NUMERICAL SOLUTIONS FOR LAMINAR AND TURBULENT VISCOUS FLOW OVER SINGLE

AND MULTI-ELEMENT AIRFOILS USING BODY-FITTED COORDINATE SYSTEMS*

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SUMMARY

The technique of body-fitted coordinate systems, whereby curvilinear coordinate systems having coordinate lines coincident with all boundaries are generated numerically as solutions of an elliptic partial differential system, is applied in numerical solutions of the complete time-dependent compressible and incompressible Navier-Stokes equations for laminar flow and to the time-dependent mean turbulent equations closed by modified Kolmogorov hypotheses for turbulent flow. Coordinate lines are automatically concentrated near to the bodies at higher Reynolds number so that accurate resolution of the large gradients near the solid boundaries is achieved. Two-dimensional bodies of arbitrary shapes are treated, the body contour(s) being simply input to the program. The complication of the body shape is thus removed from the problem.

INTRODUCTION

The use of numerically generated boundary-fitted coordinate systems has made possible the development of numerical solutions of the Navier-Stokes equations that can treat bodies of arbitrary shapes as easily as simple bodies. Codes can be written that are independent of the body or boundary shape, which may even be changing with time.

These solutions are based on a method of automatic numerical generation of a general curvilinear coordinate system with coordinate lines coincident with all boundaries of a general multi-connected region containing any number of arbitrary shaped bodies (ref. 1). The curvilinear coordinates are generated as the solution of two elliptic partial differential equations with Dirichlet boundary conditions, one coordinate being specified to be constant on each of the boundaries, and a distribution of the other being specified along the boundaries. Regardless of the shape, number, or movement of the bodies and regardless of spacing of the curvilinear coordinate lines, all numerical computations, both to generate the coordinate system and to subsequently solve the Navier-Stokes equations or any other partial differential equations on the coordinate system, are done on a rectangular grid with a square mesh, (i.e., in the transformed plane). The physical coordinate system has been, in effect, eliminated from the problem, at the expense of adding two elliptic equations to

the original system. Since the curvilinear coordinate system has coordinate lines coincident with the surface contours of all bodies present, all boundary conditions may be expressed at grid points, and normal derivatives on the bodies may be represented using only finite differences between grid points on coordinate lines, without need of any interpolation even though the coordinate system is not orthogonal at the boundary. Numerical solutions for the lifting and non-lifting potential flow about Kármán-Trefftz airfoils using this coordinate system generation show excellent agreement with the analytic solutions (refs. 1 and 2).

This method of automatic body-fitted curvilinear coordinate generation has been used to construct finite-difference solutions of the full time-dependent Navier-Stokes solutions for unsteady viscous flow about arbitrary two-dimensional airfoils (refs. 2,3,4, and 5) and submerged and partially submerged hydrofoils (refs. 4 and 5).

A method of controlling the spacing of the coordinate lines encircling the body has been developed in order to treat higher Reynolds number flow, since the coordinate lines must concentrate near the surface to a greater degree as the Reynolds number increases. The solution shows excellent comparison with the Blasius boundary layer solution for the flow past a semi-infinite plate (refs. 6 and 7). Solutions are also being developed for compressible viscous transonic flow with both subsonic and supersonic free streams, and for compressible turbulent flow.

BOUNDARY-FITTED COORDINATE SYSTEMS

The basic approach of constructing body-fitted curvilinear coordinate systems in general multi-connected regions as the solution of an elliptic boundary value problem has been discussed in previous publications (refs. 1,3 and 8), and reference to related work by others has been made therein. A detailed report of the technique and code is now available (ref. 8), together with the code, a user's manual, and instructions with illustrations of its application to the numerical solution of partial differential equations.

