gives

$$\frac{d}{dt}\,\frac{\langle X_i^2\rangle}{2} = \sum_{jk} C_{ijk}\langle X_i\, X_j\, X_k\rangle \tag{19}$$

while applying $\int dX_i\, dX_j\, dX_k\, X_i\, X_j\, X_k$ to Eq. (18) gives

$$\frac{d}{dt}\,\langle X_i\, X_j\, X_k\rangle = C_{ijk}\langle X_j^2\, X_k^2\rangle + \text{c. p.}\,. \tag{20}$$

Instead of (20), we would have in the cumulant discard approximation

$$\frac{d}{dt}\,\langle X_i\, X_j\, X_k\rangle = C_{ijk}\langle X_j^2\rangle\langle X_k^2\rangle + \text{c. p.}\,. \tag{21}$$

It is clear that at the level of the moment hierarchy, the neglect of the quintuplet correlations does not even provide closure. Any moment-development equation will always involve higher moments because of the three-coefficient Liouville operator on the left-hand side of Eq. (18).

Finally, the content of the relation (13) for f_{ijk} can be translated into an equation for T_{ijk} using Eqs. (12) and (15). The result can be written compactly as

$$\left\{\frac{\partial}{\partial t} + H\right\} T_{ijk} = S_{ijk} \tag{22}$$

where S_{ijk} is a source term involving only one-body distributions:

$$S_{ijk} \equiv -\left\{C_{ijk}\, X_j\, X_k\,\frac{\partial}{\partial X_i} + \text{c. p.}\right\} f_i\, f_j\, f_k\,. \tag{23}$$

The notation "c. p." has its usual meaning, and H is a linear operator defined by the following relation:

$$
\begin{aligned}
H\,T_{ijk} \equiv & \left\{ C_{ijk}\, X_j\, X_k \frac{\partial}{\partial X_i} + \text{c. p.} \right\} T_{ijk} \\
& + \left\{ \sum_{\substack{rs \neq jk \\ \text{or } kj}} C_{irs} \int dX_r\, dX_s\, X_r\, X_s \frac{\partial f_i}{\partial X_i} \left[f_j\, T_{krs} + f_k\, T_{jrs} \right] + \text{c. p.} \right\} \\
& - 2 \left\{ C_{ijk}\, f_j\, f_k \int dX_j\, dX_k\, X_j\, X_k \frac{\partial\, T_{ijk}}{\partial\, X_i} + \text{c. p.} \right\} \qquad (24)
\end{aligned}
$$

($H\,T_{ijk}$ involves all the various triplet correlations, and not just T_{ijk} itself.)

If we like, we can choose to regard Eqs. (15) and (22) as the basic dynamical equations. Again, the only approximations involved in them are the neglect of the quintuplet correlation functions. If we make the conventional and reasonable assumption of initially-vanishing correlation functions, it will be seen that S_{ijk} acts as a source term which generates triplet correlations and involves only the f_i. Note also that if f_i is a Maxwellian, as in Eq. (17), $S_{ijk} \equiv 0$. Thus for situations close to absolute equilibrium, the triplet-correlation-generating term in Eq. (22) is small even though the C_{ijk} are not small. This provides a dynamical argument for the weakness of the correlations that is independent of the weakness of the interaction coefficients.

It will be noted that the general structure of Eq. (22) is reminiscent of the Dupree form of the equation for the pair correlation encountered when deriving the Balescu-Lenard equation.[7]

V. APPROXIMATE SOLUTION FOR T_{ijk}; THE KINETIC EQUATION FOR TURBULENCE

We conjecture that the effect of the last two collections of terms in Eq. (24) is a large number of essentially random impulses which tend to destroy correlations T_{ijk}, and compete against the source S_{ijk} which attempts to build them up. The nature of the decay of correlations produced by the last two brackets of terms in Eq. (24) is undoubtedly very complicated, but it may not be too poorly represented by a simple exponential decay. That is, we conjecture that Eq. (22) can be represented by the approximate form

$$\left\{ \frac{\partial}{\partial t} + L_3(i,j,k) + \nu_{ijk} \right\} T_{ijk} = S_{ijk} . \tag{25}$$

In Eq. (25), $L_3(i,j,k)$ is the Liouville operator for three coefficients, and is defined by

$$L_3(i,j,k) \equiv C_{ijk} X_j X_k \frac{\partial}{\partial X_i} + c.p. .$$

ν_{ijk} is an inverse decay time for three-body correlations. For many purposes it is not necessary to assume a specific form for it. Roughly speaking, ν_{ijk} plays the role that the Landau damping decrement plays in the derivation of the Balescu-Lenard equation.

