NASTRAN NONLINEAR VIBRATION ANALYSIS OF BEAM AND FRAME STRUCTURES

Chuh Mei
LTV Aerospace Corporation, Hampton, Virginia

and

James L. Rogers, Jr.
NASA Langley Research Center, Hampton, Virginia

SUMMARY

A capability for the nonlinear vibration analysis of beam and frame structures suitable for use with NASTRAN Level 15.5 is described. The nonlinearity considered is due to the presence of axial loads induced by longitudinal end restraints and lateral displacements that are large compared to the beam height. This paper includes a brief discussion of the mathematical analysis and the geometrical stiffness matrix for a prismatic beam (BAR) element. Also included are a brief discussion of the equivalent linearization iterative process used to determine the nonlinear frequency, the required modifications to subroutines DBAR and XMPLBD of the NASTRAN code, and the appropriate DMAP ALTERS to determine the frequency. To demonstrate this nonlinear vibration capability, four example problems are presented. Comparisons with existing experimental and analytical results show that excellent accuracy is achieved with NASTRAN in all cases.

INTRODUCTION

In practical beam vibration problems, transverse deflections may be greater than those assumed for linear theory. Considerable attention has been given, therefore, to the nonlinear flexural vibration of beams. Most studies have dealt with simple uniform beams with either hinged or fixed support conditions at both ends. An excellent literature survey is given by Eisley (ref. 1) through 1964. More recent surveys are given by Ray and Bert (ref. 2), and Pandalai (ref. 3). Nonlinear vibrational behavior of non-uniform beam and frame structures found in many engineering applications, however, has not received much attention in the literature because of analytical difficulties. The use of the finite-element method overcomes these difficulties and removes the uniform beam and limited support condition restrictions.

The purpose of this paper is to describe a nonlinear vibrational analysis capability for determining fundamental frequency of beam and frame structures suitable for use with NASTRAN Level 15.5 and to present results demonstrating this capability. The paper includes a discussion of the mathematical analysis and the derivation of the geometrical stiffness matrix that represents the induced axial force in the governing equation, the appropriate modifications to the NASTRAN code, and solutions of example problems. Procedures for nonlinear vibration analysis with and without applied axial forces are available for NASTRAN Level 15.5 by means of DMAP ALTERS and modifications of the NASTRAN code given in the Appendices.

SYMBOLS

A	area
c	amplitude of vibration
d	amplitude ratio, $c/\sqrt{I/A}$
E	modulus of elasticity
$\{f\}$	element forces
h	height of beam
I	area moment of inertia of cross section
$[k]$	stiffness matrix
$[k^d]$	differential stiffness matrix
$[k^g]$	geometrical stiffness matrix
ℓ	length
$[m]$	mass matrix
P_{xo}	applied axial force on undeformed beam
P_x	axial force due to deflection
P_E	Euler load
$[p]$	inplane force matrix
$\{u\}$	nodal displacements
t	time

x,y,z	element coordinate system
w	lateral deflection
ρ	mass density
$\{\phi\}$	normal mode
ω_o	fundamental linear frequency
ω	fundamental nonlinear frequency

Subscripts:

a,aa	system
e,ee	element
eq	equivalent

MATHEMATICAL ANALYSIS

Formulation of Matrix Equation of Motion

The strain energy in a deformed bar element of uniform cross section is given by:

$$U = \frac{EI}{2}\int_0^{\ell}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + P_{xo}\int_0^{\ell}\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{P_x}{2}\int_0^{\ell}\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (1)$$

where $$P_x = \frac{EA}{\ell}\int_0^{\ell}\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (2)$$

The first two terms of the strain energy expression are due to bending and applied axial force, respectively. The last term is the nonlinear contribution of the axial force P_x induced by large deflections. If the axial force P_x is neglected from Eq. (1), the strain energy is reduced to that of the linear theory.

The kinetic energy is given by:

$$T = \frac{\rho A}{2}\int_0^{\ell}\left(\frac{\partial w}{\partial t}\right)^2 dx \qquad (3)$$

where the rotatory inertia term has been neglected.

In the finite-element approach, transverse deflection of an element is expressed in terms of generalized displacements $\{u_e\}$ and interpolation functions. The function chosen to represent the transverse deflection is given by:

$$w = \left\lfloor (1 - 3\,\bar{x}^2 + 2\,\bar{x}^3) \quad -(1 - 2\,\bar{x} + \bar{x}^2)x \quad (3\,\bar{x}^2 - 2\,\bar{x}^3) \quad (\bar{x} - \bar{x}^2) \quad x \right\rfloor \{u_e\} \tag{4}$$

where $\bar{x} = \frac{x}{\ell}$ (5)

This displacement function is the same as used for the NASTRAN bar element. The displacement vector describing bending of a bar element in the xz plane (see figure 1) is defined by:

$$\{u_e\} = \{u_{za} \quad \theta_{ya} \quad u_{zb} \quad \theta_{yb}\}^T \tag{6}$$

Substituting Eq. (4) into Eqs. (1) and (3), gives:

$$U = \frac{1}{2}\,\{u_e\}^T\,([k_{ee}] + [k^d{}_{ee}] + [k^g{}_{ee}])\,\{u_e\} \tag{7}$$

and

$$T = \frac{1}{2}\,\{\dot{u}_e\}^T\,[m_{ee}]\,\{\dot{u}_e\} \tag{8}$$

where $[k_{ee}]$, $[k^d{}_{ee}]$, $[k^g{}_{ee}]$, and $[m_{ee}]$ represent the element stiffness, differential stiffness, geometrical stiffness, and mass matrices, respectively and $[k^g{}_{ee}]$ is a function of $\{u_e\}$. Substituting Eq. (4) into Eq. (2) yields

$$P_x = \{u_e\}^T\,[p_{ee}]\,\{u_e\} \tag{9}$$

where $[p_{ee}]$ is the inplane force matrix.