Certain considerations must be taken into account in the choice of a suitable elliptic generating system for the coordinates as discussed in references 3 and 8. The system chosen in the present work allows considerable control of the coordinate line spacing as is illustrated in reference 8. Control of the spacing of the coordinate lines on the body is easily accomplished, since the points on the body are input to the program. The spacing of the coordinate lines in the field, however, must be controlled by varying the elliptic generating system for the coordinates. One method of variation is to add inhomogeneous terms to the right sides of the Laplace equation, so that the generating system becomes

$$\xi_{xx} + \xi_{yy} = P(\xi,\eta) \ , \ \eta_{xx} + \eta_{yy} = Q(\xi,\eta) \tag{1}$$

Since it is desired to perform all numerical computations in the uniform rectangular transformed plane, the dependent and independent variables must be interchanged in eq. (1). This results in the coupled system (see ref. 8 for

the transformation relations).

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = - J^2 [x_{\xi} P(\xi,\eta) + x_{\eta} Q(\xi,\eta)] \tag{2a}$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = - J^2 [y_{\xi} P(\xi,\eta) + y_{\eta} Q(\xi,\eta)] \tag{2b}$$

where

$$\alpha = x_{\eta}^2 + y_{\eta}^2 \qquad \beta = x_{\xi} x_{\eta} + y_{\xi} y_{\eta} \qquad \gamma = x_{\xi}^2 + y_{\xi}^2 \qquad J = x_{\xi} y_{\eta} - x_{\eta} y_{\xi}$$

The inhomogeneous functions $P(\xi,\eta)$ and $Q(\xi,\eta)$ allow coordinate lines to be attracted to specified lines and/or points in the field or on the boundaries as discussed in detail in reference 8. It is necessary to give some consideration to the rapidity with which the spacing varies else truncation error effects in the form of numerical diffusion, possibly negative, may be introduced. All derivatives in equation (2) are approximated by second-order central finite difference expressions. The resulting difference equations are given in reference 8. The set of nonlinear simultaneous difference equations is solved by point SOR iteration.

In the present application to the Navier-Stokes equations, all diffussive space derivatives in the transformed equations are represented by second-order, central difference expressions. Both second-order central and second-order upwind differences have been investigated for convective derivatives. Derivatives off a boundary are represented by backward difference expressions being solved simultaneously by point SOR iteration at each time step.

LAMINAR FLOW ABOUT MULTIPLE AIRFOILS

With the velocity and pressure as the dependent variables the transformed Navier-Stokes equations are (ref. 8)

$$u_t + [y_{\eta}(u^2)_{\xi} - y_{\xi}(u^2)_{\eta}]/J + [x_{\xi}(uv)_{\eta} - x_{\eta}(uv)_{\xi}]/J$$

$$+ (y_{\eta} p_{\xi} - y_{\xi} p_{\eta})/J = (\alpha u_{\xi\xi} - 2\beta u_{\xi\eta} + \gamma u_{\eta\eta})/RJ_2$$

$$+ (Qu_{\eta} + Pu_{\xi})/R \tag{3}$$

$$v_t + [y_{\eta}(uv)_{\xi} - y_{\xi}(uv)_{\eta}]/J + [x_{\xi}(v^2)_{\eta} - x_{\eta}(v^2)_{\xi}]/J$$

$$+ (x_{\xi} p_{\eta} - x_{\eta} p_{\xi})/J = (\alpha v_{\xi\xi} - 2\beta v_{\xi\eta} + \gamma v_{\eta\eta})/RJ_2$$

$$+ (Qv_{\eta} + Pv_{\xi})/R \tag{4}$$

$$\alpha p_{\xi\xi} - 2\beta p_{\xi\eta} + \gamma p_{\eta\eta} + (Qp_{\eta} + Pp_{\xi})J_2 = - (y_{\eta} u_{\xi} - y_{\xi} u_{\eta})^2$$

$$- 2(x_{\xi} u_{\eta} - x_{\eta} u_{\xi})(y_{\eta} v_{\xi} - y_{\xi} v_{\eta}) - (x_{\xi} v_{\eta} - x_{\eta} v_{\xi}) - J^2 D_t \tag{5}$$

where R is the Reynolds number

and

$$D_t \equiv (y_\eta u_\xi - y_\xi u_\eta + x_\xi v_\eta - x_\eta v_\xi)/J \tag{6}$$

Equation (5) is the transformed Poisson equation for the pressure, obtained by taking the divergence of the Navier-Stokes equations.