The formal solution of Eq. (25) for T_{ijk} is, assuming zero initial correlations,

$$T_{ijk} = \int_0^t d\tau \exp\left\{-(t-\tau)L_3(i,j,k)\right\} \exp\left\{-(t-\tau)\,\nu_{ijk}\right\} S_{ijk}. \quad (26)$$

The correctness of the solution (26) can be readily established by differentiation.

The exponential operator $\exp\{-(t-\tau)\,L_3(i,j,k)\}$ is a familiar operator[5,6,7] in BBGKY kinetic theory, and is a member of the class often called "streaming operators." Their effect on an arbitrary given function is to treat the function as an initial value for Liouville's equation and convert it into that solution of Liouville's equation which reduces to the given function initially. More specifically, for any arbitrary function $g(X_i, X_j, X_k)$,

$$\exp[-\tau\, L_3(i,j,k)]\; g(X_i, X_j, X_k)$$

$$= g\,(X_i(-\tau), X_j(-\tau), X_k(-\tau)). \quad (27)$$

In Eq. (27), $X_i(\tau)$, $X_j(\tau)$, $X_k(\tau)$ are the solutions of the ordinary differential equations

$$\frac{d\,X_i(\tau)}{d\tau} = C_{ijk}\, X_j(\tau)\, X_k(\tau)$$

$$\frac{d\,X_j(\tau)}{d\tau} = C_{jki}\, X_k(\tau)\, X_i(\tau)$$

$$\frac{d\,X_k(\tau)}{d\tau} = C_{kij}\, X_i(\tau)\, X_j(\tau) \quad (28)$$

which satisfy $X_i(0) = X_i$, $X_j(0) = X_j$, $X_k(0) = X_k$. The $X_i(\tau)$ are Lagrangian coordinates which are associated with the Eulerian coordinates X_i by the

differential equations (28) and their accompanying initial data ($X_i(0) = X_i$, etc.). The explicit solutions to Eqs. (28) are not difficult to extract, but their explicit values are not required for present purposes.

With these notations and a slight change in the variable of temporal integration, Eq. (26) becomes

$$T_{ijk} = \int_0^t d\tau \exp\{-\tau \nu_{ijk}\} S_{ijk}(X_i(+\tau), X_j(+\tau), X_k(+\tau), \overset{t-\tau}{\mathcal{H}}) \tag{29}$$

Equation (29) is our approximate solution for T_{ijk}. Its substitution in Eq. (15) gives:

$$\frac{\partial f_i(X_i,t)}{\partial t} = \frac{\partial}{\partial X_i} \sum_{jk} C_{ijk} \int dX_j\, dX_k\, X_j\, X_k \int_0^t d\tau \exp(-\tau \nu_{ijk}) \times \left[C_{ijk} X_j(+\tau) X_k(+\tau) \frac{\partial}{\partial X_i(+\tau)} + \text{c. p.} \right] f_i(X_i(+\tau), \overset{t-\tau}{\mathcal{H}}) f_j(X_j(+\tau), \overset{t-\tau}{\mathcal{H}}) f_k(X_k(+\tau), \overset{t-\tau}{\mathcal{H}}) \tag{30}$$

Equation (30) is our basic kinetic equation for turbulence. It can be further simplified, and a number of its properties can be proved. This is done in Sec. VI. (A formally exact restatement of Eq. (30), which does not assume the exponential relaxation times ν_{ijk}, is obtained by simply replacing $\exp(-\tau \nu_{ijk})$ by the exponential of the operator whose eigenvalue ν_{ijk} is conjectured to be; this formal generalization is at present of no computational value.)

VI. PROPERTIES OF THE KINETIC EQUATION FOR TURBULENCE

Three conservation laws can be proved directly from Eq. (30) without further manipulations.