Substituting Eqs. (7) and (8) into Lagrange's equations, that is,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\left(\frac{\partial U}{\partial q_i}\right) = 0 \qquad i = 1, 2, \ldots, n \tag{10}$$

where n is the number of elemental degrees of freedom, leads to the matrix equation of motion for the large amplitude free oscillations of a bar element which is given by

$$[m_{ee}]\,\{\ddot{u}_e\} + ([k_{ee}] + [k^d{}_{ee}] + [k^g{}_{ee}])\{u_e\} = \{f_e\} \tag{11}$$

The mass, stiffness, and differential stiffness matrices of the bar element are 12 x 12 matrices relating the forces and moments acting at the ends of the bar (see ref. 4). The portion of the differential stiffness matrix, for example, that describes bending in the xz plane of figure 1, is given by

$$[k^d{}_{ee}] = \begin{array}{c} \begin{array}{cccc} u_{za} & \theta_{ya} & u_{zb} & \theta_{yb} \end{array} \\ \begin{bmatrix} \frac{6\,P_{xo}}{5\ell} & -\frac{P_{xo}}{10} & -\frac{6\,P_{xo}}{5\ell} & -\frac{P_{xo}}{10} \\ & \frac{2\ell\,P_{xo}}{15} & \frac{P_{xo}}{10} & -\frac{\ell\,P_{xo}}{30} \\ & & \frac{6\,P_{xo}}{5\ell} & \frac{P_{xo}}{10} \\ & & & \frac{2\ell\,P_{xo}}{15} \end{bmatrix} \end{array} \tag{12}$$

The relations between $[k^d{}_{ee}]$, $[k^g{}_{ee}]$, and $[p_{ee}]$ can be found from Eqs. (1) and (2) and they are

$$[k^g{}_{ee}] = \frac{1}{2}\frac{P_x}{P_{xo}}\,[k^d{}_{ee}] \tag{13}$$

where

$$P_x = \{u_e\}^T [p_{ee}] \{u_e\} \tag{14}$$

and

$$[p_{ee}] = \frac{\left(\frac{EA}{2\ell}\right)}{P_{xo}} [k^d_{ee}] \tag{15}$$

Solution Technique

The goemetrical stiffness matrix $[k^g_{ee}]$ in Eq. (11) is displacement dependent (see Eqs. (13) and (14)). Therefore, the frequency for nonlinear vibration also depends on the amplitude of vibration. This phenomenon is different from the linear case, in which the frequency is independent of amplitude. In the following the frequency associated with the linear vibration problem is referred to as the linear frequency, and the frequency associated with the nonlinear vibration problem is referred to as the nonlinear frequency. To determine the nonlinear frequency, an iterative procedure with an equivalent linearization technique is used and is illustrated by the simplified flow chart shown in figure 2. The system matrices indicated in figure 2 are assembled from the element matrices by a standard finite-element procedure. The basic idea is to replace the displacement dependent geometrical stiffness matrix $[k^g_{ee}]$ by an equivalent matrix $[k^g_{ee}]_{eq}$ using the mode shape of the linear vibration problem as the first approximation to the displacement. This reduces the nonlinear system equation of motion to a linearized equation which can be solved as a standard eigenvalue problem. The mode obtained by solving this eigenvalue problem may be used to recompute $[k^g_{ee}]_{eq}$ for the next iteration in the nonlinear vibration iterative solution procedure. The solution procedure is illustrated as follows. The first step is to solve the linear vibration problem:

$$\omega_o^2 [m_{aa}] \{\phi\}_o = [k_{aa}] \{\phi\}_o \tag{16}$$

where ω_o is the fundamental frequency of the linear problem, $\{\phi\}_o$ represents the corresponding mode shape normalized by its maximum component, and the subscript aa represents the system matrices. Solving Eq. (16) provides the first approximate displacement in the form

$$\{u_a\}_1 = c \{\phi\}_o \tag{17}$$

where c is the amplitude of vibration. The equivalent geometrical stiffness matrix now can be obtained using $\{u_a\}_1$ which leads to a linearized eigenvalue equation of the form

$$\omega^2 [m_{aa}] \{\phi\}_1 = ([k_{aa}] + [k^d_{aa}] + [k^g_{aa}]_{eq})\{\phi\}_1 \tag{18}$$

where ω is the fundamental nonlinear frequency associated with amplitude c, and $\{\phi\}_1$ is the corresponding mode shape. The iterative process can be repeated by using

$$\{u_a\}_2 = c\ \{\phi\}_1 \tag{19}$$

as the second approximation, and similarly the i-th iteration approximate displacement is of the form

$$\{u_a\}_i = c\ \{\phi\}_{i-1} \tag{20}$$

The iterative process can be continued until the nonlinear frequency converges to the desired accuracy or the mode shape $\{\phi\}_i$ satisfies some convergence criterion (e.g., the modified Euclidean norm of ref. 5).

MODIFICATIONS TO THE NASTRAN CODE

To compute the geometrical stiffness matrix $[k^g_{ee}]$, subroutine DBAR was modified to take advantage of the fact that the differential stiffness matrix and the geometrical stiffness matrix are related as shown in Eq. (13). Appendix A shows these changes in CDC UPDATE format. The core storage requirement for DBAR was increased by 647_8 locations.

To avoid going through the modified section of code each time DBAR was called, a new parameter, NLVIB, was added to the DMAP calling sequence for module DSMG1. The contents of NLVIB are passed through blank common from DSMG1 to DBAR. The default value for NLVIB, set in block data subroutine XMPLBD, is zero (0). When NLVIB = 0, the new code in DBAR will not be executed. To set NLVIB = 1 and execute the new DBAR code, the following calling sequence for the DSMG1 module is used:

```
DSMG1        CASECC,,SIL,,PHIG,CSTM,MPT,ECPT,GPCT,
   DIT/KGGG/V,N,DSCOSET/C,N,1 $
```

The underlined parameter sets NLVIB to 1.

Once the changes shown in Appendix A were made to DBAR and XMPLBD, they were compiled and replaced the old DBAR and XMPLBD in the NASTRAN object library. Link 1 and Link 13 were relinked and a new executable NASTRAN was created. Although this procedure was done on a CDC computer, similar procedures will produce similar results on both the IBM and UNIVAC computers. In order to use this nonlinear vibration capability in NASTRAN, extensive alters have to be applied to either Rigid Format 5 and Rigid Format 13, depending on how the capability is needed. A summary of the applicable analyses, their governing equations and their appropriate Rigid Formats is given as follows:

ANALYSIS	EQUATION	RIGID FORMAT
Normal Modes	$\omega_o^2[m] = [k]$	3
Buckling	$[k] + \lambda[k^d] = 0$	5
Normal Modes with Differential Stiffness	$\omega_o^2[m] = [k + k^d]$	13
Nonlinear Vibration Analysis	$\omega^2[m] = [k + k^g]$	5 with ALTERS
Nonlinear Vibration Analysis with Differential Stiffness	$\omega^2[m] = [k + k^d + k^g]$	13 with ALTERS

where λ is an eigenvalue.

The appropriate DMAP alter sequences for both Rigid Formats 5 and 13 are shown in Appendix B. The alters between the statements

```
LABEL    CONV $
REPT     CONV, 1 $
```

will go through two iterations. If the user desires more iterations, the integer in the REPT statement must be increased. The only other input required to use this capability is the addition of a PARAM card in the Bulk Data deck. The parameter AMP is used to specify the amplitude of vibration of the structure.