The boundary conditions are

$$u = v = 0 \text{ on body surface} \tag{7a}$$

$$u = \cos\theta, \ v = \sin\theta, \ p = 0 \text{ on remote boundary} \tag{7b}$$

The pressure at each point on the body was adjusted at each iteration by an amount proportional to the velocity divergence evaluated using second-order one-sided differences for the η-derivative on the body.

The body force components are obtained from the integration of the pressure and shear forces around the body surface:

$$F_x = + \oint p y_\xi \ d\xi - \frac{2}{R} \oint \omega x_\xi \ d\xi \tag{8a}$$

$$F_y = - \oint p x_\xi \ d\xi - \frac{2}{R} \oint \omega y_\xi \ d\xi \tag{8b}$$

and the lift and drag coefficients are given by

$$C_L = F_y \cos\theta - F_x \sin\theta \tag{9a}$$

$$C_D = F_y \sin\theta + F_x \cos\theta \tag{9b}$$

where θ is the angle of attack.
In the velocity-pressure formulation it is necessary to calculate the body vorticity before applying equation (8) from

$$\omega = - \frac{1}{J}(y_\xi v_\eta + x_\xi u_\eta) \tag{10}$$

Figure la shows the coordinate system for a multiple airfoil consisting of two Kárman-Trefftz airfoils, one simulating a separated flap. Coordinate system control was used to attract the coordinate lines strongly to the first ten lines around the bodies and to the intersections of the cut between the bodies with the trailing edge of the forebody and the leading edge of the aftbody. Velocity vectors for the viscous flow at R = 1000 are shown in figures lb - e. Included in these plots are detail views of the slot between the airfoils, the trailing edge of the aft airfoil, and the separated region about the aft airfoil.

Results have recently been obtained by Hodge at Flight Dynamics Laboratory, Wright-Patterson Airforce Base, (ref. 5), for a single NACA 0018 airfoil at R = 41,400 which compare very well with experimental drag values.

TURBULENT SHEAR FLOW AROUND A CIRCULAR CYLINDER

The aim of the following analysis is to select a turbulence model for the computation of mean turbulent flow fields around two-dimensional bodies of arbi-

trary shapes in the body-fitted coordinate systems. As a test case the coordinate system for a circular cylinder was generated over which the model equations of mean turbulent flow were solved.

As is well known, the mean turbulent flow equations and the moment equations of the desired order can be obtained from the non-steady three-dimensional Navier-Stokes equations. The specification of the unknown correlations forms the closure problem. This line of approach for closing the system of equations was initiated by Chou (ref. 9) and Rotta (ref. 10). Since then much work has been done on the closure problem (see refs. 11 and 12).

A philosophically different approach was initiated by Kolmogorov (ref. 13) in which the turbulent transport equations were written down heuristically to model the physics rather than to model the unknown correlations resulting from the averaged Navier-Stokes equations. A set of equations similar to those of Kolmogorov were proposed by Saffman (refs. 14 and 15). The prediction capabilities of the Kolmogorov's equations have not been investigated at all, while Saffman himself has used his equations only in the cases of plane Couette flow, plane Poiseuille flow, and the two-dimensional jet and wake.

The turbulence model chosen in this paper to describe the two-dimensional mean turbulent flow around finite bodies is comprised of the energy transport equation of Kolmogorov's model and the vorticity-density transport equation of the Saffman model. Both models have been described by Saffman (ref. 15).

