NUMERICAL AND APPROXIMATE SOLUTION OF THE HIGH REYNOLDS NUMBER SMALL SEPARATION PROBLEM†

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SUMMARY

The purpose of this paper is to explore several possible methods of solving the small separation problem at high Reynolds number. In addition to using analytical methods, there are several numerical approaches which can be used and in addition there is the possibility of using approximate integral methods. We will restrict ourselves to high Reynolds number laminar two-dimensional problems for simplicity. Presumably, the same techniques can be extended to more complicated flow fields. Only a brief discussion will be given of the finite difference methods since these methods are discussed in detail by Davis and Werle (ref. 1). Most of the emphasis will be placed on developing an approximate integral method. As a model problem we will choose the supersonic compression ramp problem since several numerical solutions along with experimental data are available for this case. The techniques discussed can be modified and applied to other similar type wall geometries.

INTRODUCTION

It has been recognized for many years that the problem of computing high Reynolds number separated flows is extremely difficult. The reason for the difficulty becomes clear if one examines the results of the asymptotic theory, see Stewartson (ref. 2). Davis and Werle (ref. 1) have discussed the implications of these results and suggested how one might use the results of the asymptotic analysis in order to do efficient numerical computations.

Briefly, the asymptotic theory reveals that at high Reynolds number severe scaling problems exist around and downstream of separation. In addition the asymptotic theory for the small separation problem reveals a mechanism for upstream propagation through boundary-layer interaction, even if the external flow is supersonic. These two features require that an efficient numerical scheme use properly scaled independent variables for resolution and in addition require that the boundary-layer interaction be handled in a manner appropriate to boundary value problems. These features should be accounted for even in the solution of the full Navier-Stokes equations.

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The asymptotic theory is partially complete for some massive separation problems, see Messiter (ref. 3) for example, and the results indicate that numerical solutions will be extremely difficult to perform for this type of problem. On the other hand, the theory for the small separation problem is essentially complete, see Stewartson (ref. 2), and we can now begin to compute flows of this type with confidence. For the remainder of this paper we will therefore concentrate on the type of problem where separation is of limited extent and can be handled within the framework of boundary-layer theory.

According to the asymptotic theory, we may define a small separation problem to be one such that the scales of a bump or depression on a flat plate are the same as the length scales of the lower deck in the triple deck analysis, see Stewartson (ref. 2). This requires that the length of the bump or depression generating the separation scale as $Re^{-3/8}$ while the height must scale as $Re^{-5/8}$. If this is true, the separated region will be entirely confined to the lower (fundamental) deck and the high Reynolds number separation problem can be attacked entirely with the lower deck equations coupled with an interaction law for the outer inviscid flow. This is the approach taken by Jenson, Burggraf, and Rizzetta (ref. 4) and Rizzetta (ref. 5) in considering supersonic ramp type separations. Smith (ref. 6) has in addition solved the linear version of the small separation problem for flow over protuberances.

As an alternative, for the same type of separation problems, one may solve the ordinary Prandtl boundary-layer equations including interaction with the outer inviscid flow. It can be shown that these equations contain all of the terms in the triple deck equations plus some additional ones. The extra terms in fact provide some corrections which allow better agreement with experiment at moderately high Reynolds numbers. This is the approach taken by Werle and Vatsa (ref. 7) and Vatsa (ref. 8) in considering supersonic ramp type separations.

The supersonic ramp separation problem has also been solved by Carter (ref. 9) and others using the full Navier-Stokes equations. These calculations provide a basis for comparison with other less exact models of separation.

The high Reynolds number small separation problem may therefore be approached in a variety of ways. The most complicated method would involve the solution of the full Navier-Stokes equations. Next in complexity would be to solve interacting boundary-layer like equations or one of the sets of so-called parabolized Navier-Stokes equations. The simplest set of equations one could solve and retain all of the features of the flow would be the triple deck equations.