EXAMPLES AND RESULTS

The nonlinear vibration capability developed for use with NASTRAN has been demonstrated by solving two examples of a nonlinear vibration analysis and two examples of a nonlinear vibration analysis with differential stiffness. NASTRAN solutions are compared with previously published results.

Nonlinear Vibration Analysis

The first example is the vibration of a uniform beam with various end support conditions. Evensen (ref. 6) obtained approximate amplitude-frequency relations for uniform beams with fixed-fixed, hinged-hinged, and fixed-hinged boundary conditions using a perturbation method. Good agreement is obtained between the NASTRAN and perturbation solutions as shown in Figure 3. For the hinged support case, the two amplitude-frequency curves coincide.

The second example demonstrates the effect of the amplitude of vibration on a rectangular frame structure. The frame is 304.8 cm (10.0 ft.) wide, 609.6 cm (20.0 ft.) long, and is made of 1.27 cm (0.5 in.) diameter steel rod. There are 10 equally spaced cells lengthwise and 4 equally spaced cells along the width. All four edges of the frame are fixed. A plot of the undeformed frame is shown in figure 4. Only one-fourth of the frame is used in the analysis due to symmetry. The linear frequency and nonlinear frequencies for values of the amplitude c (see Eq. 17) up to 7.62 cm (3 in.) are as follows:

Amplitude, c		Frequency, Hertz
cm	(in.)	
0	0	4.638 linear
1.27	0.5	5.319
2.54	1.0	6.830
3.81	1.5	8.565
5.08	2.0	10.295
6.35	2.5	11.940
7.62	3.0	13.478

The results indicate that the amplitude has an important influence on the frequency. In this example, a 5.08 cm (2 in.) amplitude at the center of the frame increases the fundamental nonlinear frequency to more than twice the linear one.

Nonlinear Vibration Analysis with Differential Stiffness

The third example is a hinged rectangular beam subjected to an applied axial tensile force of 105.4 N (23.7 lbf). The same problem was solved in ref. 2 using three different approximate analytical procedures, and results from these procedures, as well as experiment results, are given. One procedure is based on an assumption for the spatial dependence of the displacement function, one is based on an assumption for the temporal dependence of the displacement function, and the third procedure is the Galerkin procedure. The beam has the following properties:

Length	50.8 cm	(20.0 in.)
Width	1.27 cm	(0.50 in.)
Height	0.081 cm	(0.032 in.)
Material	Titanium Alloy	
Elastic Modulus	100.6 GPa	(14.6 x 10^6 psi)
Specific Mass	5.15 Mg/m^3	(0.186 $lb./in.^3$)

One-half of the beam modeled by six BAR elements was used for the analysis Three analytical fundamental frequencies and an experimentally measured one fro ref. 2 and the NASTRAN solution are shown in figure 5. Comparing the results demonstrates that the NASTRAN results provide the closest comparison with the experiment.

The fourth example is a beam-column subjected to an applied compressive force with various support conditions. Based on linear theory, Lurie (ref. 7) has shown that the relation between the square of the frequency and the axial load is linear for a beam that has identically shaped vibration and buckling modes. He also showed that the condition of zero fundamental frequency corresponds to buckling. The linear vibration-stability problem studied by Lurie is actually the limiting case of a more general phenomenon of large amplitude vibrations under the influence of axial loads. Burgreen (ref. 8) obtained an exact solution in terms of elliptic functions for a uniform beam hinged at both ends and also verified his results experimentally. Srinivasan (ref. 9) used Galerkin's method to study beam-columns with both ends hinged. Table 1 shows good agreement between the NASTRAN solutions and the results given by Burgreen and Srinivasan, and NASTRAN gives better predictions than the one-term Galerkin method. The load-frequency curves for different amplitude ratios of vibration, d, for beams with various support conditions are given in figure 6, where d is the ratio of amplitude to the radius of gyration of the beam. No comparison is made for the cases of fixed-hinged and fixed-fixed because there appears to be no solution available in the literature. It has been found from this example that (1) the effect of amplitude is more pronounced for a less stiff structure, and (2) nonlinear theory shows that the frequency of a column at the Euler buckling load is not zero for finite amplitudes of vibration.

CONCLUDING REMARKS

Nonlinear vibration capability for beam and frame structures has been developed for use with NASTRAN Level 15.5 by means of DMAP ALTERS and modifications to the NASTRAN code. A geometrical stiffness matrix for a bar element has been developed for NASTRAN by modifying subroutine DBAR. An equivalent linearization technique and iterative process used to determine nonlinear frequencies are implemented into NASTRAN by the DMAP ALTERS. The versatility of the finite-element method enables the analyst to determine nonlinear frequencies of vibration for non-uniform beam and frame structures. Comparison with previously published results show that excellent accuracy is achieved with NASTRAN.

