

# TRANSONIC FLOW THEORY OF AIRFOILS AND WINGS\*

P. R. Garabedian  
Courant Institute of Mathematical Sciences

## SUMMARY

Supercritical wing technology is expected to have a significant influence on the next generation of commercial aircraft. Computational fluid dynamics is playing a central role in the development of new supercritical wing sections. One of the principal tools is a fast and reliable code that simulates two-dimensional wind tunnel data for transonic flow at high Reynolds numbers. This is used widely by industry to assess drag creep and drag rise. Codes for the design of shockless airfoils by the hodograph method have not been so well received because they usually require a lot of trial and error. However, a more advanced mathematical approach makes it possible to assign the pressure as a function of the arc length and then obtain a shockless airfoil that nearly achieves the given distribution of pressure. This tool should enable engineers to design families of transonic airfoils more easily both for airplane wings and for compressor blades in cascade.

## INTRODUCTION

There are plans to use the supercritical wing on the next generation of commercial aircraft so as to economize on fuel consumption by reducing drag. Computer codes have served well in meeting the consequent demand for new wing sections. One of the most widely adopted codes was developed at the Courant Institute to simulate two-dimensional transonic flow over an airfoil at high Reynolds numbers (ref. 1). This work is an example of the possibility of replacing wind tunnel tests by computational fluid dynamics. Another approach to the supercritical wing is through shockless airfoils. Here a novel boundary value problem in the hodograph plane will be discussed that enables one to design a shockless airfoil so that its pressure distribution very nearly takes on data that have been prescribed. An advanced design code of this kind has been written recently by David Korn and is turning out to be so successful that it may ultimately gain the same acceptance as the better established analysis code.

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Physically realistic transonic flow computations can be based on a potential equation that presupposes conservation of entropy across shock waves, but permits a jump in the normal component of momentum. However, to treat either the problem of design or of analysis for transonic airfoils in a satisfactory way from the engineering point of view, it is necessary to take into account the effect of the turbulent boundary layer. The simplest procedure is to calculate the displacement thickness of the boundary layer from the inviscid pressure distribution by a momentum integral method of Nash and Macdonald (ref. 2). For analysis one adds the displacement thickness to the profile at each cycle of an iterative scheme determining the flow. In the case of design a corresponding quantity is subtracted from the airfoil coordinates, which therefore have to be provided with a slightly open trailing edge to begin with.

It is important to eliminate separation entirely in the problem of design if there is to be no loss of lift in practice. This can be accomplished by imposing a pressure distribution at the rear of the upper surface that just avoids separation according to a criterion of Stratford (ref. 3). The boundary layer correction has been found to give satisfactory results even when its implementation only involves a primitive model of the wake in which pressure forces balance across a parallel pair of trailing streamlines. Extensive wind tunnel tests from laboratories all over the world confirm that the analysis code agrees well with experimental data when the boundary layer correction is made. Preliminary test data on a cascade airfoil that was heavily aft-loaded also inspire confidence in the concept of using a Stratford pressure distribution to avoid loss of lift in design by the hodograph method.

The transonic flow codes developed at the Courant Institute have been distributed to industry by the Langley Research Center. In the future they will also become available through the Argonne Code Center of the Argonne National Laboratory.

#### THE METHOD OF COMPLEX CHARACTERISTICS

The partial differential equations for the velocity potential  $\phi$  and stream function  $\psi$  of two-dimensional irrotational flow of a compressible fluid can be written in terms of characteristic coordinates  $\xi$  and  $\eta$  in the canonical form

$$\phi_{\xi} = i\sqrt{1-M^2} \psi_{\xi}/\rho, \quad \phi_{\eta} = -i\sqrt{1-M^2} \phi_{\eta}/\rho,$$

where the local Mach number  $M$  and the density  $\rho$  are functions of the speed  $q$  defined by Bernoulli's law. A fast and flexible numerical scheme for the construction of smooth transonic flows in the hodograph plane has been developed by continuing these equa-

tions analytically into the domain of complex values of the two independent variables  $\xi$  and  $\eta$  (ref. 4). The coordinates  $\xi$  and  $\eta$  can be specified in terms of the speed  $q$  and the flow angle  $\theta$  by the formulas

$$\log f(\xi) = \int \sqrt{1-M^2} \frac{dq}{q} - i\theta, \quad \log \overline{f(\eta)} = \int \sqrt{1-M^2} \frac{dq}{q} + i\theta,$$

where  $f$  is any complex analytic function. Prescription of a second arbitrary analytic function  $g$  serves to determine  $\phi$  and  $\psi$  as solutions of the characteristic initial value problem

$$\psi(\xi, \eta_0) = g(\xi), \quad \psi(\xi_0, \eta) = \overline{g(\bar{\eta})},$$

where  $\xi_0 = \bar{\eta}_0$  is a fixed point in the complex plane. With these conventions it turns out that  $\psi(\xi, \eta) = \overline{\psi(\bar{\eta}, \bar{\xi})}$ , whence for subsonic flow the real hodograph plane corresponds to points in the complex domain where  $\xi = \bar{\eta}$ .

