THE RELATIVE MERITS OF SEVERAL NUMERICAL TECHNIQUES FOR SOLVING THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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SUMMARY

Four explicit finite-difference techniques designed to solve the time-dependent, compressible Navier-Stokes equations have been compared. These techniques are (1) MacCormack, (2) modified Du Fort-Frankel, (3) modified hopscotch, and (4) Brailovskaya. The comparison was made numerically by solving the quasi-one-dimensional Navier-Stokes equations for the flow in a converging-diverging nozzle. Solutions with and without standing normal shock waves were computed for unit Reynolds numbers (based on total conditions) ranging from 45374 to 2269. The results indicate that all four techniques are comparable in accuracy; however, the modified hopscotch scheme is two to three times faster than the Brailovskaya and MacCormack schemes and three to six times faster than the modified Du Fort-Frankel scheme.

INTRODUCTION

Recently, considerable interest has surfaced in the numerical solution of the compressible Navier-Stokes equations (refs. 1-4). Explicit numerical techniques have been used in most of these studies, especially those involving shock waves. The limited use of implicit methods is due to (1) coding complexity associated with the Navier-Stokes equations, (2) limited success in obtaining the large time steps as predicted by linear stability analysis, and (3) limited success in capturing shock waves. Another factor involved is the apparent success of explicit methods over implicit methods for adapting to the new fourth generation computers (STAR 100 and ILLIAC IV).

The purpose of the present study is to investigate the relative merits of several explicit finite-difference techniques for solving the compressible, time-dependent Navier-Stokes equations. Some of the important aspects evaluated are (1) computational speed, (2) numerical accuracy, (3) computer storage requirements, (4) Reynolds number limitations, and (5) effects of artificial smoothing. The four numerical techniques investigated are (1) modified hopscotch, (2) MacCormack, (3) modified Du Fort-Frankel, and (4) Brailovskaya. Each of these methods has been used to solve a quasi-one-dimensional converging-diverging nozzle problem. Solutions with and without standing normal shock waves are presented for unit Reynolds numbers ranging from 45374 to 2269.

SYMBOLS

A	nozzle cross-sectional area, m^2
c	speed of sound, m/sec
E	total internal energy per unit volume, $N\text{-}m/m^3$
M	Mach number
p	pressure, N/m^2
$\bar{R}$	Reynolds number per unit length, m^{-1}
S	smoothing term
t	time, sec
T	temperature, K
u	velocity, m/sec
x	distance along nozzle axis, m
Δt	time increment, sec
Δx	space increment, m
ρ	density, kg/m^3

Subscripts:

i	space index
t	total conditions

Superscript:

n	time index

GOVERNING EQUATIONS AND TEST PROBLEM

The converging-diverging nozzle problem used in this study represents a rigorous test case for the numerical techniques. The steady-state flow field is initially subsonic, goes sonic at the throat, passes through a standing normal shock wave in the diverging portion of the nozzle, and exits the nozzle with subsonic flow. Cases which do not contain a standing normal shock wave are also computed. For these cases the flow field downstream of the throat is supersonic. Different exit boundary conditions are required for each of these cases and will be discussed in a subsequent section.

The time-dependent, quasi-one-dimensional flow of a compressible, viscous fluid is governed by a set of three partial differential equations expressing the conservation of mass, momentum, and energy. These equations in conservative form are as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} - \frac{\partial}{\partial x}\left(B\,\frac{\partial C}{\partial x}\right) + H = 0 \tag{1}$$

where

$$U = A\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix};\ F = A\begin{bmatrix} \rho u \\ \rho u^2+p \\ (E+p)u \end{bmatrix};\ B = A\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}\mu & 0 \\ 0 & \frac{4}{3}\mu u & k \end{bmatrix};\ C = \begin{bmatrix} 0 \\ u \\ T \end{bmatrix};\ H = \begin{bmatrix} 0 \\ -p\,\frac{\partial A}{\partial x} \\ 0 \end{bmatrix} \tag{2}$$

The coefficients of viscosity (μ) and thermal conductivity (k) are given by Sutherland's viscosity law and a constant Prandtl number assumption.