APPENDIX A

MODIFICATION OF CODE

```
*INSERT,DBAR.111
      DOUBLE PRECISION PXT,COEFF,HOLD(6),TEMP(36),PXP,STOR(36)
*INSERT,DBAR.185
C
C  INSERT NEW NLVIB PARAMETER INTO BLANK COMMON
C
      COMMON       ICOM,NLVIB
*INSERT,DBAR.638
      IF(NLVIB.NE.0.AND.FX.EQ.0.0D0) FX=1.0D0
*INSERT,DBAR.646
*INSERT,DBAR.766
      IF(NLVIB.EQ.0) GO TO 621
C
C  DIVIDE FX OUT OF KDGG MATRIX
C
      DO 900 KK=1,144
      KD(KK)=KD(KK)/FX
  900 CONTINUE
C
C  COMPUTE
C                        T    E*A
C              PXP = U   * --- * (KDGG) * U
C                            2*L
C
C  WHERE U IS A 6X1 VECTOR
C
C        KDGG IS A 12X12 MATRIX DIVIDE INTO FOUR 6X6 MATRICES
C
      IF(IPVT.NE.1) GO TO 621
      COEFF=DA*E/(2.0D0*L)
      PXT = 0.0D0
      DO 920 IP1 = 1,4
      JPX = 1
      JCNT=0
      IF(IP1.GE.3) GO TO 930
      ILO=1
      IHI=72
      GO TO 940
  930 ILO=73
      IHI=144
  940 DO 960 I = ILO,IHI,12
      JLO=I
      JHI=JLO+5
      DO 950 K = JLO,JHI
```

```
      JCNT = JCNT+1
      IF(IP1.EQ.1.OR.IP1.EQ.3) TEMP(JCNT)=KD(K)
      IF(IP1.EQ.2.OR.IP1.EQ.4) TEMP(JCNT)=KD(K+6)
 950  CONTINUE
 960  CONTINUE
      IWLEFT=37
      IF(IP1.GE.3) IWLEFT=73
      IWRGHT=37
      IF(IP1.EQ.2.OR.IP1.EQ.4) IWRGHT=73
      CALL GMMATD(KE(IWLEFT),6,6,1,TEMP(1),6,6,0,STOR(1))
      CALL GMMATD(STOR(1),6,6,0,KE(IWRGHT),6,6,0,TEMP(1))
 912  DO 915 IP2 = 1,36
      TEMP(IP2) = TEMP(IP2)*COEFF
 915  CONTINUE
      IF(IP1.EQ.2.OR.IP1.EQ.4) JPX = 2
      IF(IP1.LE.2)  CALL GMMATD(UA(1),6,1,1,TEMP(1),6,6,0,HOLD(1))
      IF(IP1.GE.3)  CALL GMMATD(UB(1),6,1,1,TEMP(1),6,6,0,HOLD(1))
      IF(JPX .EQ.1) CALL GMMATD(HOLD(1),1,6,0,UA(1),6,1,0,PXP)
      IF(JPX .EQ.2) CALL GMMATD(HOLD(1),1,6,0,UB(1),6,1,0,PXP)
C
C  STORE SUM INTO PXT
C
      PXT = PXT+PXP
 920  CONTINUE
C
C  PXT=.5 * PXP * KDGG
C
      PXT = PXT*.5D0
 621  CONTINUE
*INSERT,DBAR,790
      IF(NLVIB.EQ.0) GO TO 653
C
C  CALCULATE KDGGG AND STORE
C
      DO 652 IPX=1,36
      JPX=IPX+108
      KEP(JPX)=PXT*KEP(JPX)
 652  CONTINUE
 653  CONTINUE
*COMPILE,DBAR
*DELETE,XMPLBD,78
     1, 10, 4HDSMG,4H1    , 1,10, 1, 1, -1, 1, 0
*DELETE,WT133-L14,39
      DIMENSION MPL01( 48), MPL02(178), MPL03(191), MPL04(179)
```

```
*DELETE,WT155-L16.8
      COMMON/XGPI2/LMPL,MPLPNT,MPL(1621)
*DELETE,WT133-L14.42,WT133-L14.45
     3              ,(MPL( 227),MPL03(1)) ,(MPL( 418),MPL04(1))
     5              ,(MPL( 597),MPL05(1)) ,(MPL( 723),MPL06(1))
     7              ,(MPL( 858),MPL07(1)) ,(MPL(1067),MPL08(1))
     9              ,(MPL(1172),MPL09(1)) ,(MPL(1351),MPL10(1))
*DELETE,WT155-L16.9
     1              ,(MPL(1538),MPL11(1))
*COMPILE,XMPLBD
```

APPENDIX B

DMAP ALTERS

For Nonlinear Analysis

```
ID      NLVBF,BEAM FRAME
APP     DISP
SOL     5,0
TIME    5
$       NONLINEAR FREE VIBRATIONS OF BEAM AND FRAME STRUCTURES
ALTER       19,23
GP3         GEOM3,EQEXIN,GEOM2/,GPTT/C,N,123/V,N,NOGRAV/C,N,123 $
CHKPNT      GPTT $
ALTER       31,31
ALTER       54
EQUIV       MGG,MNN/MPCF1 $
CHKPNT      MNN $
ALTER       66,76
MCE2        USET,GM,KGG,MGG,,/KNN,MNN,, $
CHKPNT      KNN,MNN $
LABEL       LBL2 $
EQUIV       KNN,KFF/SINGLE /MNN,MFF/SINGLE $
CHKPNT      KFF,MFF $
COND        LBL3,SINGLE $
SCE1        USET,KNN,MNN,,/KFF,KFS,KSS,MFF,, $
CHKPNT      KFS,KSS,KFF,MFF $
LABEL       LBL3 $
EQUIV       KFF,KAA/OMIT /MFF,MAA/OMIT $
CHKPNT      KAA,MAA $
ALTER       79
SMP2        USET,GO,MFF/MAA $
CHKPNT      MAA $
ALTER       81,129
ALTER       134,141
SETVAL      //V,N,BREAK/C,N,1/V,N,LINK/C,N,-1 $
SAVE        BREAK,LINK $
LABEL       CONV $
EQUIV       KAA,KDAA/BREAK $
EQUIV       MAA,MDAA/BREAK $
READ        KAA,MAA,,,EED,USET,CASECC/LAMA,PHIA,MI,OEIGS/C,N,MODES/V,
            N,NEIG/C,N,2 $
SAVE        NEIG $
CHKPNT      LAMA,PHIA,MI,OEIGS $
OFP         LAMA,OEIGS,,,,//V,N,CARDNO $
SAVE        CARDNO $
COND        FINIS,NEIG $
SDR1        USET,,PHIA,,,GO,GM,,KFS,,/PHIG,,BQG/C,N,1/C,N,REIG $
CHKPNT      PHIG,BQG $
EQUIV       PHIG,PHIAMP/BREAK $
ADD         PHIG,/PHIAMP/V,Y,AMP $
```

```
CHKPNT     PHIAMP $
EQUIV      PHIAMP,PHIG/LINK $
DSMG1      CASECC,,SIL,,PHIG,CSTM,MPT,ECPT,GPCT,DIT/KGGG/V,N,
           DSCOSET/C,N,1 $
CHKPNT     KGGG $
ADD        KGGG,KGG/KDGGG $
CHKPNT     KDGGG $
EQUIV      KDGGG,KDNN/MPCF2 /MGG,MDNN/MPCF2 $
CHKPNT     KDNN,MDNN $
COND       LBL2D,MPCF2 $
MCE2       USET,GM,KDGGG,MGG,,/KDNN,MDNN,, $
CHKPNT     KDNN,MDNN $
LABEL      LBL2D $
EQUIV      KDNN,KDFF/SINGLE /MDNN,MDFF/SINGLE $
CHKPNT     KDFF,MDFF $
COND       LBL3D,SINGLE $
SCE1       USET,KDNN,MDNN,,/KDFF,KDFS,,MDFF,, $
CHKPNT     KDFF,KDFS,MDFF $
LABEL      LBL3D $
EQUIV      KDFF,KDAA/OMIT /MDFF,MDAA/OMIT $
CHKPNT     KDAA,MDAA $
COND       LBL5D,OMIT $
SMP1       USET,KDFF,,,/GDO,KDAA,KDOO,LDOO,UDOO,,,,, $
CHKPNT     GDO,KDAA $
SMP2       USET,GDO,MDFF/MDAA $
CHKPNT     MDAA $
LABEL      LBL5D $
EQUIV      KDAA,KAA/LINK $
EQUIV      MDAA,MAA/LINK $
REPT       CONV,1 $
ADD        MDAA,KDAA/KMAA $
ADD        PHIAMP,/PHIM $
ALTER      157,158
ENDALTER
CEND
```