Consider the nonlinear boundary value problem of designing an airfoil on which the speed  $q$  has been assigned as a function of the arc length  $s$ . To construct a solution it is helpful to view  $f$  as a function mapping the region of flow onto the unit circle  $|\xi| < 1$ . There both  $\log f$  and  $g$  have natural expansions as power series in  $\xi$  after appropriate singularities accounting for the flow at infinity have been subtracted off. The coefficients of truncations of these series can be determined by interpolating to meet boundary conditions on  $q$  and  $\psi$  at equally spaced points of the circumference  $|\xi| = 1$ . Such a numerical solution is easily calculated because the matrix of the system of linear equations for the coefficients is well conditioned. This analytical procedure has the advantage that its formulation can be extended to the case of transonic flow so as to yield a shockless airfoil nearly fitting the prescribed data even when an exact solution of the physical problem does not exist.

To calculate transonic flows by the method that has been proposed, it is necessary to circumvent the sonic locus  $M = 1$ , which becomes a singularity of the partial differential equations for  $\phi$  and  $\psi$  in canonical form. In the plane  $\xi = \bar{\eta}$  this locus separates the region of subsonic flow from a domain where  $\psi(\xi, \bar{\xi})$  is no longer real. In the latter domain it is necessary to extend in some empirical fashion the relationship between  $\phi$  and  $s$  that is imposed by assigning  $q$  as a function of  $s$ . A formulation of the boundary conditions that applies to both the subsonic and the supersonic flow regimes is given by the formulas

$$\operatorname{Re} \{ \log f(\xi) \} = h, \quad \operatorname{Re} \{ \psi(\xi, \bar{\xi}) \} + k \operatorname{Im} \{ \psi(\xi, \bar{\xi}) \} = 0$$

on  $|\xi| = 1$ , where  $k$  is a real constant and  $h$  is a function of

$\text{Re} \{ \phi(\xi, \bar{\xi}) \}$  obtained from the known relationships among  $s$ ,  $q$  and  $\phi$ . The nonlinearity of the problem makes it necessary to iterate on this relationship in finding a numerical solution.

Empirical data on the condition number of the matrix for the linear equations determining the power series coefficients of the analytic function  $g$  indicate that the boundary value problem for  $\psi$  that has been formulated is well posed even in the transonic case. In contrast with the Tricomi problem, boundary values are assigned around the whole circumference of the unit circle. The success of the procedure can be attributed to the fact that data are assigned in a suitable complex extension of the real plane.

In general limiting lines may appear in the physical plane, but it has been found that these can be suppressed by appropriate selection of the rules defining the function  $h$  and the real parameter  $k$  that occur in the specification of the boundary conditions. Thus a tool becomes available for the construction of supercritical wing sections from their pressure distributions. Figure 1 shows an example of a shockless airfoil that was obtained this way, together with its Mach lines. Observe that the input pressure coefficient  $C_p$  differs somewhat from the values calculated as output of the flow in the supersonic zone, which is rather large. The data that were assigned are based on a modification of the experimental pressure distribution on Whitcomb's original supercritical wing (ref. 5) shown in Figure 2.

The design code has been written to include the case of transonic airfoils in cascade. This model seems to offer considerable promise for improvement in the efficiency of certain stages of high speed compressors. However, to handle cascades of high solidity with adequate resolution it is desirable to replace a conformal mapping onto the unit circle  $|\xi| < 1$  by the mapping onto an ellipse, where the Tchebycheff polynomials become preferable to powers of  $\xi$  for expansion of the analytic functions  $\log f$  and  $g$ . Likewise, to achieve adequate resolution at the trailing edge in cases of heavy aft-loading it is helpful to insert a special term at the tail in the representation of the map function  $f$ .

