

PRESSURE DEFORMATION OF TIRES USING DIFFERENTIAL STIFFNESS FOR  
TRIANGULAR SOLID-OF-REVOLUTION ELEMENTS

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SUMMARY

This paper presents the derivation of the differential stiffness for triangular solid of revolution elements. The derivation takes into account the element rigid body rotation only, the rotation being about the circumferential axis. Internal pressurization of a pneumatic tire is used to illustrate the application of this feature.

INTRODUCTION

In the NASTRAN computer code, the rigid format No.4 is the so-called static analysis with differential stiffness. This method of analysis is a first approximation to large deflection effects. The differential stiffness refers to the geometry stiffness which is generated using the stresses in the element which are computed from the initial linear stress analysis. The differential stiffness is available for rods, beams, shear panels, plates, and conical shell elements. However, sometimes one desires that the differential stiffness be available to solid elements. This is particularly true when one deals with a structure having thick wall yet flexible in nature. With this in mind, the differential stiffness for a triangular solid-of-revolution element is developed and incorporated into NASTRAN.

In the case of a ring element loaded axisymmetrically, there exists only one rotation of the element which is the rotation about the circumferential axis. We will assume that the stress and circumferential rotation are uniform through the element. Furthermore, in computing the work done by the forces in an element we will neglect the work done in conjunction with the normal and shear strains and the work done by the circumferential stress and strain. This is in accord with the derivation of the differential stiffness for a triangular membrane element presented in reference 1.

## DERIVATION OF DIFFERENTIAL STIFFNESS MATRIX

The triangular ring element in NASTRAN assumes the following displacement field (Fig. 1)

$$u = \beta_1 + \beta_2 r + \beta_3 z \quad (1.a)$$

$$w = \beta_4 + \beta_5 r + \beta_6 z \quad (1.b)$$

where  $u$ ,  $w$  are the displacements in the direction of the  $r$ ,  $z$  coordinates respectively. Inverting the above yields

$$\{\beta\} = [r_{\beta q}] \{q\} \quad (2)$$

where  $\{q\}^T = [u_1 \ w_1 \ u_2 \ w_2 \ u_3 \ w_3]$

$$\{\beta\}^T = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4 \ \beta_5 \ \beta_6]$$

Due to symmetry the only rotation of the element is the circumferential rotation  $\omega_\theta$  which according to reference 1 is given as

$$\omega_\theta = \frac{1}{2} \left( \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} \right) = \frac{1}{2} (\beta_5 - \beta_3) \quad (3)$$

Substituting  $\beta_5$  and  $\beta_3$  from equation (2) into equation (3) we obtain

$$\{\omega\} = [C] \{q\} \quad (4)$$

where  $\{\omega\}^T = \{ \omega_r \ \omega_z \ \omega_\theta \}$

$$[C] = \frac{1}{2\Delta} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ r_{23} & z_{23} & r_{31} & z_{31} & r_{12} & z_{12} \end{bmatrix}$$

and

$$r_{ij} = r_i - r_j$$

$$z_{ij} = z_i - z_j$$

$$\Delta = r_{12} z_{13} - r_{13} z_{12}$$

Considering only the circumferential rigid body rotation and neglecting the effects of the normal and shear strains, then the work done by forces in an element of volume  $V$  reduces to

$$W = -\frac{V}{2} \omega_{\theta}^2 (\sigma_r + \sigma_z) \quad (5)$$

where  $\sigma_r$  and  $\sigma_z$  are the element normal stresses in the r and z directions, respectively. Hence the matrix of the differential stiffness coefficient for that element is simply

$$[K_{\omega\omega}^d] = V \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_r + \sigma_z \end{bmatrix} \quad (6)$$

We then have the differential stiffness matrix in terms of the generalized coordinate  $\{\beta\}$

$$[K_{ee}^d] = [C]^T [K_{\omega\omega}^d] [C] \quad (7)$$

The above differential stiffness matrix is then transformed to grid point coordinates by means of equation (2) as

$$[K^d] = [r_{\beta q}]^T [K_{ee}^d] [r_{\beta q}] \quad (8)$$

This geometric stiffness matrix is then added to the elastic stiffness matrix to yield the total stiffness matrix of the structure.