APPENDIX B

Nonlinear Vibration Analysis With Differential Stiffness

```
D       NLVDS,BEAM FRAME
PP      DISP
OL      13,0
IME     5
        NONLINEAR VIBRATION MODE WITH DIFFERENTIAL STIFFNESS FOR BEAM
        AND FRAME STRUCTURES
LTER        50
QUIV        MGG,MNN/MPCF1 $
HKPNT       MNN $
LTER        62,72
CE2         USET,GM,KGG,MGG,,/KNN,MNN,, $
HKPNT       KNN,MNN $
ABEL        LBL2 $
QUIV        KNN,KFF/SINGLE /MNN,MFF/SINGLE $
HKPNT       KFF,MFF $
OND         LBL3,SINGLE $
CE1         USET,KNN,MNN,,/KFF,KFS,KSS,MFF,, $
HKPNT       KFS,KSS,KFF,MFF $
ABEL        LBL3 $
QUIV        KFF,KAA/OMIT /MFF,MAA/OMIT $
HKPNT       KAA,MAA $
ALTER       75
MP2         USET,GO,MFF/MAA $
HKPNT       MAA $
ALTER       106,125
ALTER       130,130
SETVAL      //V,N,BREAK/C,N,1/V,N,LINK/C,N,-1 $
SAVE        BREAK,LINK $
ABEL        CONV $
EQUIV       KAA,KSAA/BREAK $
EQUIV       MAA,MSAA/BREAK $
READ        KAA,MAA,,,EED,USET,CASECC/LAMA,PHIA,MI,OEIGS/C,N,MODES/V,
            N,NEIGV/C,N,2 $
ALTER       136,137
SDR1        USET,,PHIA,,,GO,GM,,KFS,,/PHIG,,BQG/C,N,1/C,N,BKL1 $
CHKPNT      PHIG,BQG $
EQUIV       PHIG,PHIAMP/BREAK $
ADD         PHIG,/PHIAMP/V,Y,AMP $
CHKPNT      PHIAMP $
EQUIV       PHIAMP,PHIG/LINK $
DSMG1       CASECC,,SIL,,PHIG,CSTM,MPT,ECPT,GPCT,DIT/KGGG/V,N,
            DSCOSET/C,N,1 $
CHKPNT      KGGG $
ADD5        KGG,KDGG,KGGG,,/KSGG $
CHKPNT      KSGG $
```

```
EQUIV      KSGG,KSNN/MPCF2 / MGG,MSNN/MPCF2 $
CHKPNT     KSNN,MSNN $
COND       LBL2S,MPCF2 $
MCE2       USET,GM,KSGG,MGG,,/KSNN,MSNN,, $
CHKPNT     KSNN,MSNN $
LABEL      LBL2S $
EQUIV      KSNN,KSFF/SINGLE / MSNN,MSFF/SINGLE $
CHKPNT     KSFF,MSFF $
COND       LBL3S,SINGLE $
SCE1       USET,KSNN,MSNN,,/KSFF,KSFS,,MSFF,, $
CHKPNT     KSFF,KSFS,MSFF $
LABEL      LBL3S $
EQUIV      KSFF,KSAA/OMIT / MSFF,MSAA/OMIT $
CHKPNT     KSAA,MSAA $
COND       LBL5S,OMIT $
SMP1       USET,KSFF,,,/GSO,KSAA,KSOO,LSOO,USOO,,,,, $
CHKPNT     GSO,KSAA $
SMP2       USET,GSO,MSFF/MSAA $
CHKPNT     MSAA $
LABEL      LBL5S $
EQUIV      KSAA,KAA/LINK $
EQUIV      MSAA,MAA/LINK $
REPT       CONV,1 $
ADD        MSAA,KSAA/KMAA $
ADD        PHIAMP,/PHIM $
ENDALTER
CEND
```

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TABLE 1. FREQUENCY RATIO OF A HINGED BEAM

Axial Load, $\frac{P_{xo}}{P_E}$	Method	Amplitude Ratio, $d = \frac{c}{\sqrt{I/A}}$			
		0.0	1.0	2.0	3.0
0.0	NASTRAN	1.0000	1.0889	1.3183	1.6260
	Ref. 8	1.0000	1.0892	1.3178	1.6257
	Ref. 9	1.0000	1.0897	1.3229	1.6394
0.2	NASTRAN	.8944	.9928	1.2401	1.5631
	Ref. 8	.8944	.9930	1.2389	1.5618
	Ref. 9	.8944	.9937	1.2450	1.5772
0.4	NASTRAN	.7746	.8864	1.1566	1.4976
	Ref. 8	.7746	.8864	1.1543	1.4949
	Ref. 9	.7746	.8874	1.1619	1.5125
0.6	NASTRAN	.6325	.7653	1.0666	1.4291
	Ref. 8	.6325	.7649	1.0627	1.4246
	Ref. 9	.6325	.7665	1.0724	1.4448
0.8	NASTRAN	.4472	.6210	.9682	1.3570
	Ref. 8	.4472	.6194	.9617	1.3502
	Ref. 9	.4472	.6225	.9747	1.3739
1.0	NASTRAN	.0026	.4309	.8586	1.2810
	Ref. 8	.0000	.4236	.8472	1.2708
	Ref. 9	.0000	.4330	.8660	1.2990
1.2	NASTRAN			.7329	1.2000
	Ref. 8			.7105	1.1851
	Ref. 9			.7416	1.2196

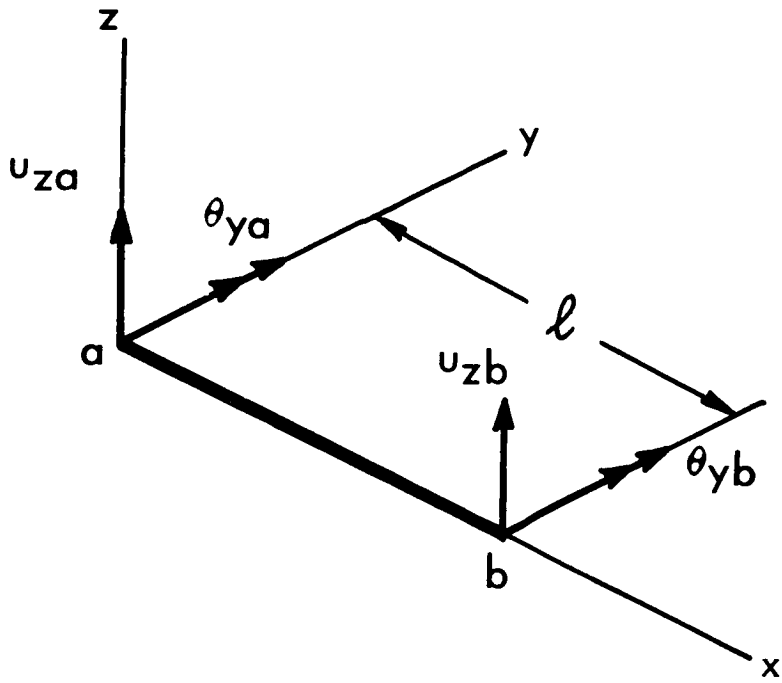

Figure 1. Bar coordinate system, showing displacements due to bending in the xz plane.