Differencing Schemes

Several characteristics are common to each of the numerical schemes evaluated in this study. They are all second-order-accurate (for the steady-state solution) finite-difference techniques which solve the time-dependent form of the governing equations in search of a final steady-state solution. The methods are explicit, and hence, easily programmed. In particular, the methods evaluated here have been chosen especially with regard to programming simplicity for the Navier-Stokes equations.

Modified Hopscotch

The current version of hopscotch was first introduced in reference 5 where it was applied to the compressible Navier-Stokes equations for a shear layer mixing problem. This modified hopscotch technique (applied to eq. (1)) is expressed in two sweeps given by

first sweep (i+n even)

$$\begin{aligned} U_i^{n+1} = U_i^n &- \frac{\Delta t}{2\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right) - \frac{\Delta t}{2}\left(H_{i+1}^n + H_{i-1}^n\right) \\ &+ \frac{\Delta t}{\Delta x}\left[\frac{B_{i+1}^n + B_i^n}{2}\left(\frac{C_{i+1}^n - C_i^n}{\Delta x}\right) - \frac{B_i^n + B_{i-1}^n}{2}\left(\frac{C_i^n - C_{i-1}^n}{\Delta x}\right)\right] \end{aligned} \tag{3}$$

second sweep (i+n odd)

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^{n+1} - F_{i-1}^{n+1}\right) - \frac{\Delta t}{2}\left(H_{i+1}^{n+1} + H_{i-1}^{n+1}\right) + \frac{\Delta t}{\Delta x}\left[\frac{B_{i+1}^{n+1}+B_i^n}{2}\left(\frac{C_{i+1}^{n+1} - C_i^n}{\Delta x}\right) - \frac{B_i^n+B_{i-1}^{n+1}}{2}\left(\frac{C_i^n - C_{i-1}^{n+1}}{\Delta x}\right)\right] \tag{4}$$

Gottlieb and Gustafsson (ref. 6) have investigated the stability of the current version of hopscotch and have found it to be governed by the following CFL condition:

$$\Delta t_{CFL} \leqq \frac{\Delta x}{|u| + c} \tag{5}$$

In addition reference 6 found the viscous stability condition for the modified hopscotch technique to be

$$\frac{\mu \Delta t}{\rho (\Delta x)^2} \leqq 1 \tag{6}$$

Gourlay (ref. 7) suggested a simplification to the standard two-sweep hopscotch scheme which almost entirely removed the first sweep where equation (3) is replaced by

$$U_i^{n+1} = 2U_i^n - U_i^{n-1} \qquad \text{(i+n even)} \tag{7}$$

Numerical tests were performed with and without the use of equation (7) yielding identical results. The use of equation (7) increases the speed of the modified hopscotch technique by a factor of two without requiring additional storage.

Modified Du Fort-Frankel

The current version of the Du Fort-Frankel scheme was introduced by Gottleib and Gustafsson (ref. 8) as follows:

$$U_i^{n+1} = U_i^{n-1} - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right) - 2\Delta t\, H_i^n + \frac{2\Delta t}{\Delta x}\left[\frac{B_{i+1}^n + B_i^n}{2}\left(\frac{C_{i+1}^n - C_i^n}{\Delta x}\right) - \frac{B_i^n+B_{i-1}^n}{2}\left(\frac{C_i^n - C_{i-1}^n}{\Delta x}\right)\right] - \frac{2\Delta t\, \omega}{(\Delta x)^2} B_i^n \left(U_i^{n+1} - 2U_i^n + U_i^{n-1}\right) \tag{8}$$

where the last term is a stabilizing term. The value of ω (stabilizing coefficient) must be determined by numerical experiment. The stabilizer takes the place of the time averaging appearing in the viscous terms of the standard Du Fort-Frankel scheme. This simplifies the resulting numerical code, especially for the Navier-Stokes equations in multiple dimensions.

In addition to equation (8) an additional dissipative term must be added for stable operation

$$S_i^n = \frac{\varepsilon}{16}\left(U_{i+2}^n - 4U_{i+1}^n + 6U_i^n - 4U_{i-1}^n + U_{i-2}^n\right) \tag{9}$$

where the constant ε is determined numerically.