If one wishes to do for example a full Navier-Stokes calculation for flow over a complicated configuration one may not wish to provide the mesh refinements in small separated regions as is indicated as being necessary by the asymptotic theory. These regions can be excluded from the overall calculation and accounted for by a local calculation. This local calculation can then provide a slip type boundary condition on the edge of the small separated region. Thus by excluding the small separation bubble and replacing it with a slip type boundary condition, the overall calculation can be made accurately with a much larger mesh size than would be required if the separated bubble were included.

One important application of triple deck theory could thus be to provide local solutions in small separated regions to match into an overall calculation. It therefore seems important that we be able to solve the triple deck equations in as efficient a manner as possible.

The simplest and therefore fastest type of approximate solutions to the boundary-layer equations are obtained with integral methods. To test the applicability of the integral technique to triple deck theory, we will develop an integral method for solving the lower deck equations and compare solutions obtained from this method with more exact solutions. This is perhaps the fastest method for solving the small separation problem if one is willing to accept the inaccuracies associated with an approximate integral method. The method is attractive for doing local calculations, especially if one considers that the errors obtained from the integral method would probably be much smaller than those which would exist from a course mesh finite difference calculation which might be used as an alternative.

GOVERNING EQUATIONS FOR SMALL SEPARATIONS

The small separation problem is by definition a separation generated by a bump or depression on a flat plate such that the scaling of the bump or depression is the same as that given by the lower deck analysis of Stewartson (ref. 2). Thus such a bump or depression scales as ε^3 in the streamwise direction and ε^5 in the normal direction, as $\varepsilon \to 0$, where $\varepsilon = Re^{-1/8}$. Such a bump or depression generates an interaction which falls within the framework of triple deck analysis and the problem can thus be handled by solving the lower deck equations coupled with an interaction law.

In the lower deck variables defined by equations (4.4) in Stewartson (ref. 2), the small separation problem is governed by the following equations, see figure 1

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{1}$$

and

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{dP}{dx} + \frac{\partial^2 u}{\partial y^2} \quad , \tag{2}$$

with boundary and matching conditions

$$u = v = 0 \quad \text{at} \quad y = f(x) \quad , \tag{3}$$

$$u \sim y - f(x) - \delta(x) \quad \text{as} \quad y \to \infty \quad , \tag{4}$$

and

$$\delta(\pm \infty) = 0 \quad . \tag{5}$$

According to linear theory the interaction law is given by

$$P = f'(x) + \delta'(x) \qquad \text{for supersonic flow} \tag{6}$$

or

$$P = - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f'(x_1) + \delta'(x_1)}{x - x_1} dx_1 \qquad \text{for subsonic flow.} \tag{7}$$

The quantity f(x) denotes the dimensionless surface measured from the Cartesian coordinate system on the flat plate surface and δ is the dimensionless displacement thickness. Both of these quantities are nondimensionalized in the same manner as the y coordinate.

In order to solve the lower deck equations, it is convenient to shift the coordinate system such that the body surface lies along a constant coordinate curve. This can be accomplished with the use of Prandtl's transposition theorem, see Jenson, Burggraf, and Rizzetta (ref. 4) for example.

With the change of variables

$$z = y - f(x) \tag{8}$$

and

$$w = v - \frac{df}{dx} u \tag{9}$$

and all other variables remaining unchanged we obtain from equations (1)-(4)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad , \tag{10}$$

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{dP}{dx} + \frac{\partial^2 u}{\partial z^2} \quad , \tag{11}$$

$$u = w = 0 \quad \text{at} \quad z = 0 \quad , \tag{12}$$

and

$$u \sim z - \delta(x) \quad \text{as} \quad z \to \infty \quad . \tag{13}$$

The remaining equations (5)-(7) are unchanged by the transformations.

INTEGRAL FORMULATION FOR THE LOWER DECK EQUATIONS

The lower deck equations (10)-(13) and (5)-(7) can be put into a form similar to the von Karman momentum integral equation for two-dimensional boundary layers. The advantage in doing this is that a simple approximate solution technique can be developed for the lower deck equations along the same lines as approximate solution methods for non-interacting boundary layers.

