A THEORY OF FLIGHT FLUTTER TESTING

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Abstract

Flight flutter testing is considered as a method for finding generalized aerodynamic forces. The coefficients determined from flight flutter tests are used in flutter calculations, using a simple expansion in frequency and Mach number. The errors in the procedure are discussed, and expressions for the error in flutter prediction are given. Methods of testing procedure are discussed.

INTRODUCTION

This paper considers flutter testing and flight flutter testing a part of flutter analysis. Very often nowadays, tests which were originally intended as proof tests or acceptance tests inadvertently became exploration of the unknown. This situation will persist until flutter analysis can be used with confidence, to the extent that the accuracy of a flutter prediction can be computed as part of the analysis.

Since this situation exists, one might as well consider such tests as links in the flutter analysis, and squeeze out as much information as possible from the test results, rather than rest content with say, flutter frequency, speed and Mach number as the only result of a wind tunnel flutter test, which usually cannot be repeated using the same model.

We shall, therefore, consider the equations of motion on a wing vibrating in an airstream, examine which quantities can be measured, which quantities can be found from a simpler test, and attempt to assay the accuracy of data obtained from static, ground vibration, flight vibration and flutter tests.

Finally, we shall look at the accuracy of a flutter prediction, in terms of the precision of the data used in the computation.

The Equations of Motion

We assume the wing to be perfectly elastic, and assume the motion of the wing to be small, such that the aerodynamic loads are proportional to some linear integral-differential transform of the deflections.

This integro-differential dependence of airloads on deflection is proportional to dynamic pressure, but may depend upon flight altitude, and depends upon Mach number and frequency of oscillation.

The equation of motion can then be written:

$$z(x, y) = \iint_{\substack{Wing\\Area}} c(x, y|\xi, \eta)\{\omega^2 m(\xi, \eta) z(\xi, \eta) +$$

$$F(\xi, \eta) + \tfrac{1}{2}\rho U^2 D_1(\xi, \eta; k, M) \iint_{\substack{Wing\\Area}} q(\xi, \eta; r, s, k, M)$$

$$D_2(r, s; k, M) z(r, s) dr ds\} d\xi d\eta$$

where:

$$R_e [z(x, y) e^{i\omega t}]$$

is the deflection of the wing at (x, y) at time t.

$$2\pi\omega$$

is the frequency of vibration.

$$m(x, y)$$

is the mass per unit wing area at (x, y).

$$c(x, y|\xi, \eta)$$

is the deflection at (x, y) due to a unit load applied at (ξ, η).

$$R_e\,[F(x,\ y)\,e^{i\omega t}]$$

is the force applied to the wing by shakers, or ground supports.

$$D_1(\xi,\eta;\ k,\ M)\iint_{Wing} q(\xi,\eta;\ r,\ s,\ k,\ M)\,D_2(r,\ s,\ k,\ M)\,z(r,\ s)\,dr\,ds$$

is the operator which yields the lift per unit area at $(\xi,\ \eta)$ divided by the dynamic pressure for a deflection amplitude distribution z(x, y) at Mach number M and reduced frequency $\frac{\omega b}{U}$.

At zero airspeed and frequency, this is the equation for a ground static test, for zero airspeed only it is the equation of a ground vibration test, and for zero impressed force, it is the equation for flutter, while the whole equation describes a flight vibration test.

To be able to use the equation, one must rewrite it using some kind of approximation. One can use an approximation in natural modes, but that seems pointless unless they are known precisely. The alternative is to use an approximation in discrete ordinates, or if one is in a fancy mood, to use station functions, or an approximation in terms of surface stresses.

We shall use an approximation in discrete ordinates, namely the deflections Z_ν at the points $(x_\nu,\ y_\nu)$ where ν refers to the number of the point in some kind of ordered sequence.

Equation 1 then becomes:

$$\{z_\mu\} = [c_{\mu\nu}][m_\nu]\omega^2[H]\ \{z_\nu\} + [c_{\mu\nu}][H]\ \{F_\nu\} + \frac{1}{2}\rho U^2\,[c_{\mu\nu}][q_{\nu\alpha}(k,\ M)]\{z_\alpha\} \qquad (2)$$

where $[q_{\nu\alpha}]$ is the matrix corresponding to the linear integro-differential operator which yields the lift distribution. $[H]$ is a diagonal matrix of integration weights, it has been lumped with the $[q_{\nu\alpha}(k,\ M)]$ in the last term on the right hand side.