Using Cartesian tensor rotation and the summation convention the equations of continuity and momentum for an incompressible flow are

$$\frac{\partial U_i}{\partial x_j} = 0 \tag{11}$$

$$\frac{\partial U_i}{\partial t_1} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla_1^2 U_i - \frac{\partial}{\partial x_j}(\overline{u_i u_j}) \tag{12}$$

where t_1 is the physical time, ρ the density, ν the kinematic viscosity, U_i the Cartesian components of the mean velocity vector and u_i the fluctuating components. An overbar denotes the average.

To close the system of equations (11) and (12), the eddy viscosity formulation due to Boussinesq is used

$$\overline{u_i u_j} = \frac{2}{3}\overline{e}\delta_{ij} - 2\nu_T S_{ij} \tag{13}$$

where

δ_{ij} is the Kronecker delta

$$S_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i})$$

$$\overline{e} = \frac{1}{2}\overline{u_i u_i} \text{ , the mean turbulence energy}$$

$$\nu_T = \text{kinematic eddy viscosity}$$

The Kolmogorov-Saffman model now depends in specifying ν_T as

$$\nu_T = \frac{\alpha_o^2 \bar{e}}{\theta} \tag{14}$$

where α_o is a constant and θ is the square root of the turbulence vorticity-density defined as

$$\theta = (\bar{\phi})^{1/2} = (\frac{1}{2}\overline{\omega_i \omega_i})^{1/2}$$

ω_i being the vorticity fluctuations. The quantities $\bar{e}$ and $\bar{\phi}$ are to be determined from their own transport equations which are

$$\frac{\partial \bar{e}}{\partial t_1} + U_j \frac{\partial \bar{e}}{\partial x_j} = \nu\nabla_1^2\bar{e} + \nu_T\bar{Q} - \bar{e}\theta + \frac{\partial}{\partial x_j}(\frac{\nu_T}{2}\frac{\partial \bar{e}}{\partial x_j}) \tag{15}$$

$$\frac{\partial \bar{\phi}}{\partial t_1} + U_j \frac{\partial \bar{\phi}}{\partial x_j} = \nu\nabla_1^2\bar{\phi} + \bar{P}\,\bar{\phi} - b(\bar{\phi})^{3/2} + \frac{\partial}{\partial x_j}(\frac{\nu_T}{2}\frac{\partial \bar{\phi}}{\partial x_j}) \tag{16}$$

where
$$\bar{P} = \frac{\alpha_o}{2}[(\frac{\partial U_i}{\partial x_j})^2]^{1/2}, \quad \alpha_o, \text{ b constants}$$

$$\bar{Q} = 2(S_{ij})^2$$

$$\nabla_1^2 = \frac{\partial^2}{\partial x_j \partial x_j}$$

The boundary conditions for equations (11), (12), (15), and (16) are
at the wall:
$$U_i = 0, \quad \bar{e} = 0, \quad \bar{\phi} = \frac{\alpha_o^2 \, S^2 u_*^4}{\nu^2} \tag{17}$$

at infinity:
$$U_i = U_{\infty i}, \quad \bar{e} = o \text{ or } \bar{e}_\infty, \quad \bar{\phi} = 0$$

where u_* is the friction velocity and S a non-dimensional constant having the value of about 8000 and $\alpha_o = 0.3$, $b = 5/3$.

The required equations to predict the mean turbulent flow in two dimensions are equations (11)-(16). For the numerical computation the vorticity-stream function formulation is adopted in which

$$U_1 = \frac{\partial \bar{\psi}}{\partial x_2}, \quad U_2 = -\frac{\partial \bar{\psi}}{\partial x_1},$$

so that in place of (11) and (12) the equations are

$$\nabla_1^2\bar{\psi} = -\bar{\omega} \tag{18}$$

and

$$\frac{\partial \bar{\omega}}{\partial t_1} + U_j \frac{\partial \bar{\omega}}{\partial x_j} = \nu\nabla_1^2\bar{\omega} + \frac{\partial^2}{\partial x_1 \partial x_2}(\overline{u_1^2} - \overline{u_2^2}) + (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2})(\overline{u_1 u_2}) \tag{19}$$