(1) Conservation of probability follows from the obvious conclusion that

$$\frac{\partial}{\partial t} \int f_i \, dX_i = \int dX_i \, \frac{\partial}{\partial X_i} \left\{ \begin{array}{l} \text{a function of } X_i \\ \text{which} \to 0 \text{ as } X_i \to \pm\infty \end{array} \right\} = 0.$$

(2) Conservation of enstrophy is proved by considering the expression

$$\frac{\partial}{\partial t} \sum_i \int f_i \, \frac{X_i^2}{2} \, dX_i =$$

$$- \sum_{ijk} C_{ijk} \int dX_i \, dX_j \, dX_k \, X_i \, X_j \, X_k \int_0^t d\tau \, \exp(-\tau \, \nu_{ijk})$$

$$\times \left\{ C_{ijk} \, X_j(+\tau) \, X_k(+\tau) \, \frac{\partial}{\partial X_i(+\tau)} + \text{c. p.} \right\}$$

$$\times \; f_i \, (X_i(+\tau), t-\tau) \; f_j \, (X_j(+\tau), t-\tau) \; f_k \, (X_k(+\tau), t-\tau) \; . \qquad (31)$$

Permuting the dummy indices i, j, k in Eq. (31) and adding the three equivalent expressions together, then dividing by 3, gives the same expression as Eq. (31) with the first C_{ijk} on the right-hand side replaced by $\frac{1}{3} (C_{ijk} + C_{jki} + C_{kij})$, which by property (4) of Section II, is zero for all i, j, k.

(3) Conservation of energy is proved in the same way as conservation of enstrophy, with $\frac{1}{3} (C_{ijk} + C_{jki} + C_{kij})$ being replaced by $\frac{1}{3}\left(\frac{C_{ijk}}{|i|^2} + \frac{C_{jki}}{|j|^2} + \frac{C_{kij}}{|k|^2}\right)$.

A fourth obvious property is:

(4) Solution by a Maxwellian. Since the right-hand side of Eq. (30) vanishes identically when f_i is given by Eq. (17), it is clear that the absolute equilibrium f_i is a time-independent solution of Eq. (30).

The kinetic equation (30) as it stands is not Markovian. The time derivative of the f_i depends not only upon f_i at the present instant, but on its integral over a time of the order of the ν_{ijk}^{-1}, the decay time for freely decaying triplet correlations. A Markovian limit can be obtained by making the following assumption, which is the analogue of the Bogolyubov[5] "adiabatic hypothesis." We may assume that the free decay of triplet correlations is sufficiently rapid that neither the f_i nor the three-body orbit variables of Eqs. (28) change appreciably in a time of the order of ν_{ijk}^{-1}. This hypothesis either is or is not true, but experimental or numerical data that would shed light on it would appear to be scarce. If we make it, we may freeze the τ-dependent values of the variables in the integrand of Eq. (30) at their $\tau = 0$ values, and perform the τ-integrations to get:

$$\frac{\partial f_i}{\partial t} = \frac{\partial}{\partial X_i} \sum_{jk} \frac{C_{ijk}}{\nu_{ijk}} \int dX_j \, dX_k \, X_j \, X_k \left\{ C_{ijk} \, X_j \, X_k \frac{\partial}{\partial X_i} + C_{jki} \, X_k \, X_i \frac{\partial}{\partial X_j} + C_{kij} \, X_i \, X_j \frac{\partial}{\partial X_k} \right\} \times f_i(X_i,t) \, f_j(X_j,t) \, f_k(X_k,t) \, . \tag{32}$$

From the Markovian form (32), it is immediately possible to prove the positive semi-definiteness of f_i and that all f_i approach a Maxwellian

as $t \to \infty$ (H-theorem). It can also be shown that a moment closure has now been achieved.

(5) Positive-definiteness of f_i. Suppose that $f_i\ (X_i, 0)$ is everywhere > 0, all i, and first becomes negative for some $X_i = X_{ic}$ at $t = t_o$ as a consequence of Eq. (32). We will show that a contradiction is implied. For at $t = t_o$, the stated conditions imply:

$$\text{(i)} \quad f_i\ (X_{io}, t_o) = 0$$

$$\text{(ii)} \quad \frac{\partial\ f_i\ (X_{io}, t_o)}{\partial\ t_o} < 0$$

$$\text{(iii)} \quad \frac{\partial\ f_i\ (X_{io}, t_o)}{\partial\ X_{io}} = 0$$

$$\text{(iv)} \quad \frac{\partial^2\ f_j\ (X_{io}, t_o)}{\partial\ X_{io}^2} > 0\ .$$

Evaluating the right-hand side of Eq. (32) under these conditions gives

$$\frac{\partial\ f_i\ (X_{io}, t_o)}{\partial t_o} =$$

$$\sum_{jk} \frac{C_{ijk}^2}{\nu_{ijk}} \int dX_j\ dX_k\ X_j^2\ X_k^2\ \frac{\partial^2 f_i\ (X_{io}, t_o)}{\partial\ X_{io}^2}\ f_j\ (X_j, t_o)\ f_k\ (X_k, t_o)\ ,$$

which is manifestly positive, contradicting (ii). Thus no intrinsically positive moment, such as $\langle X_i^2 \rangle$, can ever become negative as a consequence of Eq. (32).