MacCormack

The version of the two-step Lax-Wendroff scheme used in this study was first introduced by MacCormack (ref. 9). Using the MacCormack technique to difference equation (1) yields

<u>predictor step</u>

$$U_i^{\overline{n+1}} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_i^n\right) + \frac{\Delta t}{\Delta x}\left[B_{i+1}^n\,\frac{C_{i+1}^n - C_i^n}{\Delta x} - B_i^n\,\frac{C_i^n - C_{i-1}^n}{\Delta x}\right] - H_i^n\Delta t \tag{10}$$

<u>corrector step</u>

$$U_i^{n+1} = \frac{1}{2}\left(U_i^{\overline{n+1}} + U_i^n\right) - \frac{\Delta t}{2\Delta x}\left(F_i^{\overline{n+1}} - F_{i-1}^{\overline{n+1}}\right) - \frac{\Delta t}{2}H_i^{\overline{n+1}} + \frac{\Delta t}{2\Delta x}\left[B_i^{\overline{n+1}}\,\frac{C_{i+1}^{\overline{n+1}} - C_i^{\overline{n+1}}}{\Delta x} - B_{i-1}^{\overline{n+1}}\,\frac{C_i^{\overline{n+1}} - C_{i-1}^{\overline{n+1}}}{\Delta x}\right] \tag{11}$$

where the overbar on the n superscript indicates a predicted value. The stability requirement for this scheme is the CFL condition (eq. (5)). In addition, a stability condition due to viscous effects is also present:

$$\frac{\mu\,\Delta t}{\rho(\Delta x)^2} \leqq \frac{1}{2} \tag{12}$$

To surpress pointwise oscillations an artificial smoothing term can be added to the right-hand side of equations (10) and (11) as follows:

<u>predictor step</u>

$$S_i^n = C_x\,\frac{\left|\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n\right|}{\rho_{i+1}^n + 2\rho_i^n + \rho_{i-1}^n}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) \tag{13}$$

corrector step

$$S_i^{\overline{n+1}} = C_x \frac{\left|\rho_{i+1}^{\overline{n+1}} - 2\rho_i^{\overline{n+1}} + \rho_{i-1}^{\overline{n+1}}\right|}{\rho_{i+1}^{\overline{n+1}} + 2\rho_i^{\overline{n+1}} + \rho_{i-1}^{\overline{n+1}}} \left(U_{i+1}^{\overline{n+1}} - 2U_i^{\overline{n+1}} + U_{i-1}^{\overline{n+1}}\right) \tag{14}$$

where C_x is an adjustable constant. In regions of smooth flow these terms will be negligible and will not influence the solution. In regions of pointwise oscillations these terms will provide the effect of solution smoothing.

Brailovskaya

The two-step finite-difference scheme introduced by Brailovskaya in 1965 (ref. 10) is second-order accurate in space and first-order accurate in time. Using this technique to difference equation (1) yields

predictor step

$$\begin{aligned} U_i^{\overline{n+1}} &= U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right) - \Delta t\, H_i^n \\ &+ \frac{2\Delta t}{\Delta x}\left[\frac{B_{i+1}^n + B_i^n}{2}\left(\frac{C_{i+1}^n - C_i^n}{\Delta x}\right) - \frac{B_i^n + B_{i-1}^n}{2}\left(\frac{C_i^n - C_{i-1}^n}{\Delta x}\right)\right] \end{aligned} \tag{15}$$

corrector step

$$\begin{aligned} U_i^{n+1} &= U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^{\overline{n+1}} - F_{i-1}^{\overline{n+1}}\right) - \Delta t\, H_i^n \\ &+ \frac{2\Delta t}{\Delta x}\left[\frac{B_{i+1}^n + B_i^n}{2}\left(\frac{C_{i+1}^n - C_i^n}{\Delta x}\right) - \frac{B_i^n + B_{i-1}^n}{2}\left(\frac{C_i^n - C_{i-1}^n}{\Delta x}\right)\right] \end{aligned} \tag{16}$$

The viscous terms in the predictor step are identical with the viscous terms in the corrector step and, therefore, need to be computed only once per time step. This feature reduces the required amount of computer time. The stability requirement for the Brailovskaya scheme is the usual CFL condition (eq. (5)). An additional viscous stability condition is required and is given by equation (12). The artificial smoothing applied to the MacCormack scheme (eqs. (13) and (14)) was also applied to the Brailovskaya scheme.