#### APPLICATION TO PNEUMATIC TIRE INFLATION

We will apply the above formulation to the problem of inflating a pneumatic tire. This is an example which can be considered as an axisymmetric problem. Fig. 2 shows one-half of a typical meridian cross-section of a bias pneumatic tire. Its structural elements consist of (1) rubber which spreads from the tread area to shoulder area and to sidewall area and to the bead area, (2) casing or carcass which is the main load carrying member, and (3) bead, which seats the tire to the wheel rim. The carcass is a multi-layer composite which consists of a number of plies stacked together. The ply is a unidirectional composite made of many flexible high modulus cord of natural textile, synthetic polymer, glass fiber embedded in and bonded to a matrix of low-modulus polymeric material such as rubber. The plies are usually stacking together in a pair of symmetric bias cord angle with respect to the circumferential axis.

If we know the properties of the matrix and cord materials and their volume ratios, then the properties of the composite ply in its principal directions, directions 1 and 2 in Fig. 3, can be determined by the well-known Halpin-Tsai equation, reference 2. The ply is assumed to be transversely isotropic, i.e. Young's modulus  $E_{33} = E_{22}$  and Poisson's ratios  $\nu_{13} = \nu_{12}$ ,  $\nu_{32} = \nu_{23}$ . The well-known transformation matrix for elastic constants through a rotation is then applied to obtain the ply stiffness matrix in the laminate coordinate system x, y. For the present case, they are the meridional and circumferential directions. The

cord angle  $\alpha$  which is also the angle of rotation of the transformation is calculated from the following equation known as "lift equation" in the tire industry.

$$\frac{r}{r_c} \frac{1}{1 + \epsilon} = \frac{\cos \beta}{\cos \alpha} \quad (9)$$

where  $\alpha$  = cord angle in the body of tire which takes the molded shape  
 $\beta$  = cord angle in the initial cylinder before molding  
 $r$  = radius coordinate of the location in the body of tire  
 $r_c$  = radius of the initial cylinder before molding  
 $\epsilon$  = estimated cord strain

The total effective carcass stiffness matrix is then obtained by the method in reference 3. In general, the laminate can be satisfactorily represented by two-ply laminate of  $(+\alpha)$  and  $(-\alpha)$  cord orientations.

The material properties used in our example are

cord modulus  $E_c = 0.52376 \times 10^5 \text{ kg/cm}^2$  ( $7.45 \times 10^5 \text{ psi}$ )

cord Poisson's ratio  $\nu_c = 0.3$

rubber modulus  $E_r = 52.7278 \text{ kg/cm}^2$  (750 psi)

rubber Poisson's ratio  $\nu_r = 0.48$

Other data used are

$\beta = 57 \text{ degrees}$

$r_c = 21.4846 \text{ cm}$  (8.4585 in.)

The cord strain  $\epsilon$  is assumed to be constant and its value at the crown is used. The radius of the ply center line at the crown is 32.8308 cm (12.9255 in.) and the final cord angle at that point is assumed to be 32 degrees with reference to circumferential circle. The tire is subjected to internal pressure of

1.6873  $\text{kg/cm}^2$  (24 psi). For simplicity, the tire is assumed to be fixed at a section close to the bead. In all, 488 elements and 317 grid points are used for one-half of the tire cross-section. Figure 4 shows the shape of a deformed and undeformed tire section for the linear solution and for the differential stiffness solution. The radial displacement of the outer ply line using NASTRAN is 1.3785 cm (0.54277 in.) compared with 1.2217 cm (0.481 in.) from a shell theory. The shell theory neglects the rubber in the tread, shoulder, sidewall, and bead and only considers the carcass as a thin elastic shell of orthotropic material. The increase in the tire outer radius due to inflation is computed by NASTRAN to be 1.3419 cm (0.52829 in.) as compared with 1.0976 cm (0.448 in.) from measurement. The radial displacement of the same point from NASTRAN linear solution is 1.7499 cm (0.68895 in.). The inflated dimension of the outer carcass line in the z direction is 10.3045 cm (4.0569

.) from NASTRAN and 9.9847 cm (3.931 in.) from shell theory. The section  
dth of the tire (the maximum dimension of the tire outer profile in the z  
rection) after inflation is 21.116 cm (8.3134 in.) from NASTRAN as compared  
th 20.7772 cm (8.18 in.) from the measurement. As can be seen, the inclusion  
the differential stiffness results in a great improvement over the linear  
lution. However, there are still discrepancies in the various results. The  
screpancies can be attributed to factors such as material characterization,  
sumptions on the boundary conditions, initial shape, cord angles, cord strain,  
d the possibility of the residual stresses in the actual tire.