First let $z \to \infty$ in the momentum equation (11) and substitute the outer edge condition for u given by equation (13). This results in

$$v \sim (z - \delta)\delta' - P' \quad \text{as} \quad z \to \infty \quad . \tag{14}$$

Next integrate the continuity equation (10) with respect to z to find another expression for v as $z \to \infty$. Thus results in

$$v \sim \delta' z - \frac{d}{dx} \int_0^\infty (u-U_e)dz \quad \text{as} \quad z \to \infty \tag{15}$$

where we have defined U_e as

$$U_e = z - \delta \ . \tag{16}$$

Equating the two expressions for v as $z \to \infty$ from equations (14) and (15) we find

$$\delta\delta' + P' = \frac{d}{dx} \int_0^\infty (u-U_e)dz \quad . \tag{17}$$

Integrating this expression with respect to x and using the condition that all quantities in the equation die out as $x \to -\infty$ we obtain

$$\frac{\delta^2}{2} + P = \int_0^\infty (u - U_e)dz \quad . \tag{18}$$

We next integrate the momentum equation (11) with respect to z from z equals zero to infinity. After some fairly straightforward algebra, this results in

$$\frac{d}{dx} \int_0^\infty (u^2 - U_e^2)dz + \delta \frac{dD}{dx} = 1 - u_z(x,0) \quad . \tag{19}$$

We therefore have three integral quantities which must be evaluated. The displacement thickness δ, from equation (13) can be written as

$$\delta = \int_0^\infty (1 - \frac{\partial u}{\partial z})dz \quad . \tag{20}$$

The remaining two integral quantities are defined as

$$D = \int_0^\infty (u - U_e)dz \tag{21}$$

and

$$M = \int_0^\infty (u^2 - U_e^2)dz \quad . \tag{22}$$

With these definitions the equations (18) and (19) become

$$\frac{\delta^2}{2} + P = D \tag{23}$$

and

$$\frac{dM}{dx} + \delta \frac{dD}{dx} = 1 - \tau_w \tag{24}$$

where

$$\tau_w = u_z(x,0) \quad . \tag{25}$$

The simplest possible approximate solution method to the integral equations (23) and (24) is to assume a linear shear profile of the following form

$$\frac{\partial u}{\partial z} = \tau_w + (1 - \tau_w) \frac{z}{\ell(x)} \qquad \text{for} \qquad z \leq \ell$$

and (26)

$$\frac{\partial u}{\partial z} = 1 \qquad \text{for} \qquad z > \ell$$

where $\ell(x)$ is the boundary-layer thickness function. Substituting the above equations into equation (20) results in

$$\delta = (1 - \tau_w) \frac{\ell}{2} \quad . \tag{27}$$

Integrating equation (26) results in a parabolic velocity profile of the form

$$u = \tau_w z + (1 - \tau_w) \frac{z^2}{2\ell} + C(x) \qquad \text{for} \qquad z \leq \ell$$

and (28)

$$u = U_e = z - \delta \qquad \text{for} \qquad z > \ell \ .$$

We choose to satisfy the conditions that u = 0 and z = 0 and u = U_e at z = ℓ. The first condition results in C(x) = 0 and the second reproduces equation (27). Thus the velocity profile is given by

$$u = \tau_w z + (1 - \tau_w) \frac{z^2}{2\ell} \qquad \text{for} \qquad z \leq \ell$$

and (29)

$$u = z - \delta \quad \text{for} \quad z > \ell \quad .$$

This profile satisfies the no slip condition, is continuous and has a continuous first derivative at $z = \ell$.

Using the above velocity profile, the expression (21) for D gives

$$D = \frac{\ell\delta}{3} \tag{30}$$

while the expression (22) for M gives

$$M = \frac{\ell^2\delta}{6} - \frac{7}{15}\ell\delta^2 \quad . \tag{31}$$

Substituting these expressions into the integral equations (18) and (19) gives

$$P = \frac{\ell\delta}{3} - \frac{\delta^2}{2} \tag{32}$$

and

$$\frac{d}{dx}\left[\frac{\ell^2\delta}{6} - \frac{7}{15}\ell\delta^2\right] + \delta\frac{d}{dx}\left[\frac{\ell\delta}{3}\right] = \frac{2\delta}{\ell} \quad . \tag{33}$$

In order to close the problem, equation (6) or equation (7) for P is used depending upon whether the flow is supersonic or subsonic. Initial and downstream boundary conditions are prescribed in the form of equation (5).