Boundary Conditions

The boundary conditions described in this section were used for each numerical method. Three boundary conditions at both the inflow and outflow boundaries must be specified. At the subsonic inflow total pressure and total temperature were specified and held fixed. The third inflow boundary condition was obtained by requiring a zero gradient on static pressure.

At the outflow boundary $(i = N)$ for the supersonic case the boundary conditions were

$$p_N^n = p_{N-1}^n \ , \quad u_N^n = u_{N-1}^n \ , \quad \rho_N^n = \rho_{N-1}^n \tag{17}$$

At the outflow boundary for the normal shock wave case the flow is subsonic; consequently, the boundary conditions are modified as follows

$$p_N^n = p_{exit}, \quad u_N^n = u_{N-1}^n \ , \quad \rho_N^n = \rho_{N-1}^n \tag{18}$$

where p_{exit} is specified and held fixed. Obtaining accurate results with such simple boundary conditions is made possible by adding constant area duct segments at the inflow and outflow stations of the nozzle.

These boundary conditions when applied to the modified Du Fort-Frankel code resulted in unstable oscillations at the boundaries. These oscillations were eliminated by two different methods. The first method was to apply second-order damping given by

$$S_i^n = \frac{\varepsilon}{16} \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n \right) \tag{19}$$

at $i = 2$ for the inflow and $i = N - 1$ for the outflow.

The second method of removing the oscillations consisted of replacing the original boundary conditions with a new set given by

$$p_t = \text{const}, \quad T_t = \text{const}, \quad p_1^n = p_2^{n-1} \tag{20}$$

$$p_N^n = p_{N-1}^{n-1} \ , \quad u_N^n = u_{N-1}^{n-1} \ , \quad \rho_N^n = \rho_{N-1}^{n-1} \tag{21}$$

$$p_N^n = p_{exit}, \quad u_N^n = u_{N-1}^{n-1} \ , \quad \rho_N^n = \rho_{N-1}^{n-1} \tag{22}$$

where equations (20) and (21) were used for the no shock wave case and equations (20) and (22) for the normal shock wave case. For shock-free flow solutions the second method of removing the oscillations produced the best results and are presented in the next section. For the normal shock wave case, both methods produced similar results.

DISCUSSION OF RESULTS

Shock-Free Solution

The initial condition solution for the isentropic calculation was established by first computing the inflow and outflow endpoints from one-dimensional isentropic theory. Then linear distributions for all the flow variables were computed between the endpoints.

Table I summarizes the computing statistics of all the results presented. The results of the isentropic (shock free) calculation are presented in figure 1. Included are Mach number, pressure, and temperature distributions along the nozzle axis for all four numerical techniques. Overall the agreement is very good. In particular, all four numerically predicted values of pressure at the throat lie within 0.5 percent of the theoretical value. The largest disagreement occurs at the outflow where theory predicts an exit Mach number of 1.925. The numerically predicted exit Mach numbers are below this and lie between 1.907 and 1.920.

The maximum error (ERR) versus the central processor unit time (CPU time) is presented in figure 2 where

$$\mathrm{ERR} = \max_i \frac{|\rho_i^{n+1} - \rho_i^n|}{\rho_i^n \quad \Delta t/\Delta t_{CFL}} \tag{23}$$

The CPU time required for computing initial conditions and solution input/output has been subtracted from the CPU time displayed in figure 2. The curves have been continued until the maximum error dropped below 0.001 although the actual calculations were carried to 0.0001 accuracy. For the test problem the modified hopscotch technique is clearly the fastest of the four techniques tested, being 2.2 times faster than Brailovskaya, 2.5 times faster than MacCormack, and 4.0 times faster than modified Du Fort-Frankel.