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The pertinent equations are now non-dimensionalized as follows

$$E = \bar{e}/V^2, \quad \omega = L\bar{\omega}/V, \quad \chi = L^2\bar{\phi}/V^2K^2,$$

$$T = \nu_T/2VL, \quad \psi = \bar{\psi}/VL, \quad t = Vt_1/L, \quad P = L\bar{P}/V,$$

$$Q = L^2\bar{Q}/V^2, \quad R = VL/\nu, \quad x = x_1/L, \quad y = x_2/L;$$

$$u = U_1/V, \quad v = U_2/V, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{20}$$

where V and L are the characteristic velocity and length respectively.

In the expression for χ the function K is assumed to be a non-dimensional function of the friction velocity u_*. This function has been introduced to absorb the variation of $\bar{\phi}$ at the surface as indicated in equation (17) so as to make χ = a constant at the wall. Now at the wall

$$\chi = \alpha_o^2 S^2 R^2 U_*^4/K^2$$

where $U_* = \dfrac{u_*}{V}$, so that the choice $\chi = 1$ at the wall gives

$$K = \alpha_o SR \, U_*^2 \tag{21}$$

From the analysis of Saffman (ref. 15) the value of S for smooth walls is about 8000. Since R is the free stream Reynolds number the numerical values of K are quite large. Neglecting the gradients of K in the χ equation, the non-dimensional equations for numerical computation are

$$\nabla^2\psi = -\omega \tag{22}$$

$$\lambda_t + \psi_y\lambda_x - \psi_x\lambda_y = \frac{1}{H}\nabla^2\lambda + M(T_x\lambda_x + T_y\lambda_y) + A\lambda + B \tag{23}$$

where

$$H = \frac{R}{1+CR\,T}$$

$$T = \frac{\alpha_o^2\,E}{2K\chi^{1/2}} \tag{24}$$

and the variable subscripts denote partial differentiations. The values of A, B, M and C depend on the choice of the surrogate variable λ which are described below.

$$\lambda = \omega: \quad A = 2\nabla^2 T, \quad B = 4(T_{xx}u_y - T_{yy}v_x + 2T_{xy}v_y),$$

$$M = 4, \quad C = 2. \tag{25}$$

$$\lambda = E: \quad A = -K\chi^{1/2} \quad B = 2TQ, \quad M = 1, \quad C = 1 \tag{26}$$

$$\lambda = \chi: \quad A = P - bK\chi^{1/2}, \quad B = 0, \quad M = 1, \quad C = 1 \tag{27}$$

The boundary conditions are

at the body surface: $\psi=0$, $\psi_y=0$, $\chi=1$, $E=0$,

at infinity: $\omega \to 0$, $\chi \to 0$, $E \to 0$. \qquad\qquad (28)

Equations (22) and (23) have been transformed to the body-fitted coordinates and subjected to the finite difference approximations similar to that used in the laminar calculation. Velocity vector plots and the energy distribution for a circular cylinder at $R = 5\times10^5$ are shown in figures 2a and 2b. In figure 2b

$$\ln r = .0576\eta(1.2)^{\eta-40}, \quad r = \text{non-dimensional radial distance.}$$