### CONCLUDING REMARKS

We have presented here the derivation of the differential stiffness matrix  
r triangular solid-of-revolution elements. Its application is illustrated  
ing the inflation of a bias tire as an example. The differential stiffness  
a ring element is particularly useful when dealing with structure which has  
ick wall yet flexible enough to undergo large displacement and rotation.  
e example of the tire inflation demonstrates this point. The numerical  
ults in the present example are quite satisfactory.

As mentioned earlier, the derivation of the differential stiffness present-  
in this paper is based on the information contained in the manual for Level  
reference 1, since it was available to this author at that time. This  
thor has learned since then that Level 15.5 (reference 4) has a refined  
rivation of the differential stiffness which includes the effects of the  
ormal and shear strains in computing the energy. Hence, if a new derivation of  
e differential stiffness for a triangular ring element were made to include  
e effects of the normal and shear strains and also to include the effect of  
e circumferential strain, it would probably give a better solution than the  
esent one.

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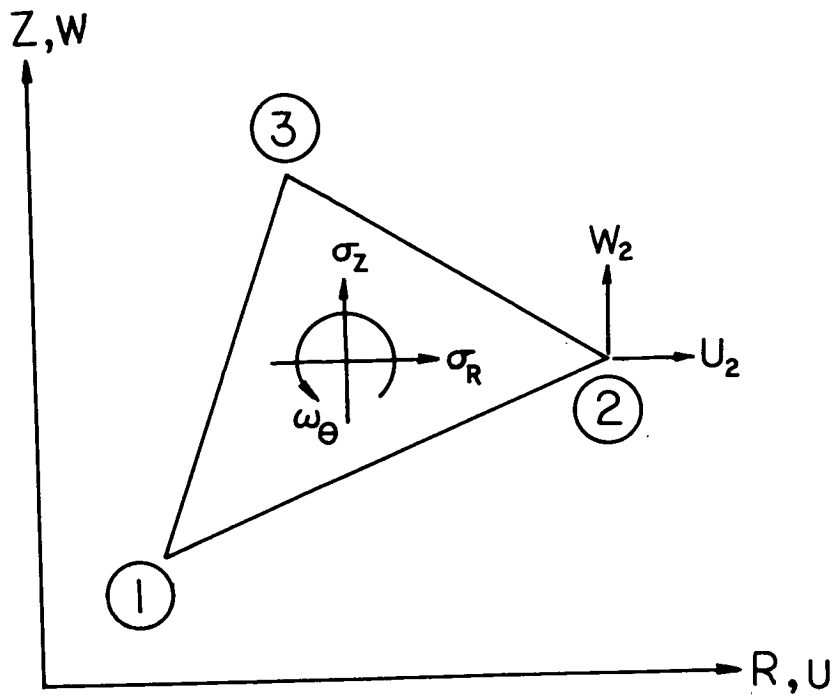


Figure 1.- Triangular solid of revolution element.