The new code represents a major advance over what was achieved in earlier versions, whose use required excessive trial and error (ref. 4). A typical run takes about six minutes on the CDC 6600 computer. Closure of the airfoil is readily attained by adjusting the pressure at the trailing edge and the relative lengths of arc over the upper and lower surfaces between the stagnation point and the trailing edge. A general principle to be observed when using shockless airfoils to design supercritical wing sections is that drag creep can be reduced by diminishing the size of the supersonic zone of flow, especially toward the rear of the profile. In practice the best way to assess the performance of a new design is to run it through the analysis code, which will be discussed next.

## ESTIMATION OF THE DRAG

Analysis of the transonic flow past an airfoil can be based on a partial differential equation for the velocity potential  $\phi$ . Weak solutions modelling shock waves are calculated by adding artificial viscosity. This can be accomplished with a full conservation form (FCF) of the equation, but a simpler version not in conservation form (NCF) is sometimes more useful (ref. 1). To handle the boundary conditions it is convenient to map the region of flow conformally onto the exterior of the unit circle. If  $r$  and  $\theta$  stand for polar coordinates there, the quasilinear equation for  $\phi$  can be written as

$$a \phi_{\theta\theta} + 2b \phi_{\theta r} + c \phi_{rr} + d = 0$$

when artificial viscosity is omitted. The simplest way of introducing artificial viscosity numerically, suggested first by Murman and Cole in a fundamental paper (ref. 6), is to use finite difference approximations that are retarded in the direction of the flow, which for practical purposes can be taken as the direction of  $\theta$ . This does not perturb the Neumann boundary condition on  $\phi$ .

The finite difference equations for transonic flow can be solved iteratively by a variety of relaxation schemes, all of which take the form of marching processes with respect to an artificial time parameter. Antony Jameson has found that the rate of convergence can be accelerated by substituting a fast solver over the subsonic flow region between every few cycles of relaxation (ref. 7). Such a procedure has been programmed by Frances Bauer using fast Fourier transform with respect to the periodic variable  $\theta$ . This reduces the calculation time by a factor of three even when a boundary layer correction is included in the computation. A standard run of her airfoil code now takes less than three minutes on the CDC 6600 computer.

Detailed comparisons with experimental data show that the NCF transonic equation gives significantly better simulation of shock wave-boundary layer interaction than does the FCF equation, especially in cases with a shock at the rear of the profile where the turbulent boundary layer is relatively thick. It would appear that the NCF method leads to less radical gradients in the pressure behind the shock, which is consistent with the observations. The NCF and experimental speeds both tend to jump down barely below the speed of sound behind a shock. Figure 2 shows the kind of agreement between theoretical and test data that is usually seen. Wall effect is accounted for by running the computer code at the same lift coefficient  $C_L$  that occurs in the experiment.

Because of erroneous positive terms in the artificial viscosity, the shock jumps defined by the NCF method create mass instead

of conserving it. However, the amount of mass produced is only of the order of magnitude of the square of the shock strength for nearly sonic flow. The resulting errors are therefore negligible except for their effect on the calculation of the wave drag, which has the order of magnitude of the cube of the shock strength. A correct estimate of the drag can be obtained from NCF computations by working with the path-independent momentum integral

$$D = \int [p \, dy + (\phi_x - c_*) \, d\psi] ,$$

where  $p$  and  $c_*$  stand for the pressure and the critical speed, respectively. The integrand has been arranged so that across a normal shock wave parallel to the  $y$ -axis it jumps by an amount of the third order in the shock strength. Therefore integration around the shocks gives a reasonable measure of the wave drag even when mass is not conserved.

The path of integration can be deformed onto the profile to define a standard integral of the pressure there, but a correction term evaluated over a large circle should be added because of a sink at infinity accounting for the mass generated by the NCF method. Let  $l$ ,  $\rho_\infty$  and  $q_\infty$  denote the chord length of the airfoil, the density at infinity and the speed at infinity, respectively. The corrected formula for the wave drag coefficient  $C_{DW}$  becomes

$$C_{DW} = \frac{2}{l \rho_\infty q_\infty^2} \int p \, dy - 2 \frac{c_* - q_\infty}{l \rho_\infty q_\infty^2} \int d\psi ,$$

where the first integral is extended over the profile and the second integral is extended over a large circle separating the profile from infinity. In Figure 3 a comparison is presented between experimental, corrected NCF, uncorrected NCF and FCF values of the total drag coefficient  $C_D$  for a shockless airfoil tested at Reynolds number  $R = 20 \times 10^6$  by Jerzy Kacprzinski at the National Aeronautical Establishment in Ottawa. The corrected drag formula is seen to give a fairly reliable assessment of the performance of the airfoil.