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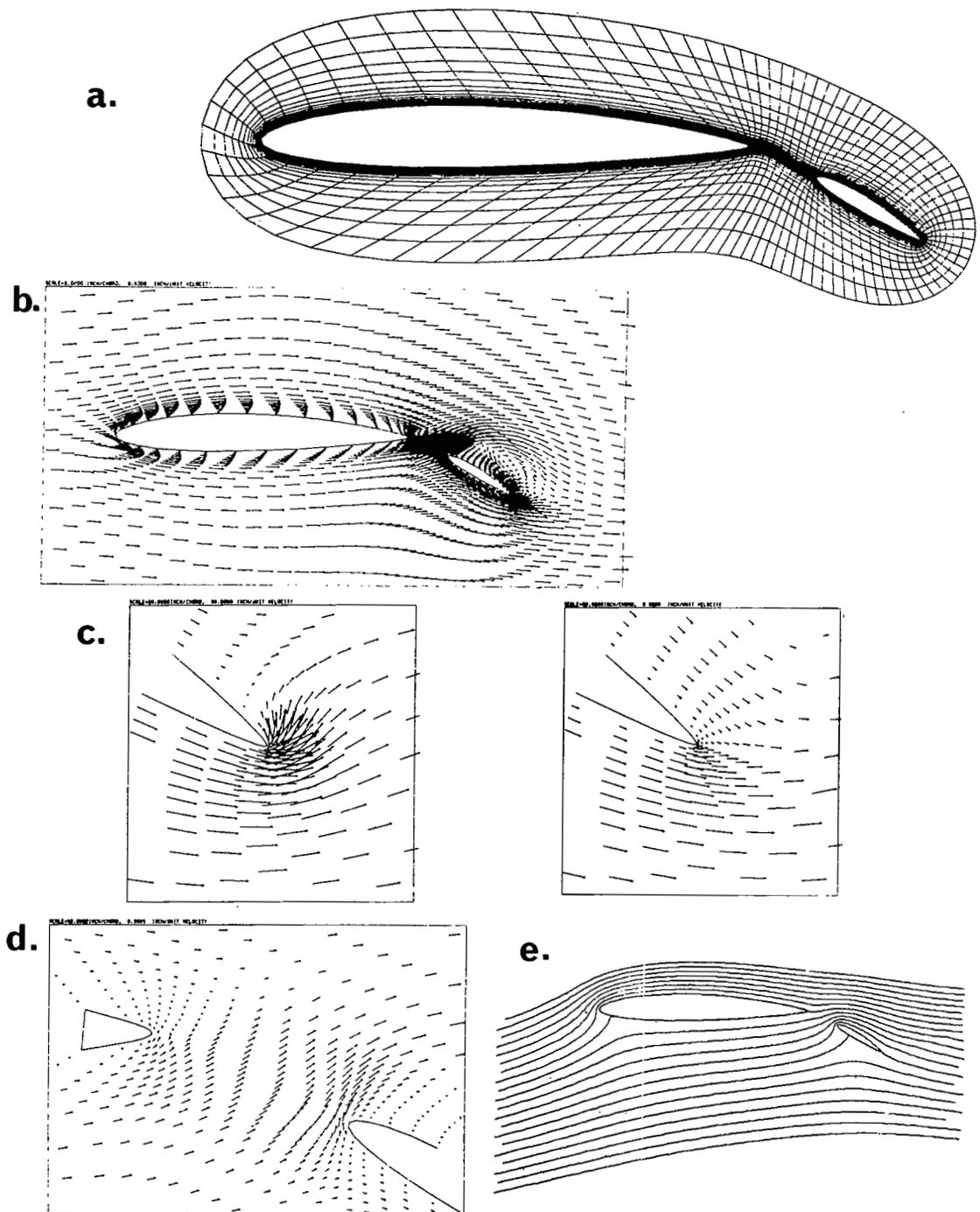
Figure 1. Multiple Airfoil (R = 1000, t = 1.2): (a) coordinate system, (b) velocity vectors (c) aft trailing edge, (d) slot detail, (e) potential
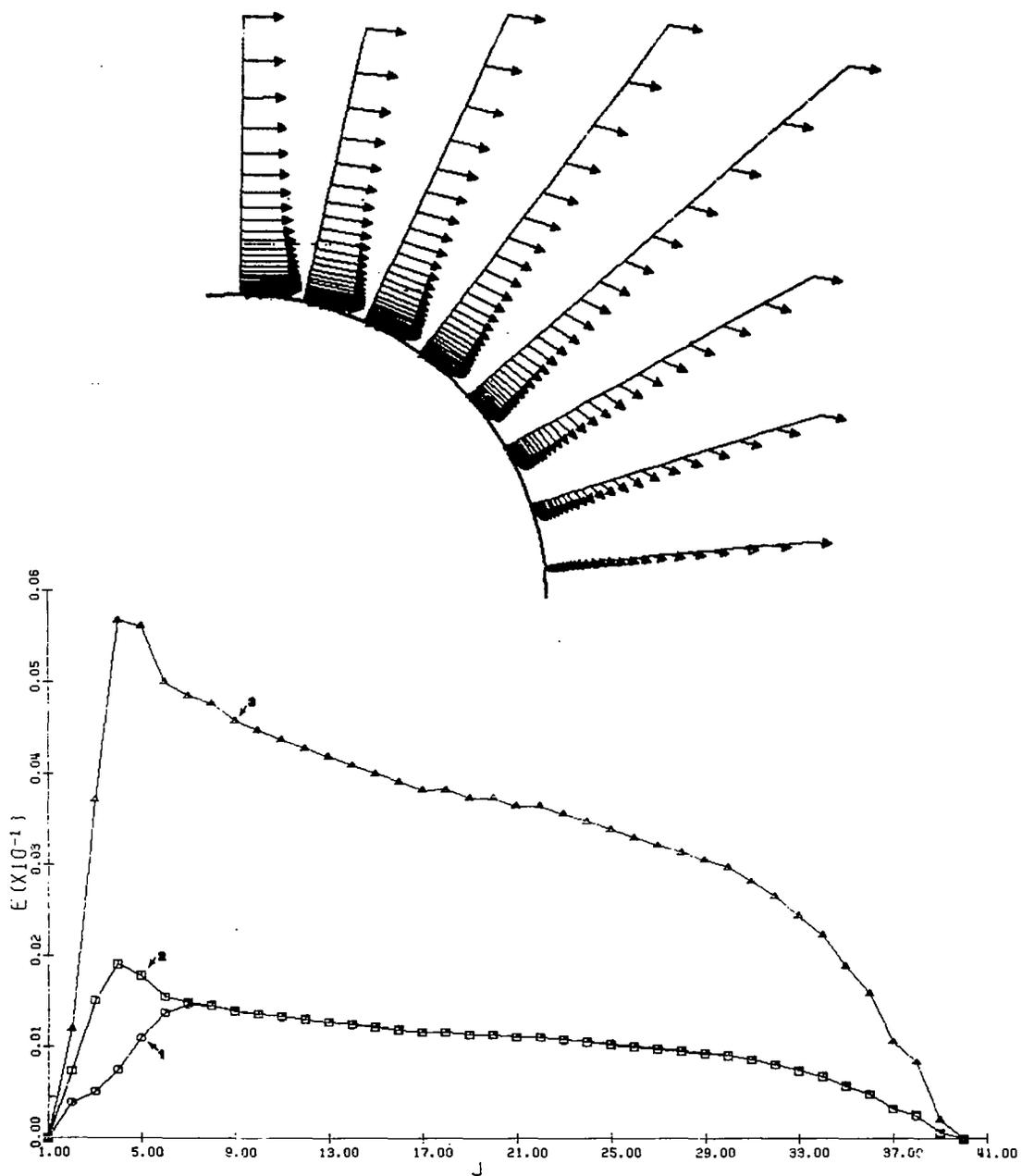
Figure 2. Turbulent flow past a circular cylinder (R = 500,000, t = 0.29; transition introduced at t = 0.12): (a) velocity vectors from 90° to the aft stagnation point, (b) turbulence energy distribution (1. front stagnation, 2. aft stagnation, 3. 90°)