The integral formulation therefore results in the solution of two nonlinear first-order ordinary differential equations for ℓ and δ for the supersonic case and the solution of one nonlinear integral equation and one nonlinear first-order ordinary differential equation for the subsonic case.

The present choice of profile shapes is the simplest possible. However, more complicated profiles can easily be chosen. The purpose of the present analysis is to show how an integral method may be formulated without paying attention at this point in time to accuracy of the method.

We can easily find an approximate solution corresponding to Lighthill's (ref. 10) analysis of the initiation of a free interaction process in supersonic flow. If we consider the possible emergence of a sublayer at a point x_p^* on a flat plate, see Stewartson (ref. 2), we can study the initiation of the sublayer using the linearized version of equations (32) and (33). For supersonic flow, if we consider δ to be small, these equations result in

$$\frac{d\delta}{dx} = \frac{\ell\delta}{3} \tag{34}$$

and

$$\frac{d}{dx}\left(\frac{\ell^2\delta}{6}\right) = \frac{2\delta}{\ell} \quad (35)$$

Dividing the first equation by the second and integrating, it is easy to show that the approximate solution corresponding to Lighthill's exact solution is

$$\ell = \sqrt{6} \quad (36)$$

and

$$\delta = \delta_o e^{kx} \quad (37)$$

where

$$k = \frac{\sqrt{6}}{3} = 0.8165 \quad . \quad (38)$$

The approximate value of k given by this analysis compares favorably with the exact value of 0.8272 given by Lighthill's analysis. From the expression (27) and the linear version of (32), assuming $\ell = \sqrt{6}$, δ can be eliminated to give

$$\tau_w = 1 - P \quad (39)$$

which also compares favorably with $\tau_w = 1 - 1.209P$ given by Lighthill's analysis.

Next we consider the case of compressive free interactions, see Stewartson and Williams (refs. 11 and 12), Stewartson (ref. 2) and Williams (ref. 13). The full approximate equations (32) and (33) for supersonic flow, i.e. $P = d\delta/dx$, were integrated numerically using a fourth-order Runge Kutta method.* The results were adjusted such that the zero shear point occurs at $x = 0$ in order to compare with Stewartson and William's results, see Stewartson (ref. 2) and Williams (ref. 13).

Figure 2 shows that the approximate results agree quite favorably up to and through the separation point. Far downstream of separation the approximate results produce a shear which asymptotically approaches -1/3 rather than zero from the exact results and a pressure which goes to zero rather than 1.800 [see Williams (ref. 13)] from the exact results. These deficiencies are due to the complicated nature of the lower deck free interaction solution as $x \to \infty$, see Williams (ref. 13). A more elaborate and physically meaningful assumption for a shear profile to handle the region for large x should overcome this deficiency. However, it is important to note that the present approximate integral method preserves the qualitative features of the exact results, except when the interaction region is too long.

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As a final example of the application of the approximate method, we consider flow past a compression ramp. Exact numerical solutions have been provided to the lower deck equations for this problem by Jenson, Burggraf and Rizzetta (ref. 4) and Rizzetta (ref. 5).

Consider supersonic flow along a flat plate which abruptly encounters a wedge type compression ramp. Jenson, Burggraf and Rizzetta (ref. 4) have considered the formulation and numerical solution of this problem within the framework of triple deck analysis. In terms of their formulation, the appropriate problem to be solved with the approximate set of equations is given by equations (32) and (33) with the supersonic interaction law (6) given by

$$P = \frac{d\delta}{dx} \qquad \text{for} \quad x < 0$$

and (40)

$$P = \frac{d\delta}{dx} + \bar{\alpha} \qquad \text{for} \quad x \geq 0$$

where $\bar{\alpha}$ is related to the physical angle α through

$$\alpha = \lambda^{1/2} \left[\frac{C(M_\infty^2 - 1)}{Re}\right]^{1/4} \bar{\alpha} \quad . \tag{41}$$

The governing equations were again integrated using a fourth-order Runge Kutta method. The initial conditions were applied at $x = -20$ with $\ell = \sqrt{6}$. A shooting method was used to find the correct initial condition on δ at $x = -20$ to produce a δ which goes to zero as $x \to \infty$. The solutions branch as downstream infinity is approached and therefore become very sensitive to initial guess. A more appropriate way to solve the problem is to recognize that it is boundary value in nature and therefore solve it as a time relaxation process using central differences on all of the space variables. This type of technique has been used by Werle and Vatsa (ref. 7) to solve the interacting boundary-layer equations.