There are examples where the results of the NCF code agree well with experimental data right up to the onset of buffet. Shock locations are predicted with remarkable accuracy over a wide range of conditions, although some improvement would be desirable at lower Reynolds numbers where transition becomes important. Thus the analysis code has been adequately validated for simulation of experimental data in two-dimensional flow. In particular, it models the trailing edge in a satisfactory way even for heavily aft-loaded airfoils. It is therefore of some interest that the code predicts no loss of lift for airfoils designed by the hodograph method when a Stratford distribution is used to eliminate separation completely over the whole profile. It would nevertheless be desirable to confirm this result experimentally by further

testing of shockless airfoils such as the one shown in Figure 1.

There is need for more research on computational methods for transonic flow. The progress in supercritical wing technology should be extended to cascades of airfoils and flows in turbomachinery. For the immediate future, the most challenging problem is analysis of the flow past wing-body combinations modelling an airplane in three dimensions. As a first step it would seem that the NCF equation for a velocity potential furnishes the most feasible mathematical formulation. Perhaps the Bateman variational principle asserting that the volume integral of the pressure is a stationary functional of the velocity potential, applied in the context of the finite element method, offers the best prospect of deriving convenient difference equations, provided artificial viscosity can be added successfully.

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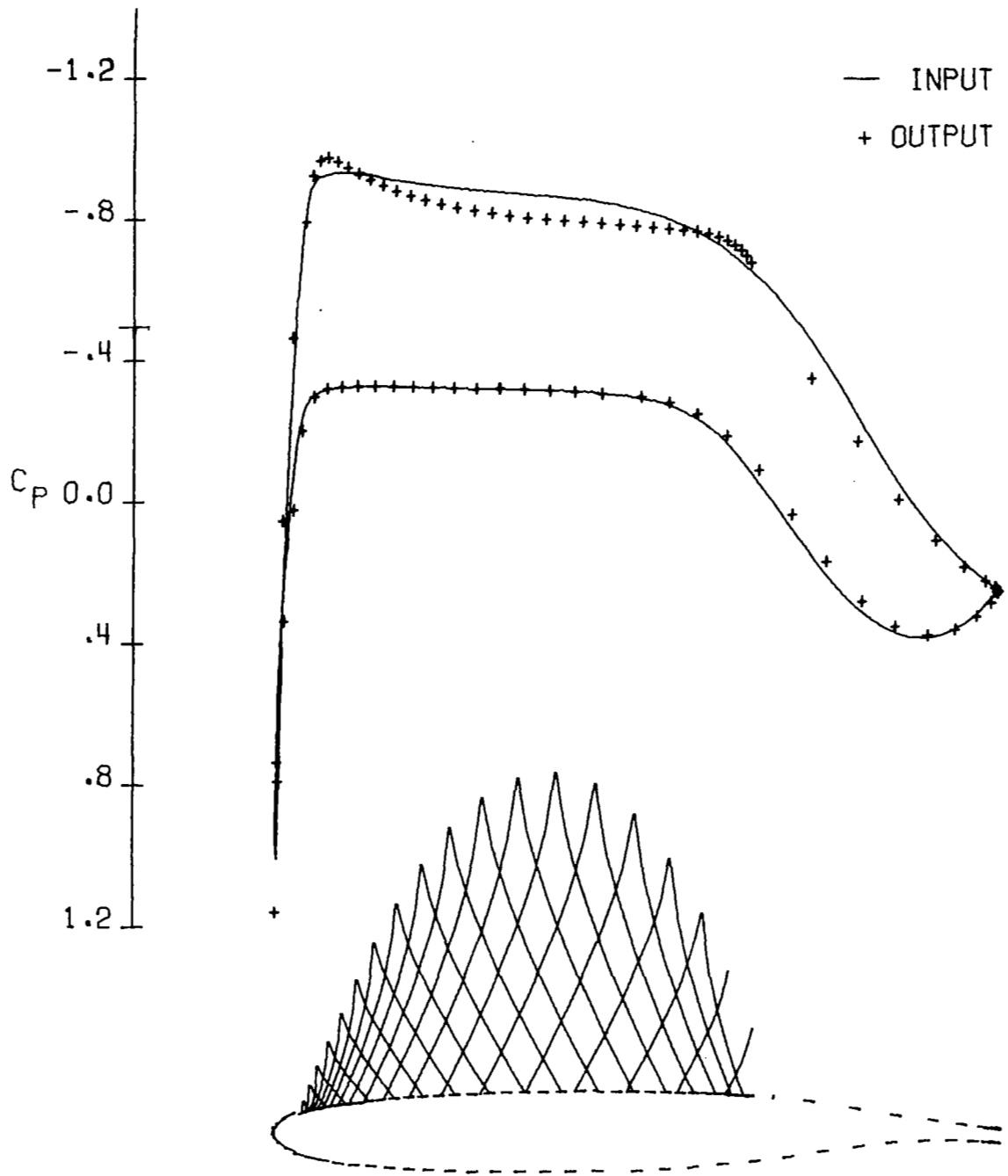
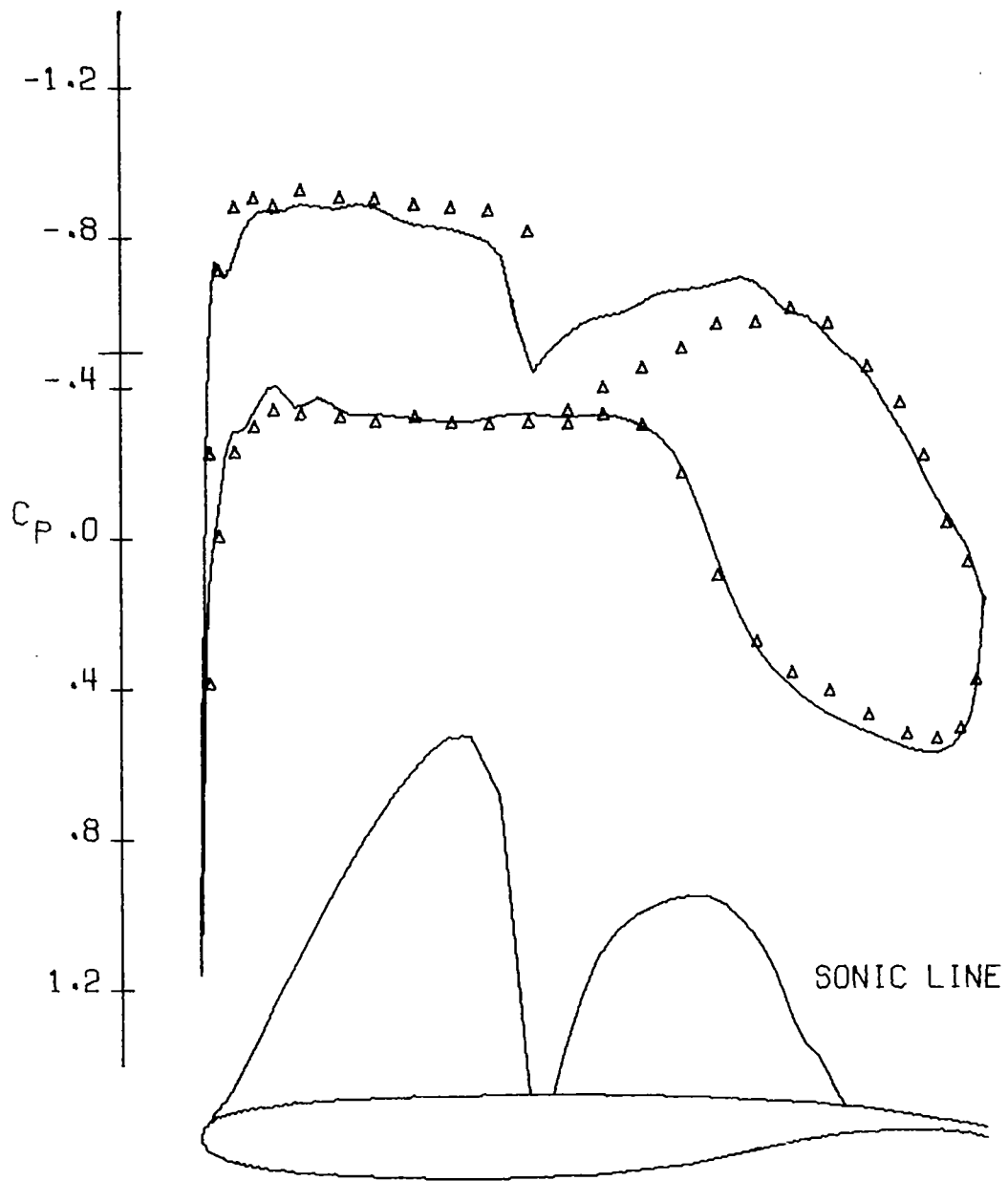


Figure 1.- Modification of Whitcomb wing at  $M = 0.78$ ,  $C_L = 0.47$ .