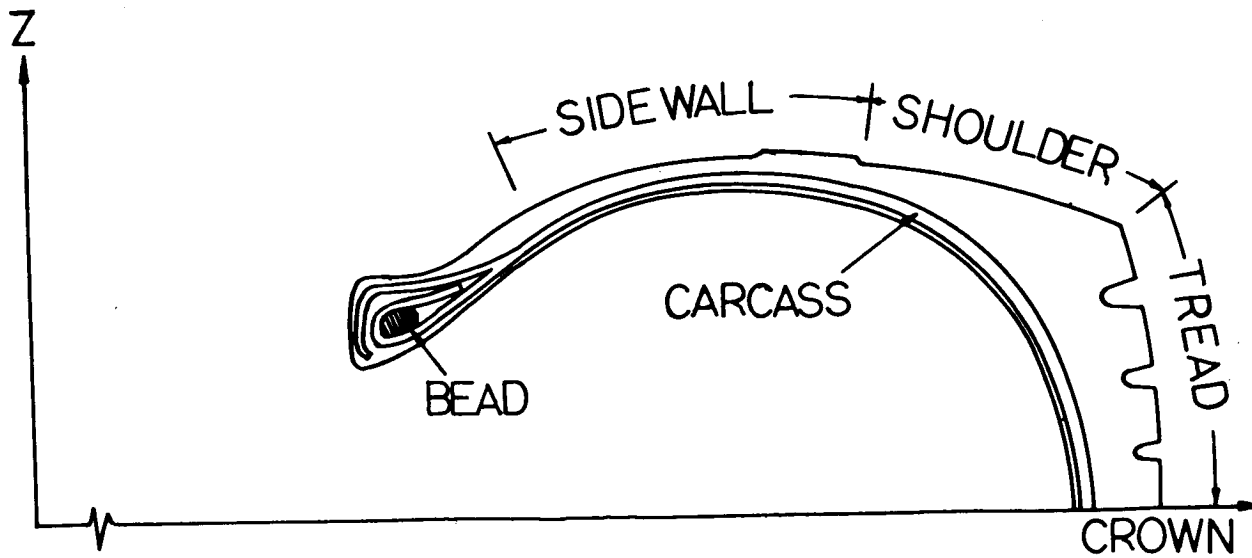


Figure 2.- Meridian cross section of a tire.