The equation for flutter states that the determinant of (2) must vanish for $\{F_\nu\} = 0$ in order to obtain a non-trivial solution:

$$D \equiv \left|\ [-1] + \omega^2[c][m][H] + \frac{1}{2}\rho U^2[c][q(k,\ M)]\ \right| = 0 \qquad (3)$$

We shall now proceed to write down the equations for a set of tests, and to examine the rate of change of flutter speed with changes in the elements of the flutter determinant. The latter will enable us to assess the first order error in the flutter prediction due to errors in wing parameters and aerodynamic coefficients.

The Equation for a Set of Tests

If one repeats a flight vibration test N times, one obtains N equations like equation (1), which can be written as a single equation. If all these tests are performed at the same reduced frequency k and Mach number M, the combined equation becomes especially simple, and takes on the form:

$$\begin{aligned}[z_{\mu n}] = {} & [c_{\mu\nu}][H_\nu][H_\nu][z_{\nu n}][\omega_n^2] + \\ & i\,[c_{\mu\nu}][d_\nu][H_\nu][z_{\nu n}][\omega_n] + \\ & [c_{\mu\nu}][F_{\nu n}] + [c_{\mu\nu}][q_{\mu\nu}(k,\ M)][z_{\nu n}][(\tfrac{1}{2}\rho U^2)_n]\end{aligned} \qquad (4)$$

where we have included a structural damping term with $[d]$ as the matrix of damping coefficients. n is the test number, so $z_{\nu n}$ is the deflection amplitude at $(x_\nu,\ y_\nu)$ in the n'th test, ω_n is the frequency, and $(\frac{1}{2}\rho U^2)_n$ is the dynamic pressure in the n'th test.

After having performed a set of N tests, where all but N columns of one of the matrices in Equation (4) are either measured in the tests or known from previous tests or analysis, it is possible to compute the unknown columns. As examples, we shall consider a set of static tests, a set of ground vibration tests, a set of flutter tests and a set of flight vibration tests.

A set of static tests should obey the equation:

$$[z_{\mu n}] = [c_{\mu\nu}][F_{\nu n}]$$

which can be inverted to yield:

$$[c_{\mu\nu}] = [z_{\mu n}][F_{\nu n}]^{-1}$$

where $[F_{\nu n}]^{-1}$ is

$$[F_{\nu n}]^{-1} = \left[\frac{\bar{F}'_{\nu n}}{|F|}\right]$$

$\bar{F}'_{\nu n}$ is the cofactor of the element $F'_{\nu n}$ in the transpose of $[F_{\nu n}]$. The first order error in $[c_{\mu\nu}]$ due to error in the measurement of $z_{\mu\nu}$ and $[F_{\nu n}]$ can be evaluated as follows:

$$\Delta[c_{\mu\nu}] = [\Delta c_{\mu\nu}] = [\Delta z_{\mu n}][F_{\nu n}]^{-1} +$$

$$[z_{\mu n}]\sum_r\sum_s \frac{1}{|F|^2}\left(-\frac{\partial|F|}{\partial F_{rs}}\,[\bar{F}'_{\nu n}] + |F|\,\frac{\partial[\bar{F}'_{\nu n}]}{\partial F_{rs}}\right)\Delta F_{rs}$$

$$= \sum_r\sum_s [K'_{rs}]\ \Delta z_{rs} + \sum_r\sum_s [K''_{rs}]\ \Delta F_{rs}$$

If the errors are given in terms of standard deviations, $\sigma_{F_{rs}}$ and $\sigma_{z_{\mu n}}$, the standard deviation in the element $c_{\mu\nu}$ is:

$$\sigma_{c_{\mu\nu}}^2 = \sum_r\sum_s K'^{\,2}_{rs}\,\sigma_{z_{rs}}^2 + \sum_r\sum_s K''_{rs}\sigma_{F_{rs}}^2$$

The coefficients K''_{rs} can be seen to be large when the determinant $|F|$ is small, i.e. when one of the columns or rows in the loading matrix is nearly a linear combination of the other columns or rows.