Normal Shock Wave Solution

The initial conditions for the cases with a standing normal shock wave were obtained from one-dimensional isentropic theory. The initial solution was entirely subsonic with the standard expansion in the converging portion in the nozzle, the sonic condition at the throat, and subsonic compression in the diverging portion of the nozzle. This condition was chosen because the use of initial conditions with supersonic outflows caused difficulties when the outflow pressure was specified.

Mach number and pressure distributions are presented in figures 3 and 4. A standing normal shock wave with a pressure ratio of approximately 3.7 has been captured by all four methods at $i \simeq 38$. The shock wave is spread over two to three grid points with minimal overshoots and no undershoots. The

Reynolds number was not low enough for any significant viscous effects to appear. The grid Reynolds numbers $(\rho u \, \Delta x/\mu)$ were between 10 and 20. In general the agreement is quite good between the four techniques.

Both the MacCormack and Brailovskaya results were computed with artificial smoothing added; however, the smoothing was not required for a stable solution. Instead, it was used to improve the characteristics of the captured shock by reducing the overshoot and undershoot oscillations.

Again the modified hopscotch technique is the fastest of the four methods tested (see table I, case 2) being 1.7 times faster than Brailovskaya, 1.8 times faster than MacCormack, and 3.9 times faster than modified Du Fort-Frankel.

Reynolds Number Effects

Three of the techniques (modified hopscotch, MacCormack, and Brailovskaya) have viscous stability conditions and therefore, should exhibit smaller time steps and longer CPU times for lower Reynolds numbers. Two test cases were computed with lower Reynolds numbers by decreasing the total pressure (see table I, cases 3 and 4). Mach number distributions for these two cases are shown in figures 5 and 6. The effect of the reduced Reynolds number is clearly evident. In figure 5 ($R = 11345 \text{ m}^{-1}$) the shock wave is spread across five to six grid points, and in figure 6 ($R = 2269 \text{ m}^{-1}$) the solution is so smeared by the physical viscosity that a shock wave cannot be recognized. The grid Reynolds numbers are between 2 and 5 for case 3 and between 0.5 and 1.0 for case 4.

The shock position predicted by the modified Du Fort-Frankel technique for case 3 (see fig. 5) is in slight disagreement with the shock position predicted by the other three methods. This is due to the different outflow boundary conditions used by the modified Du Fort-Frankel technique (see eqs. (19) - (22)). The effect is to alter the value of exit pressure and thus, change the shock position. The modified Du Fort-Frankel scheme failed to converge for case 4.

As expected, the viscous stability condition was more restrictive and therefore dominated the low Reynolds number calculations, especially case 4. Modified hopscotch seemed to have a slightly more severe viscous stability condition than MacCormack or Brailovskaya, but, possibly due to the added physical viscosity, actually reached a converged solution sooner in physical time. For instance, in case 3, modified hopscotch was 2.8 times faster than Brailovskaya, 3.2 times faster than MacCormack, and 6.2 times faster than modified Du Fort-Frankel.

The lack of a viscous stability limit for modified Du Fort-Frankel could not be fully tested due to its failure to converge for case 4. The reduced time step ratio exhibited by modified Du Fort-Frankel for all cases is due to the artificial dissipation which must be added for stable operation. Hence, even if modified Du Fort-Frankel is not restricted by a viscous stability condition, it must pay the price of a reduced time step for another reason, regardless of Reynolds number.

Artificial Smoothing

Artificial smoothing has been used in this study on three of the four methods tested (MacCormack, Brailovskaya, and Du Fort-Frankel). To investigate the effect of smoothing, a series of Mach number distributions for three different values of C_x (smoothing constant) are presented in figure 7. The three curves correspond to no smoothing (C_x = 0.0), moderate smoothing (C_x = 0.2), and massive smoothing (C_x = 1.0). All three curves were computed by the same numerical technique (Brailovskaya) and at the same flow conditions (R = 45374 m^{-1} and p_{exit}/p_t = 0.7). Enlargements of the Mach number profiles around the standing normal shock wave are presented in figure 7. The no smoothing case spreads the shock wave across three grid points and exhibits pre-shock oscillations. The moderate smoothing case, likewise, spreads the shock over three grid points, almost identically matching the no smoothing shock, but without pre-shock oscillations. The massive smoothing case spreads the shock over four or five grid points and effectively causes a position shift in the shock wave. All three profiles away from the shock are in good agreement regardless of how much smoothing is applied. Therefore, it is clear that artificial smoothing in a limited amount has helped the quality of the solution.