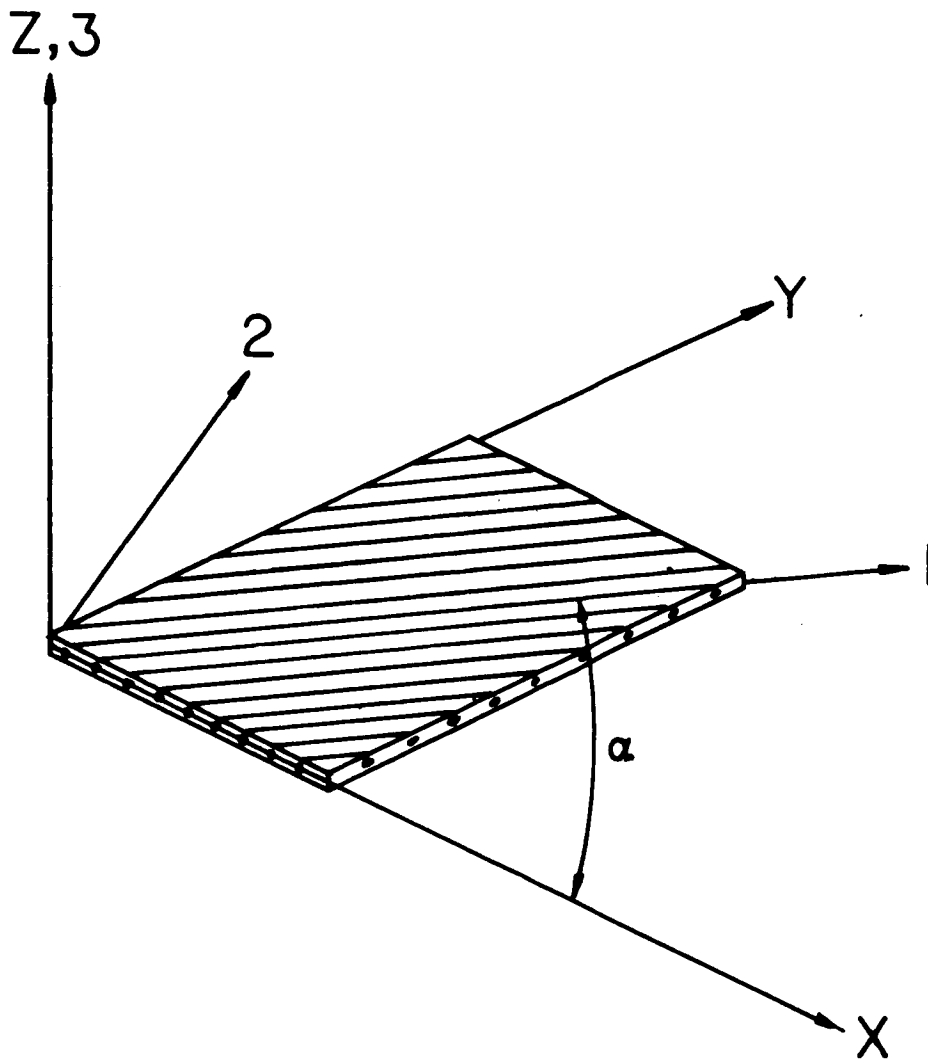
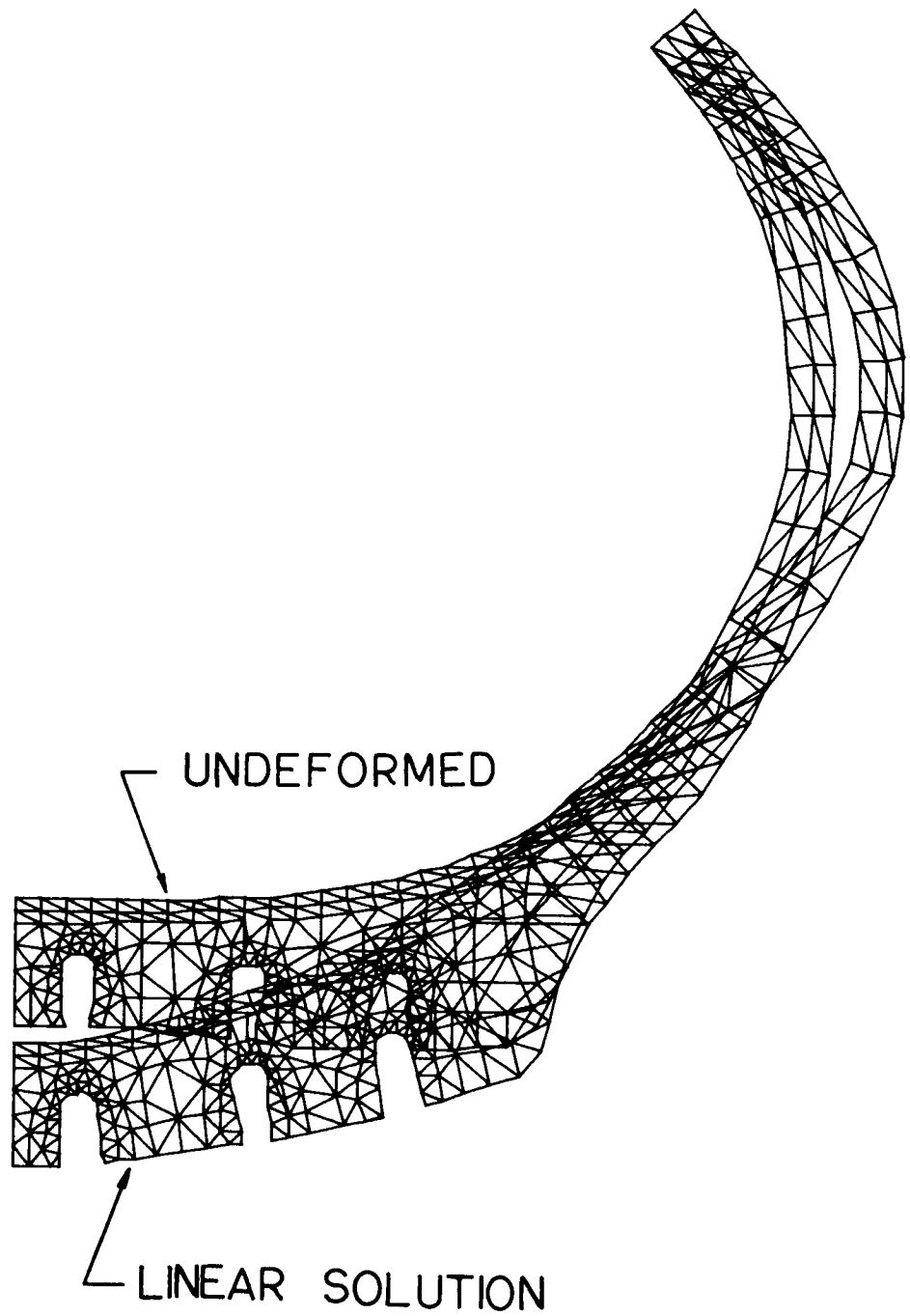