— THEORY            ANGLE= - .28     $C_D = .0088$   
 Δ EXPERIMENT       ANGLE= 1.00     $C_D = .0098$

Figure 2.- Whitcomb wing at  $M = 0.78$ ,  $C_L = 0.58$ .

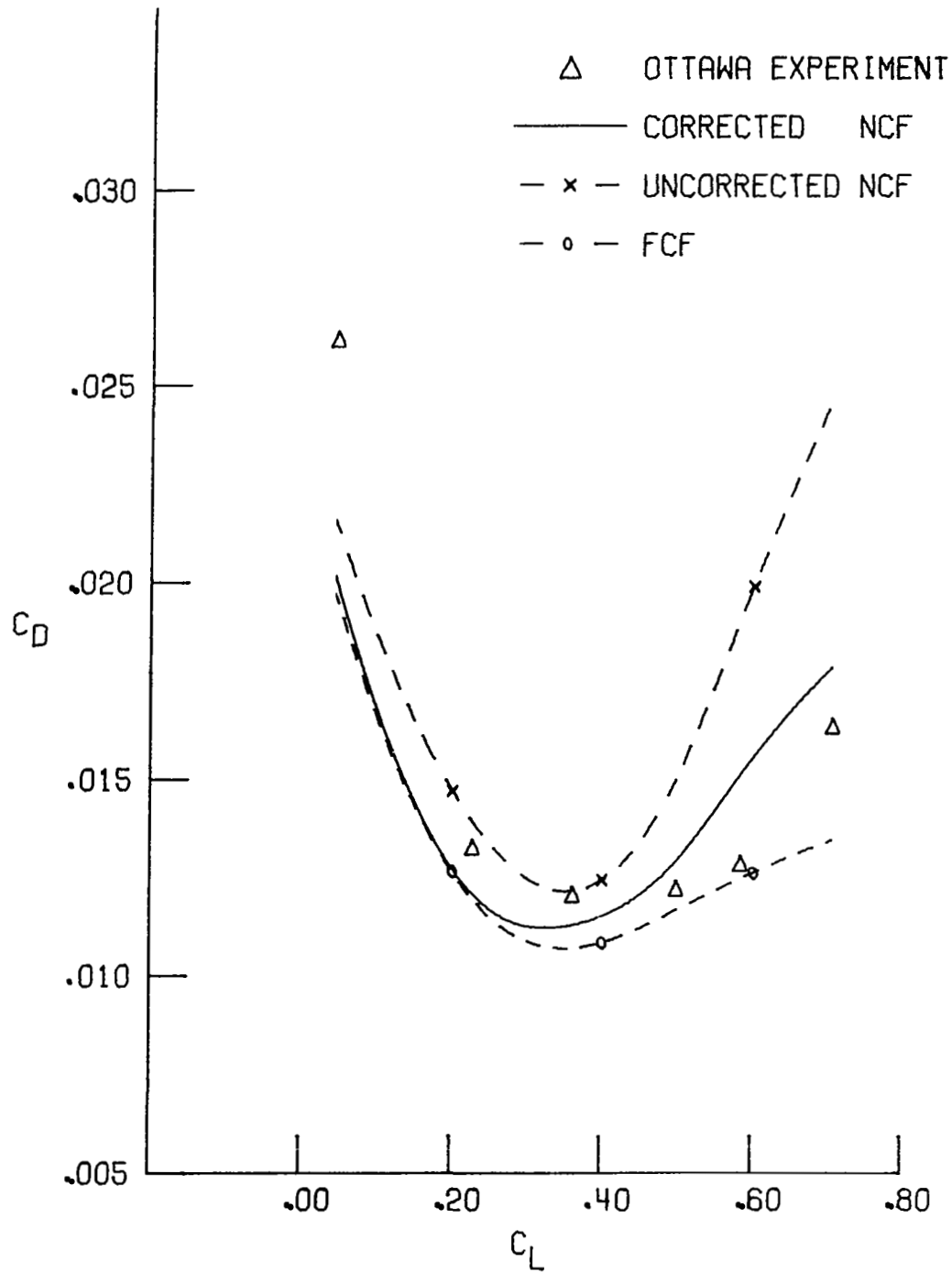


Figure 3.- Drag polar for transonic airfoil at  $M = 0.76$ .