CONCLUDING REMARKS

The modified hopscotch technique was superior in speed for all cases tested, being 1.7 to 2.8 times faster than the Brailovskaya technique, 1.8 to 3.2 times faster than the MacCormack technique, and 3.9 to 6.2 times faster than the modified Du Fort-Frankel technique.

All methods tested were comparable in accuracy for the cases tested, with or without shock waves.

The modified hopscotch scheme seemed to have a slightly more severe viscous stability condition than the MacCormack or Brailovskaya schemes. However, for the viscous stability restricted cases, solutions computed by the modified hopscotch technique actually reached steady state sooner in physical time than any of the other techniques tested.

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Table I.- Summary of results.

	Case 1 Isentropic Supersonic R =45374				Case 2 Normal Shock R =45374 p_{exit}/p_t=.7				Case 3 Normal Shock R =11345 p_{exit}/p_t=.7				Case 4 Normal Shock R =2269 p_{exit}/p_t=.7			
	$\frac{\Delta t}{\Delta t_{CFL}}$	n	Physical Time (μsec)	CPU Time (sec)	$\frac{\Delta t}{\Delta t_{CFL}}$	n	Physical Time (μsec)	CPU Time (sec)	$\frac{\Delta t}{\Delta t_{CFL}}$	n	Physical Time (μsec)	CPU Time (sec)	$\frac{\Delta t}{\Delta t_{CFL}}$	n	Physical Time (μsec)	CPU Time (sec)
MacCormack	1.0	345	143	5.8	.9	338	140	5.3	.9	546	223	8.5	.5	896	217	14.0
Modified Hopscotch	.9	395	147	2.2	1.0	565	246	3.0	.8	514	188	2.7	.3	1193	175	6.2
Brailovskaya	1.1	334	152	5.2	1.1	323	161	5.2	1.0	485	222	7.5	.4	1110	221	17.6
Modified Du Fort-Frankel	.5	858	177	10.7	.5	990	216	11.8	.4	1350	243	16.9				