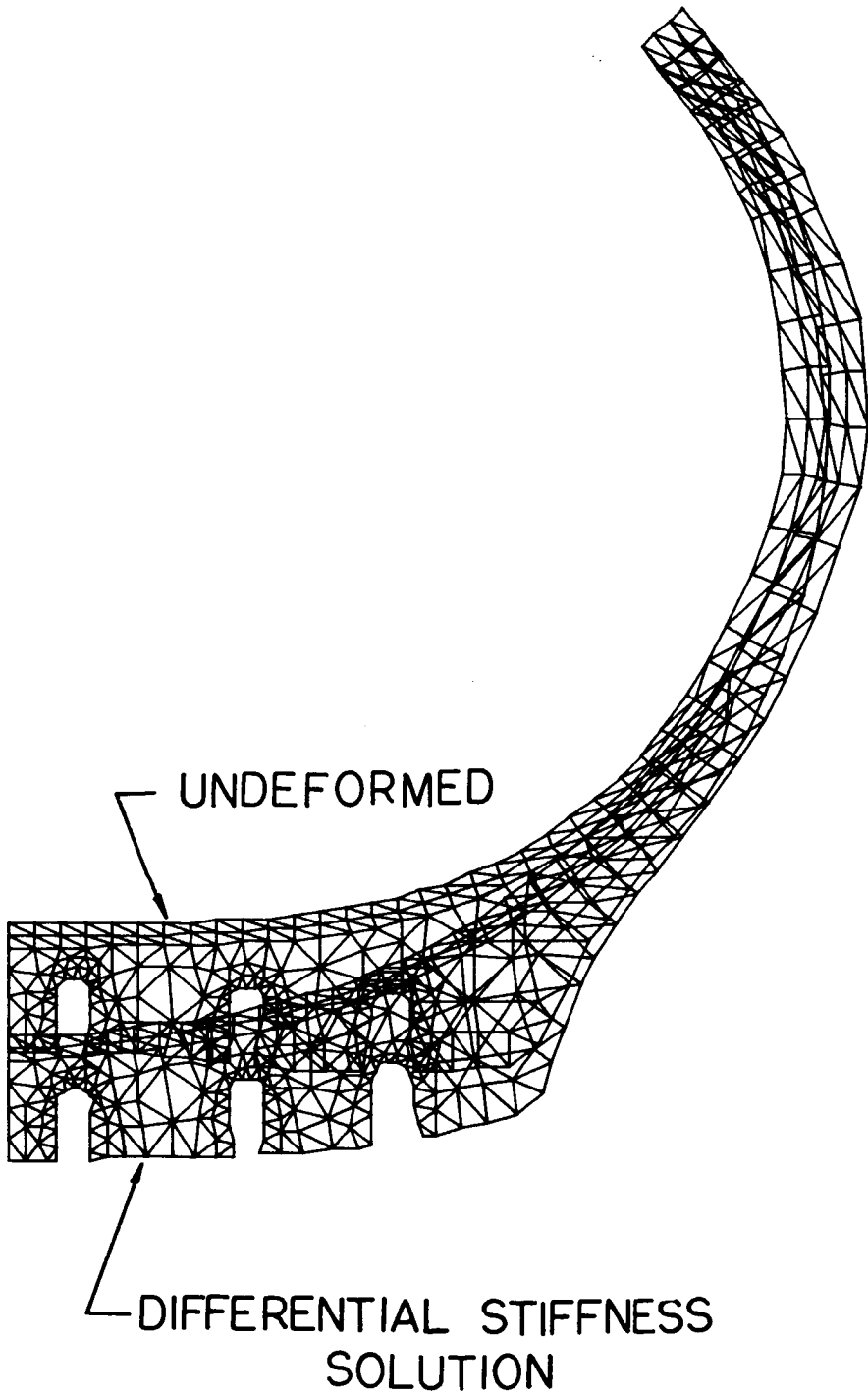
Figure 3.- Cord-reinforced composite ply.



(a) Linear solution.

Figure 4.- Deformed and undeformed tire cross section.





(b) Differential stiffness.

Figure 4.- Concluded.