Figures 3 and 4 show the results for pressure and wall shear as a function of the reduced angle $\bar{\alpha}$. Rizzetta (ref. 5) has presented exact numerical results for the same problem. For small $\bar{\alpha}$ there is good agreement between the present results and those of Rizzetta. A direct comparison is given for $\bar{\alpha} = 2.5$. The comparison between the present results and Rizzetta's becomes poorer as $\bar{\alpha}$ increases. This is at least partially due to the fact that the free interaction portion of the solution has extended far enough upstream for values of $\bar{\alpha}$ greater than 2.5 that the difficulties of the present approximate method with the free interaction plateau region are beginning to appear. However, overall this simple integral method gives the main features of the flowfield and with improvements would seem to be a reasonably accurate and extremely fast method for finding lower deck solutions.

NUMERICAL FINITE DIFFERENCE SOLUTION OF THE SMALL SEPARATION PROBLEM

Jenson, Burggraf, and Rizzetta (ref. 4) have developed a finite difference numerical scheme for solving the lower deck equations for the supersonic case. As was mentioned in the previous sections, they have applied their numerical method to the problem of flow past a compression ramp. Rizzetta (ref. 5) gives more extensive numerical results for the same problem using essentially the same finite difference technique.

It can easily be shown that the ordinary Prandtl boundary-layer equations contain all of the terms indicated as being important in the triple deck analysis as long as one takes into account the interaction of the boundary layer with the outer inviscid flow. It is not a simple matter to solve the resulting set of interacting boundary-layer equations since, like the lower deck equations, they are boundary value in nature.

A very natural way to solve the interacting boundary-layer equations is by the use of an alternating direction implicit (ADI) method. This is the approach taken by Werle and Vatsa (ref. 7) and Vatsa (ref. 8) in their solution of the same type of compression ramp problems as were considered by Jenson, Burggraf and Rizzetta.

Figure 5 shows a comparison of the results for skin friction obtained using Werle and Vatsa's method with those obtained from the triple deck analysis by Rizzetta, see also Burggraf et al. (ref. 14). The results show that the interacting boundary-layer model slowly approaches the triple deck asymptotic solution as Reynolds number goes to infinity.

Because of the slow approach to the infinite Reynolds number limit indicated in the comparison, the triple deck results do not tend to agree well with experimental data at high but finite Reynolds numbers. On the other hand, interacting boundary-layer results tend to agree well. Figure 6 shows a comparison of Navier-Stokes and interacting boundary-layer results with the experimental data of Lewis, Kubota and Lees (ref. 15).

Tu and Weinbaum (ref. 16) have suggested that the principle cause of the poor agreement between triple deck results and experimental data lies in the fact that triple deck analysis neglects streamtube divergence in the middle deck region. Since interacting boundary-layer results contain this effect, they tend to show much better agreement with experiment.

CONCLUSION

The idea of solving separation problems using triple deck theory is still relatively new. It is anticipated that with time, the theory will be modified to incorporate the additional terms which will allow better agreement with experimental data. Even if this were not done, the insight gained from triple deck theory into the mechanism of high Reynolds number separation is in itself extremely valuable.