(6) H-theorem. The irreversibility can be demonstrated by considering a slight generalization of Boltzmann's H function.

$$\frac{d}{dt}\sum_i \int dX_i\, f_i \ln f_i = \sum_i \int \ln f_i \frac{d f_i}{dt}\, dX_i$$

$$= -\sum_{ijk} \int dX_i\, dX_j\, dX_k\, \frac{1}{f_i} \frac{\partial f_i}{\partial X_i} \frac{C_{ijk}}{\nu_{ijk}} X_j X_k$$

$$\times \left\{ C_{ijk} X_j X_k \frac{\partial}{\partial X_i} + \text{c. p.} \right\} f_i f_j f_k$$

$$= -\frac{1}{3}\sum_{ijk} \frac{1}{\nu_{ijk}} \int dX_i\, dX_j\, dX_k \left\{ \frac{C_{ijk} X_i X_k}{f_i} \frac{\partial f_i}{\partial X_i} \right.$$

$$\left. + \text{c. p.} \right\} \left\{ \frac{C_{ijk} X_i X_k}{f_i} \frac{\partial f_i}{\partial X_i} + \text{c. p.} \right\} f_i f_j f_k \leq 0 ,$$

since the two { }'s are identical and the rest of the integrand is > 0. The equality sign holds if and only if the { } = 0, or

$$\left\{ C_{ijk} X_j X_k \frac{\partial}{\partial X_i} + \text{c. p.} \right\} f_i f_j f_k = 0 ,$$

which is satisfied by Eq. (17). Since $\Sigma_i \int dX_i\, f_i \ln f_i$ is bounded from below, this completes the proof of the H-theorem. All initial distributions must approach Eq. (17).

At the level of Eq. (32), in contrast to Eq. (30), a closure in terms of moments has been achieved. Applying $\int dX_i\, (X_i^2/2)$ to Eq. (32) yields, after some manipulations,

$$\frac{d}{dt}\frac{\langle X_i^2\rangle}{2} = \sum_{jk}\frac{C_{ijk}}{\nu_{ijk}}\Big\{C_{ijk}\langle X_j^2\rangle\langle X_k^2\rangle + C_{jki}\langle X_k^2\rangle\langle X_i^2\rangle + C_{kij}\langle X_i^2\rangle\langle X_j^2\rangle\Big\}, \tag{33}$$

and we have already proved that $\langle X_i^2\rangle$ can never go negative, though a similar proof can be given directly from Eq. (33). A somewhat similar set of equations is due to Leith[38] and Orszag.[39] Except for the slightly different form and interpretation of the ν_{ijk}, the content of Eq. (33) is that of the eddy-damped Markovian model.[38,39] Relaxation to equilibrium for Eqs. (33) is guaranteed by our H-theorem.

Equation (32) can be given a somewhat more simple looking form in terms of moments:

$$\frac{\partial f_i}{\partial t} = -\frac{\partial}{\partial X_i}(\lambda_i X_i f_i) + \frac{\partial^2}{\partial X_i^2}(q_i f_i) \tag{34}$$

where

$$\lambda_i \equiv \sum_{jk}\frac{C_{ijk}}{\nu_{ijk}}\left[C_{jki}\langle X_k^2\rangle\langle X_i^2\rangle + C_{kij}\langle X_i^2\rangle\langle X_j^2\rangle\right] \tag{35}$$

and

$$q_i \equiv \sum_{jk}\frac{C_{ijk}^2}{\nu_{ijk}}\langle X_j^2\rangle\langle X_k^2\rangle > 0 \quad . \tag{36}$$

Both λ_i and q_i are independent of all the phase space coordinates and depend only upon the time Equation (34) has the classical form of the Fokker-Planck equation derived from the Langevin equation[35] for the case of one-dimensional

Brownian motion with (time-varying) friction and diffusion coefficients, $-\lambda_i$ and q_i.