The Equation for a Set of Ground Vibration Tests

For a set of ground vibration tests, one obtains:

$$[z_{\mu n}] = [c_{\mu\nu}][H_\nu][m][z_{\nu n}][\omega_n^2] + [c_{\mu\nu}][F_{\nu n}] + i\,[c_{\mu\nu}][d][z_{\nu n}][\omega_n]$$

Equating real parts:

$$Re\,[z_{\mu n}] = [c_{\mu\nu}]\,[H_\nu]\,[m]\,Re\,[z_{\nu n}]\,[\omega_n^2] + [c_{\mu\nu}]\,Re\,[F_{\nu n}]$$

or

$$[m] = [H_\nu]^{-1}[c_{\mu\nu}]^{-1}(Re\,[z_{\mu\nu}] - [c_{\mu\nu}]\,Re\,[F_{\nu n}])\,[\omega_n^2]^{-1}[Re\,z_{\nu n}]^{-1}$$

and we see that if the determinants of $[c_{\mu\nu}]$ and $[z_{\nu n}]$ are small, the first order errors may become large. However, in the flutter equation, $[m]$ only occurs in the combination:

$$[c_{\mu\nu}]\,[H_\nu]\,[m_\nu]$$

and therefore only this combination is of interest:

$$[c_{\mu\nu}]\,[H_\nu]\,[m_\nu] = (Re\,[z_{\mu n}] - [c_{\mu\nu}]\,[Re F_{\nu n}])\,[\omega_n^2]^{-1}[Re\,z_{\mu n}]^{-1}$$

which shows that these errors in $[z_{\mu n}]$ and $[F_{\nu n}]$ are really important. The matrix of first order errors of the left hand side is $[\Delta]$, where:

$$[\Delta] = ([\Delta Re\,z_{\mu\nu}] - [c_{\mu\nu}]\,[\Delta Re F_{\nu n}])\,[\omega_n^2]^{-1}[Re\,z_{\mu n}]^{-1}$$
$$+ ([Re\,z_{\mu\nu}] - [c_{\mu\nu}]\,[Re F_{\nu n}])\,((\Delta[\omega_n^2]^{-1}[Re\,z_{\mu n}]^{-1})$$
$$+ [\omega_n^2]^{-1}\Delta([Re\,z_{\mu n}]^{-1}))$$

The error will therefore be proportional to the inverse square of the determinant of $[Re\,z_{\mu u}]$; this determinant should be maximized by arranging the test such that the columns of the determinant are orthogonal if possible. This means that each test should be performed at a natural frequency.

Equation for a Set of Flight Vibration Tests or Flutter Tests

The information obtainable from a set of flight vibration tests or flutter tests which cannot be obtained from tests where there are no aerodynamic forces are, of course, the aerodynamic forces.

Since the aerodynamic coefficients depend upon Mach number, M, and reduced frequency, k, the tests must either be performed at constant M and k, or one must somehow approximate this dependence.

One can for example use a Taylor series expansion of $[q_{\mu\nu}(k, M)]$ in k and M about some value of M, M_{ref} and zero reduced frequency. One obtains:

$$[q_{\mu\nu}(k, M)] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left[\left(\frac{\partial^r}{\partial M^r}\frac{\partial^s}{\partial k^s}q_{\mu\nu}(k, M)\right)_{\substack{k=0\\ M=M_{ref}}}\right]\frac{(M-M_{ref})^r k^s}{r!\ s!}$$

Instead of expanding in power of $(M-M_{ref})$, one can expand in powers of (M^2-1) for transonic Mach numbers and $(M^2-1)^{-1/2}$ for supersonic Mach numbers.

As an engineering approximation one would only use the first and zero order terms.

$$\frac{1}{r!\ s!}\left[\left(\frac{\partial^r}{\partial M^r}\frac{\partial^s}{\partial k^s}q_{\mu\nu}(k, M)\right)_{M=M_{ref}}\right] = [A_{\mu\nu}^{(r,\ s)}]$$

The equation for a set of tests is then Eq. (4), solving for the aerodynamic terms, one obtains:

$$[c_{\mu\nu}]\,[H_\nu]\,[q_{\mu\nu}(k, M)] =$$
$$[c_{\mu\nu}]\,[H_\nu]\sum_r\sum_s[A_{\mu\nu}^{(r,\ s)}]\,k_n^s(M_n - M_{ref})^r =$$
$$\{[z_{\mu u}] - [c_{\mu\nu}]\,([m]\,[H]\,[z_{\nu u}]\,[\omega_u^2] + i\,[d]\,[H]\,[z_{\nu u}]\,[\omega_u] +$$
$$[F_{\nu u}])\}\,[(\tfrac{1}{2}\rho U^2)_u]^{-1}[z_{\nu u}]^{-1}$$

This set of equations may be insufficient to determine $[q_{\mu\nu}(k, M)]$. However, some of the $q_{\mu\nu}(k, M)$ are not very important as far as the flutter speed is concerned, the zero order terms in k can be determined by wind tunnel tests on stationary but deformed wings (tied down), others can again be guessed at, at least, from linearized aerodynamic theory. The purpose of a flight vibration test or a flutter test is then to determine the remaining aerodynamic coefficients. Without going into a discussion of which aerodynamic coefficients are to be chosen as those which neither theory nor wind tunnel static tests can yield, we shall consider the precision obtainable in $q_{\mu\nu}(k, M)$ when it is determined from tests.