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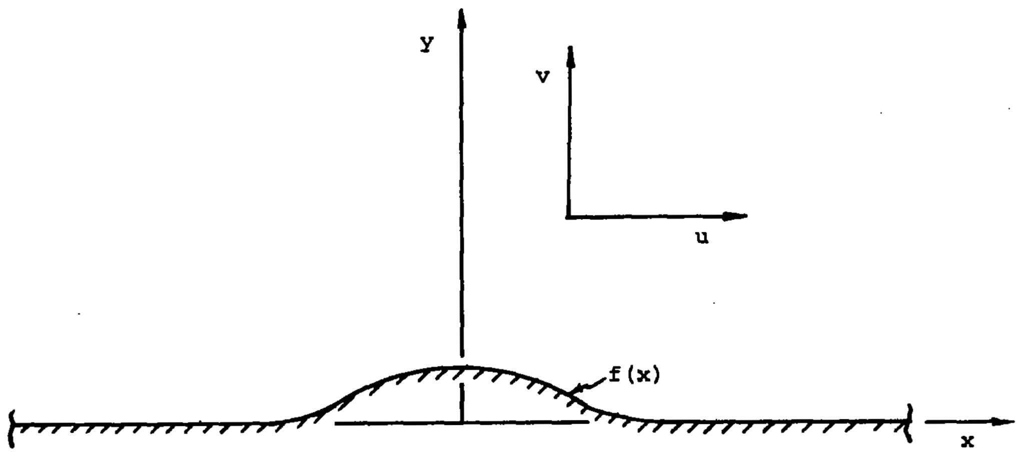

Figure 1.- Coordinate system and bump in lower deck variables.

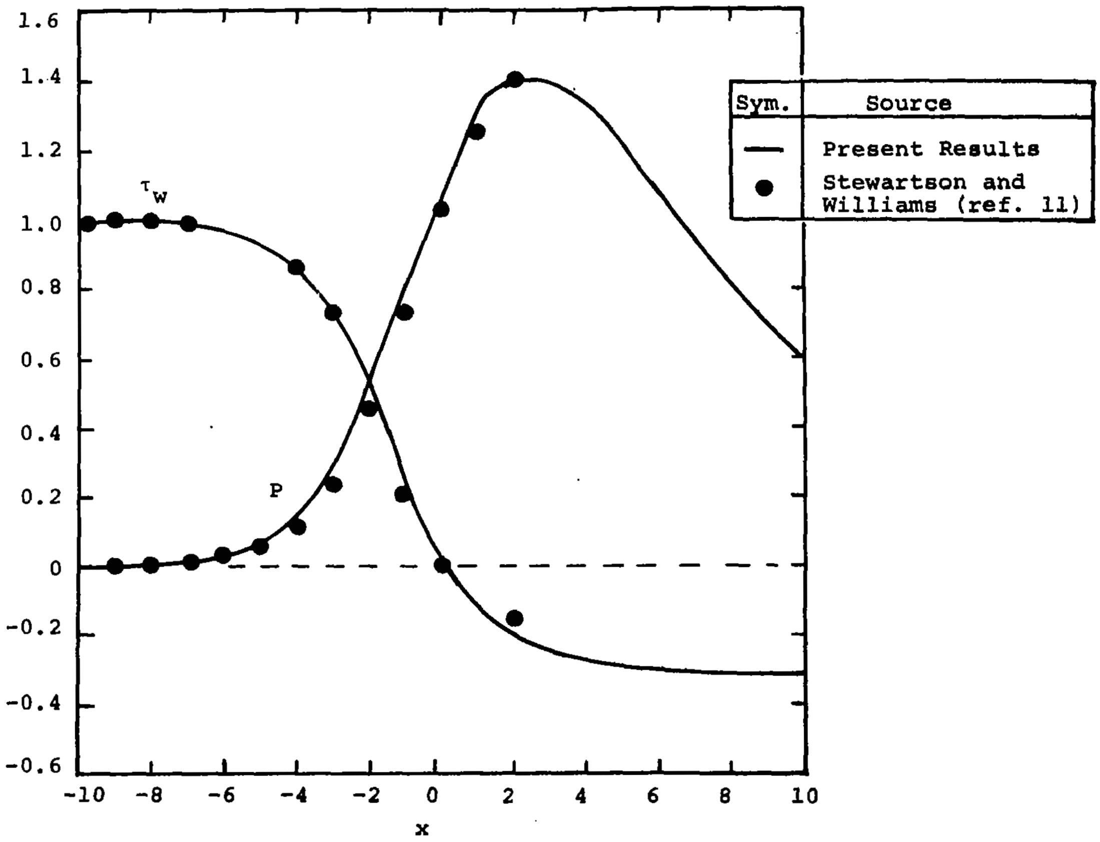

Figure 2.- Compressive free interaction results.