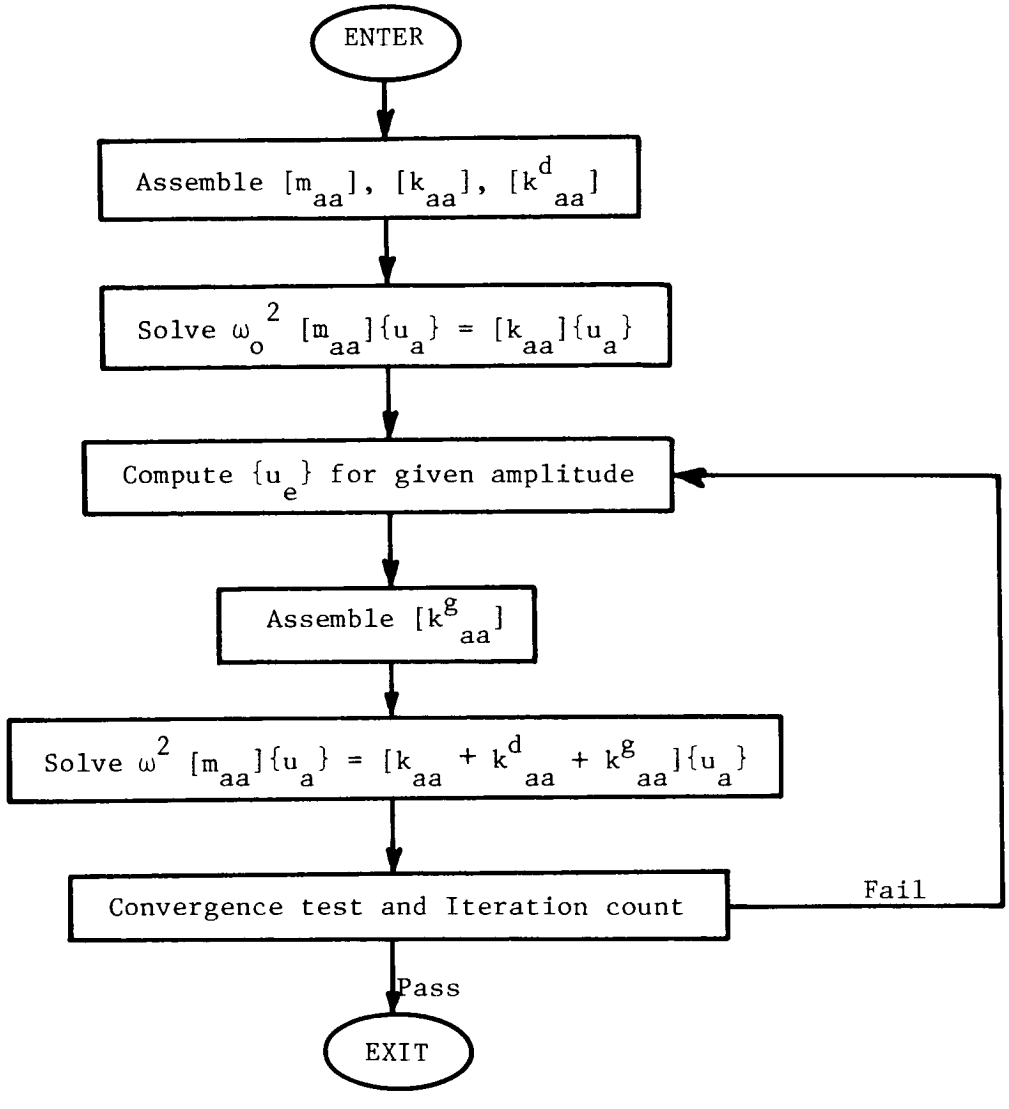

Figure 2. Simplified flow diagram for nonlinear vibration analysis.