A relaxation time for the i^{th} mode can be estimated by $\beta_i^{-1} = -\lambda_i^{-1}$ where

$$\lambda_i \cong \sum_{jk} \frac{C_{ijk}}{\nu_{ijk}} \langle X_i^2 \rangle \left[C_{jki} \langle X_k^2 \rangle + C_{kij} \langle X_j^2 \rangle \right].$$

Using thermal equilibrium estimates for the expectation values of the moments, this is approximately

$$\lambda_i \cong - \sum_{jk} \frac{C_{ijk}^2}{\nu_{ijk}} \langle X_j^2 \rangle \langle X_k^2 \rangle .$$

If typical $\langle X_i^2 \rangle$ are just $\langle X^2 \rangle$, this gives a relaxation time of

$$\beta_i^{-1} \cong -\lambda_i^{-1} \cong \frac{\nu}{C^2 \langle X^2 \rangle^2 N^2} \tag{37}$$

where C^2 is a "typical" C_{ijk}^2, and ν is a "typical" ν_{ijk}.

The condition for validity of the Markovian assumption is that this relaxation time be long compared to the triplet decay time, or

$$\frac{\nu}{C^2 \langle X^2 \rangle N^2} \gg \frac{1}{\nu}$$

or

$$\nu^2 \gg C^2 \langle X^2 \rangle N^2 . \tag{38}$$

The estimate for ν^2 (which undoubtedly depends upon the C_{ijk}, the $\langle x_i^2 \rangle$, and N) which would validate or invalidate the inequality (38) is lacking.

VII. VISCOSITY AND EXTERNAL FORCES

Viscous damping is relatively straightforward. One adds a term $-\nu_i X_i$ to the right-hand side of Eq. (3), where $\nu_i = |i|^2 \nu$, and ν is the kinematic viscosity. The system is of course no longer conservative, and approach to a Maxwellian (17) is no longer expected. The new terms are reflected in new viscous damping terms in Eqs. (9), (10), (15), and (18). In particular, the operator H in Eq. (24) has added to it a term $\{\nu_i X_i \partial/\partial X_i + \text{c. p.}\} T_{ijk}$ on the right-hand side, and each of Eqs. (28) has a damping term like $-\nu_i X_i(\tau)$ on the right-hand side. The three-body orbits defined by Eqs. (28) are no longer conservative. Again, further progress requires conjectures about relaxations of triplet correlations in order to write down an explicit expression for a kinetic equation such as Eq. (32). If we again assume that these triplet correlations' free relaxation rate is faster than any other characteristic time in the system, the net effect of the addition of viscosity is to add a term $\partial(\nu_i X_i f_i)/\partial X_i$ to the right-hand side of Eq. (32). The relaxation process then becomes a complicated competition between the non-dissipative "collision term" in Eq. (32) with the additional viscous term which leads to the decay of the energy and the concentration of the distributions at lower values of X_i. Numerical investigation would be required.

External forces are more difficult to add. The inclusion of a driving force $F_i(t)$ on the right-hand side of Eq. (3) will, if random, affect the evolution of D in a non-Markovian way. Simplifying assumptions which would permit the addition of an external random force while still obtaining a closed-form kinetic equation (Markovian or otherwise) are under investigation, but no results have as yet been achieved.

VIII. DISCUSSION

The main point of the article is considered to be an indication that it is possible to derive a systematic kinetic theory of fluid turbulence from the Liouville equation for the Fourier coefficients of the fluid variables, a line of inquiry begun by Edwards.[13] A kinetic equation has been derived and is shown to possess a number of the requirements that any reasonable kinetic equation must have: Conservation laws, positive-definite spectral densities, and an H-theorem. Some value of the kinetic equation obtained may derive from its use in a phenomenological description with ν_{ijk}^{-1} relaxation times obtained by fitting decay data.

The major lack in the theory is any reliable information about the relaxation predicted by the complicated linear operator H (Eq. (24)). There are enough relaxation times ν_{ijk}^{-1} at our disposal, in the absence of quantitative estimates of their values, to match virtually any relaxation process with the numerical solutions of Eq. (33). Other quantitative theories of relaxing two-dimensional turbulence have shown[40] a gross insensitivity to the details of the dynamical description, and it may be that a similar insensitivity exists here.

Comparisons with data and with other analytical theories of decaying turbulence await numerical investigation of Eqs. (31), (32), and (33). We have devoted little attention to the inclusion of viscosity or stirring forces, though some calculations of viscous, forced equilibrium solutions to the Liouville equation have been reported by Thompson.[41]

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FOOTNOTES

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