The term which is most liable to magnify the errors is the errors in the inverse of $[z_{\mu u}]$. The value of $[z_{\mu u}]^{-1}$ is

$$\frac{[\bar{z}'_{\mu u}]}{|z_{\mu u}|} = [z_{\mu u}]^{-1}$$

When differentiating to evaluate the error, one obtains an expression with $|z|^2$ in the denominator. To minimize errors, one must try to make $|z|$ as large as possible, i.e., the columns in $[z_{\mu u}]$ should be as different as possible. Vibration is natural modes only will go far towards the accomplishment of precision.

Errors in Flutter Prediction due to Errors in Structural, Mass and Aerodynamic Parameters

Before the obtainable precision in experimental determination of structural, mass and aerodynamic information can be meaningful in terms of resulting accuracy in flutter prediction, we have to analyze the sensitivity of a flutter point to such errors.

Flutter occurs whenever the determinant (Eq. (3)) vanishes:

$$D(k,\ M, \tfrac{1}{2}\rho U^2,\ m_1, \ldots, m_N,\ d_1, \ldots, d_\nu,$$
$$q_{11}, \ldots, q_{NN},\ c_{11}, \ldots, c_{NN}) = 0$$

Vary one of the parameters, which we shall call P. Both the real and imaginary parts of the flutter determinant will then change, and k and $\frac{1}{2}\rho U^2$ must then be changed to compensate, such as to maintain the value of the flutter determinant at zero at constant M. Instead of changing k and $\frac{1}{2}\rho U^2$, k and M can be changed, at constant $\frac{1}{2}\rho U^2$, or ρ and k can be changed only, at constant M and U.

We shall only consider changes in k and $\frac{1}{2}\rho U^2$ at constant M.

To maintain flutter for a change in P, one must have:

$$Re(\Delta D) = Re(\frac{\partial D}{\partial P}\Delta P) + Re(\frac{\partial D}{\partial(\frac{1}{2}\rho U^2)})_M \Delta(\frac{1}{2}\rho U^2) + Re(\frac{\partial D}{\partial k})_M \Delta k = 0$$

$$Im(\Delta D) = Im(\frac{\partial D}{\partial P}\Delta P) + Im(\frac{\partial D}{\partial(\frac{1}{2}\rho U^2)})_M \Delta(\frac{1}{2}\rho U^2) + Im(\frac{\partial D}{\partial k})_M \Delta k = 0$$

Solving for $\Delta(\frac{1}{2}\rho U^2)$ and Δk, one obtains:

M = const.

$$\Delta(\frac{1}{2}\rho U^2)_M = \frac{Im\{\overline{(\frac{\partial D}{\partial P}\Delta P)}\frac{\partial D}{\partial k}\}}{Im\{\overline{(\frac{\partial D}{\partial(\frac{1}{2}\rho U^2)})}\frac{\partial D}{\partial k}\}}$$

$$(\Delta k)_M = \frac{Im\{\overline{(\frac{\partial D}{\partial P}\Delta P)}\frac{\partial D}{\partial(\frac{1}{2}\rho U^2)}\}}{Im\{\overline{(\frac{\partial D}{\partial k})}\frac{\partial D}{\partial(\frac{1}{2}\rho U^2)}\}}$$

where the bars denote the complex conjugate and the derivative with respect to k is taken at constant M and $\frac{1}{2}\rho U^2$, and the derivative with respect to $\frac{1}{2}\rho U^2$ is taken at constant k and M.

It is, of course, complicated to evaluate these derivatives, but it seems to be necessary for finding the sensitivity of a flutter point to parameter changes. With modern computers it may, however, be possible.

A rough knowledge of the precision of a flutter prediction will always be useful; one must keep firmly in mind, however, that the estimate of precision is in terms of a given numerical approximation, and can give no information about the remainder term of the numerical approximation.

In practice, when a flutter point proves very insensitive to parameter changes, it should not be allowed to cause unalleviated elation, since then it will take a major design change to move the flutter point out of the flight envelope of the airplane, for example.

Conclusion

A viewpoint and a method of approach to flight flutter testing and to flutter in general has been outlined. It is realized that only the practising flutter analyst can choose the method of analysis and the tests to be performed, knowing the limitations of his facilities and his personnel.

The method which has been outlined is clearly impractical; however, if some of its elements are used, or if nothing else, its viewpoint is adopted, the paper will have accomplished its purpose.

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