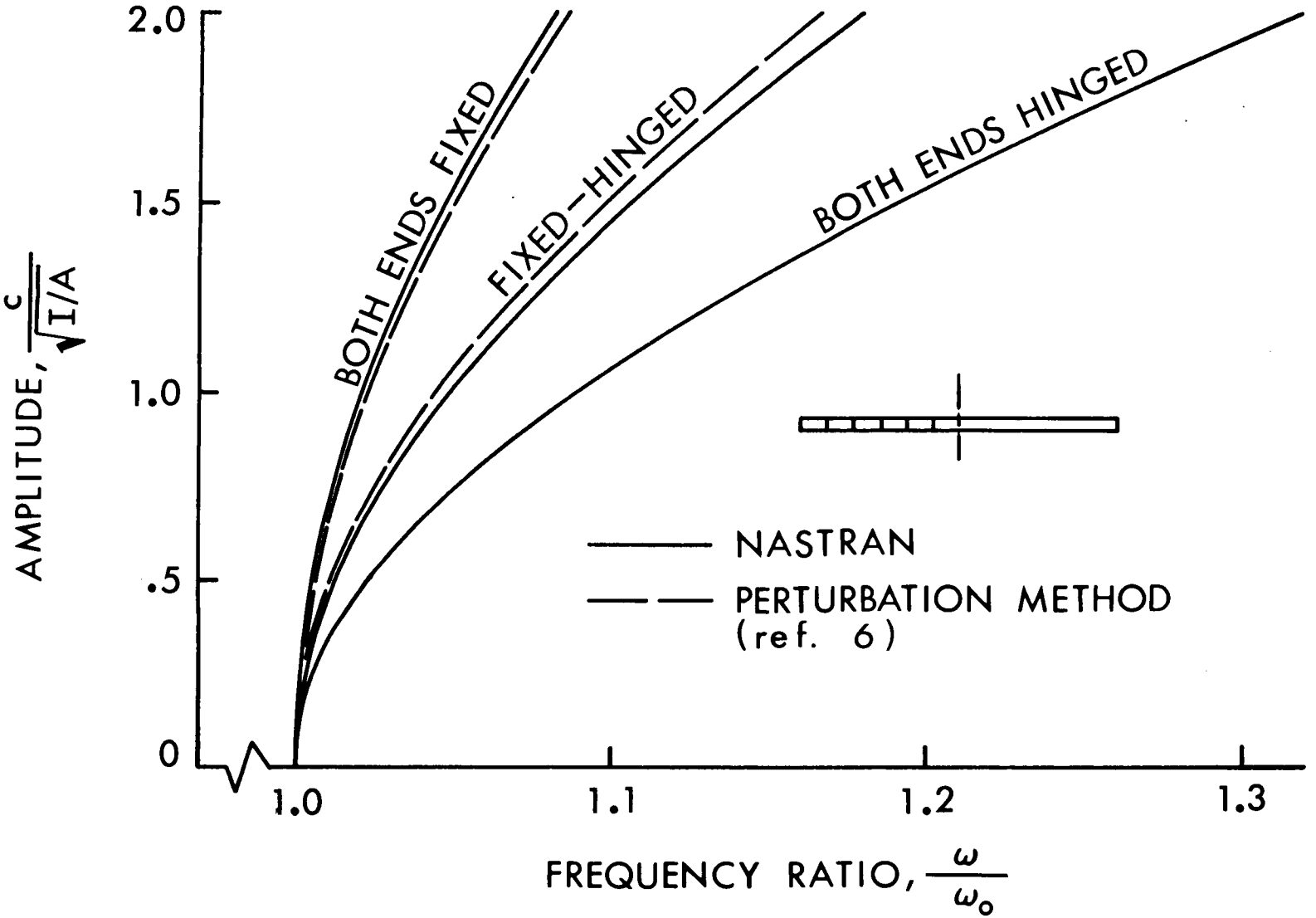

Figure 3. Amplitude-frequency curves for various support conditions.

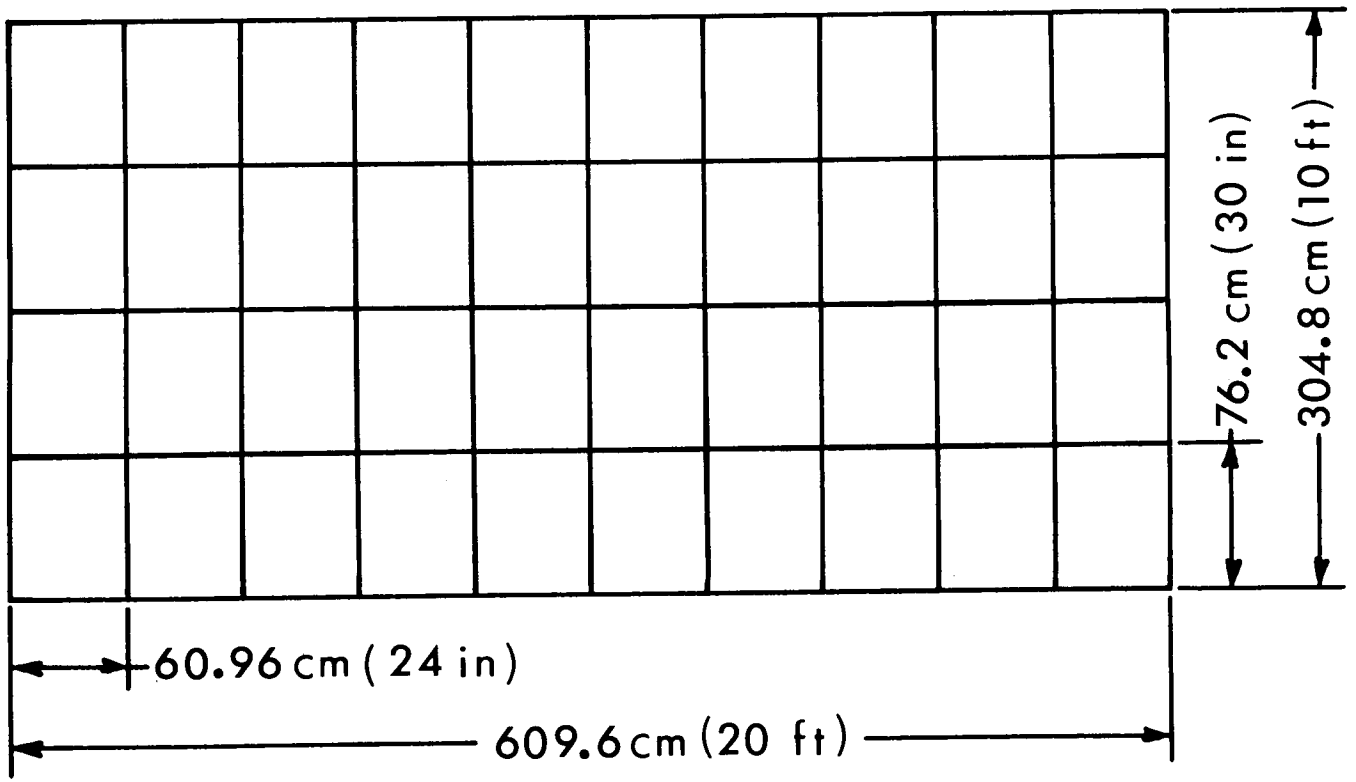

Figure 4. A rectangular frame constructed from bars with each edge fixed.