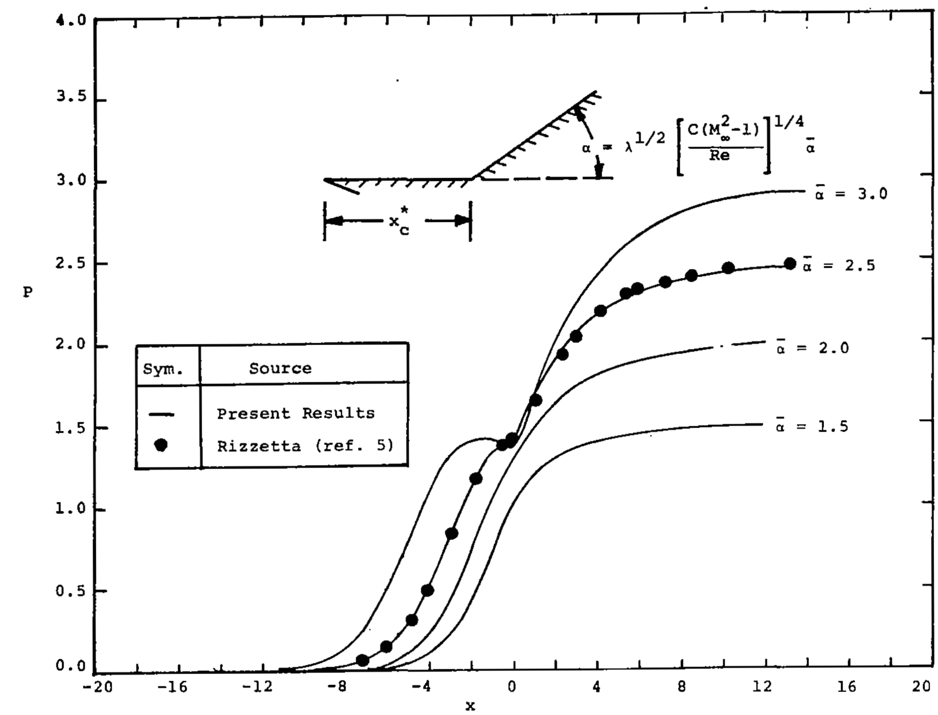

Figure 3.- Comparison of asymptotic and approximate pressure distribution.

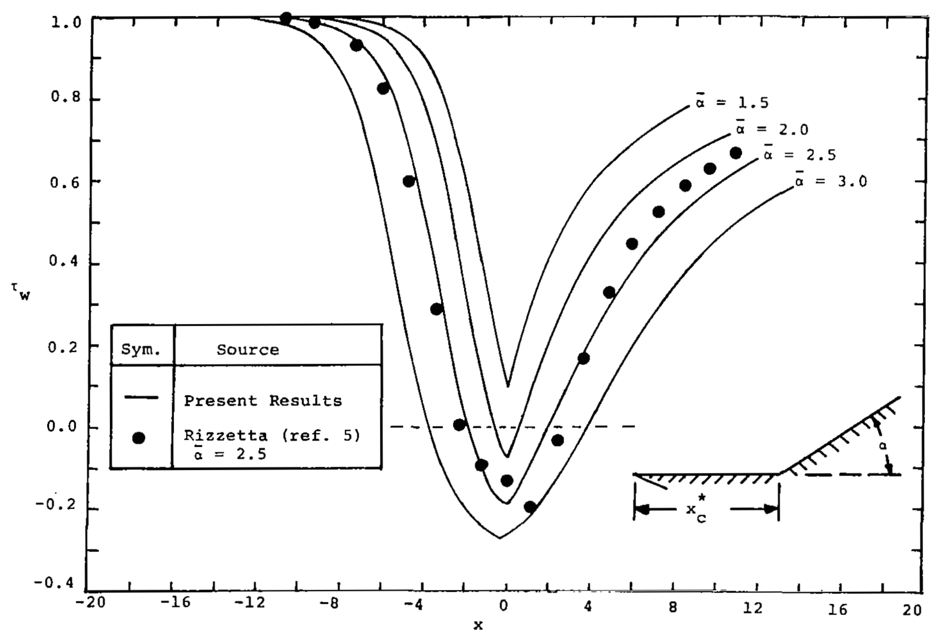

Figure 4.- Comparison of asymptotic and approximate shear distribution.