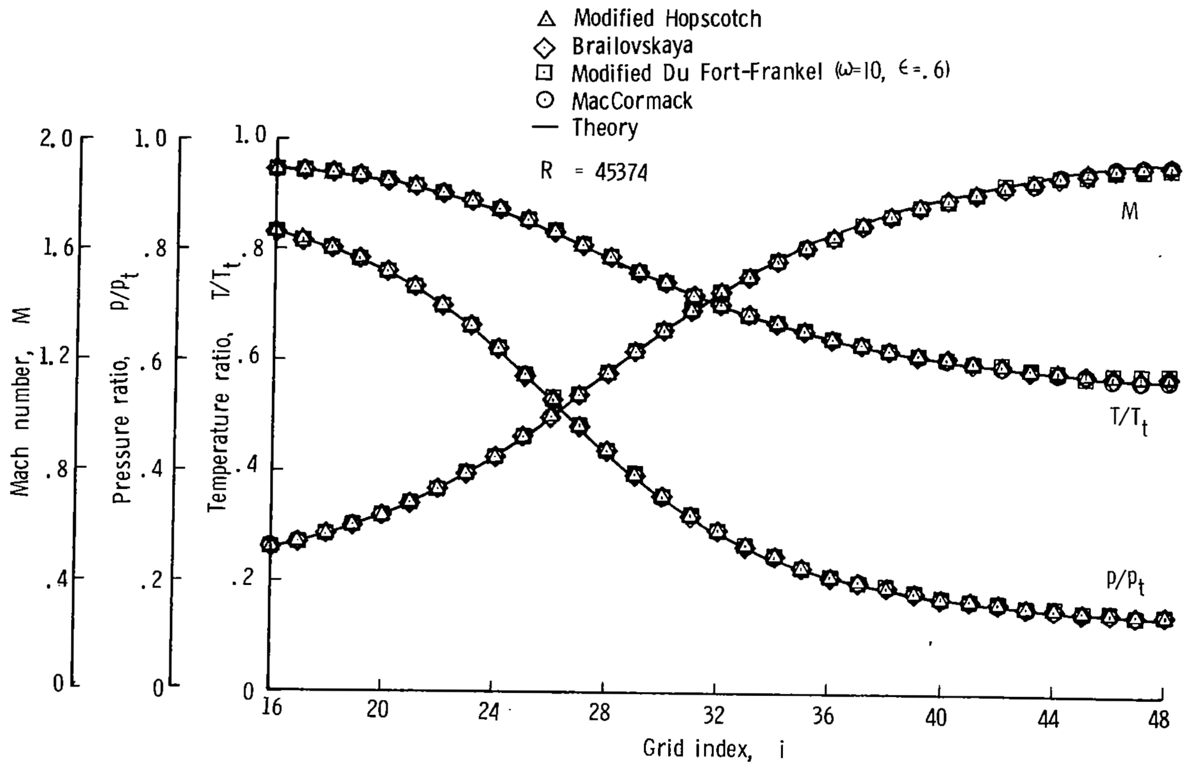

Figure 1.- Shock-free calculation.

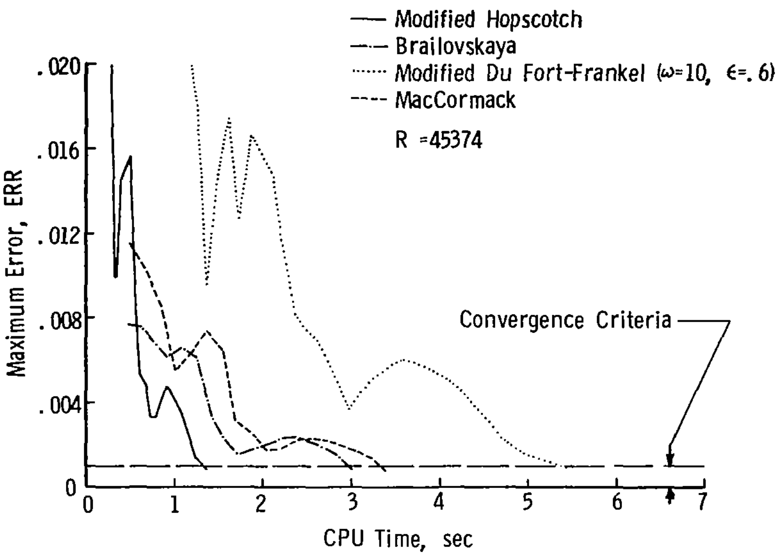

Figure 2.- Convergence rate comparison.