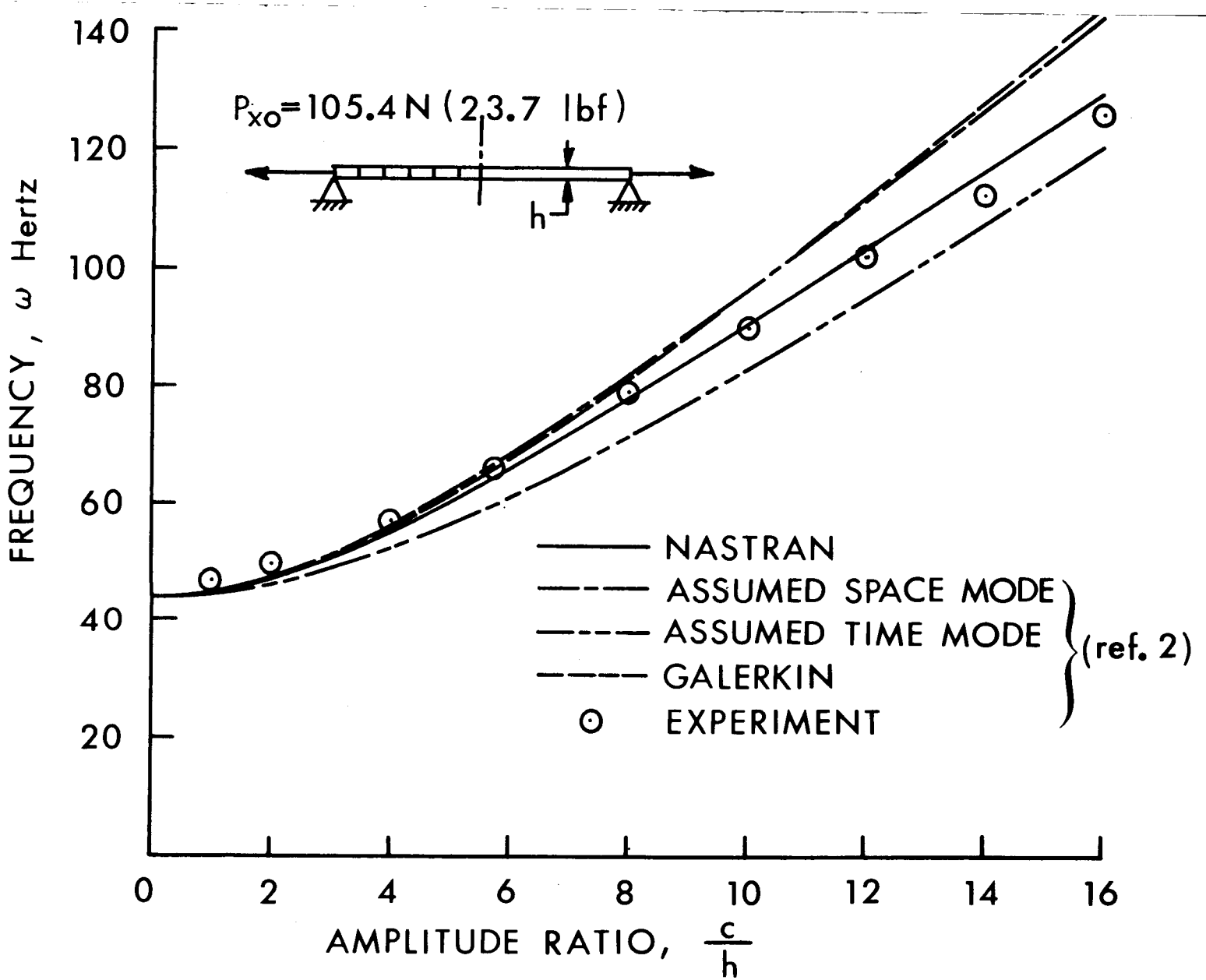

Figure 5. Comparison of experimental and analytical frequencies.

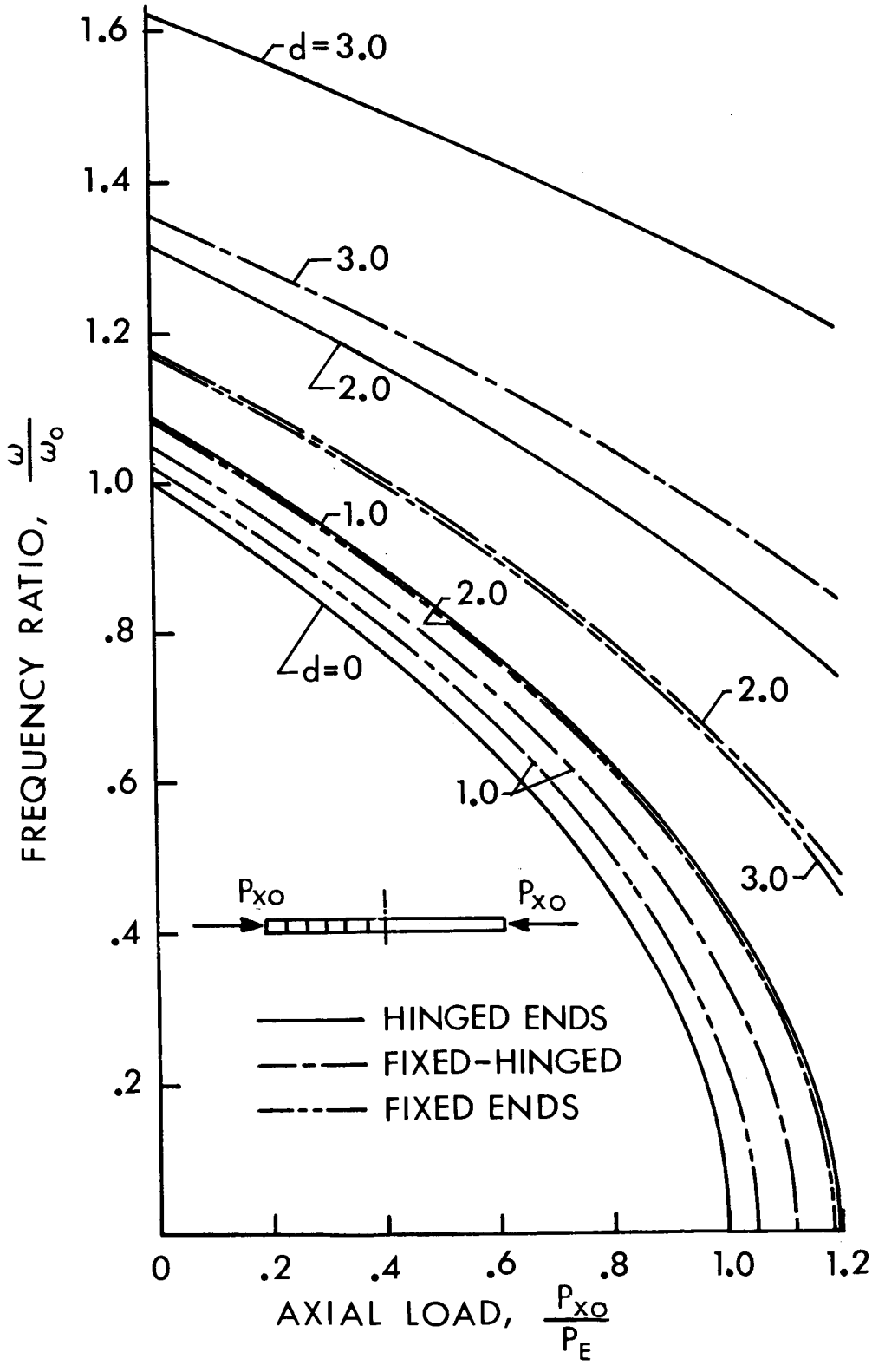

Figure 6. Variation of frequency with axial load for various support conditions.