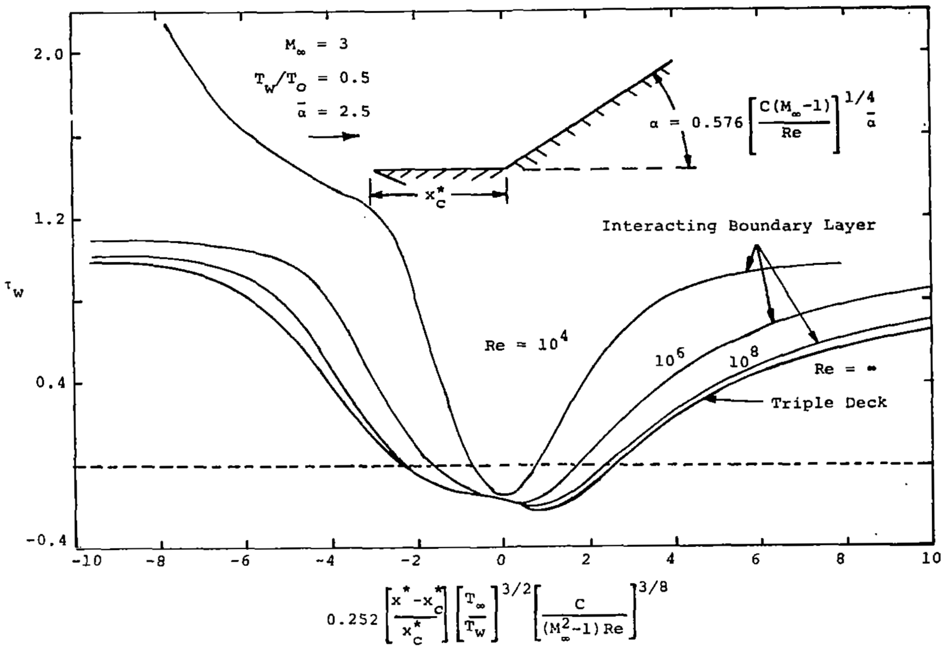

Figure 5.- Comparison of asymptotic and interacting boundary-layer solutions, after Burggraf et al. (ref. 14).

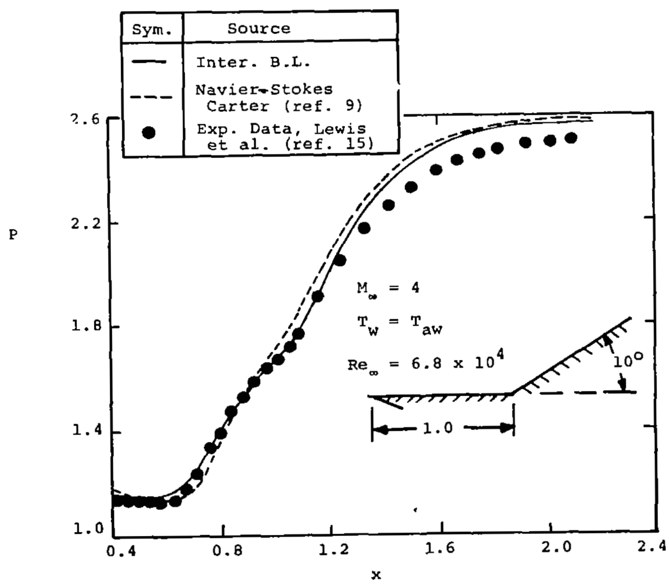

Figure 6.- Comparison of supersonic interacting boundary-layer with Navier-Stokes solutions for a compression ramp, after Vatsa (ref. 8).