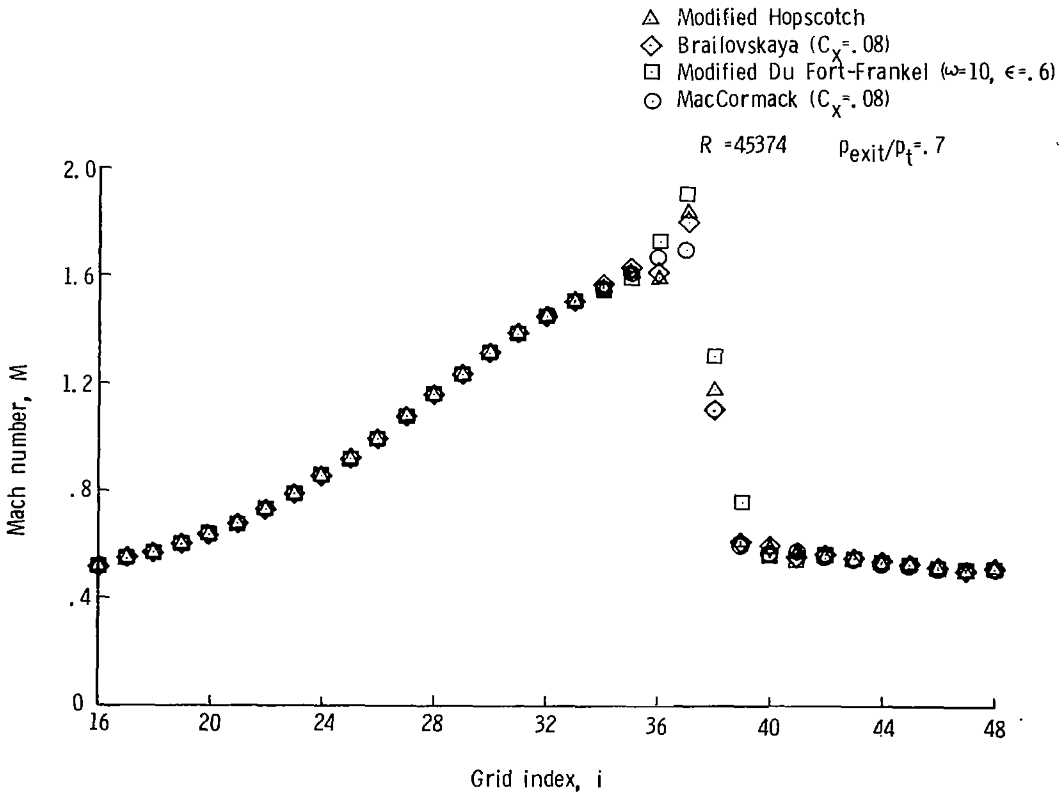

Figure 3.- Mach number distribution (case 2).

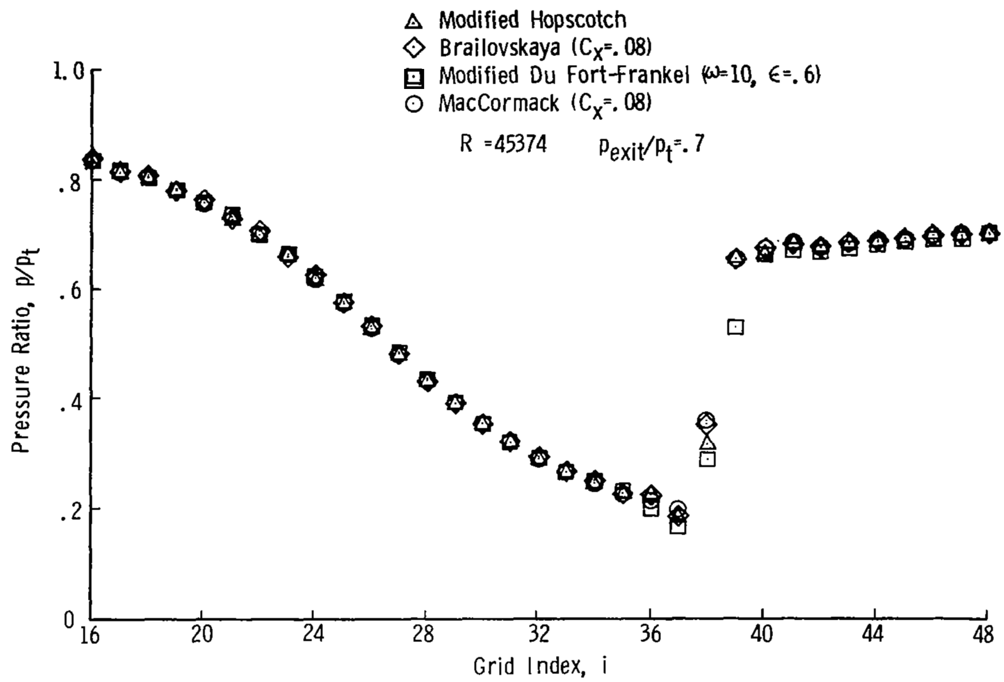

Figure 4.- Pressure ratio distribution (case 2).

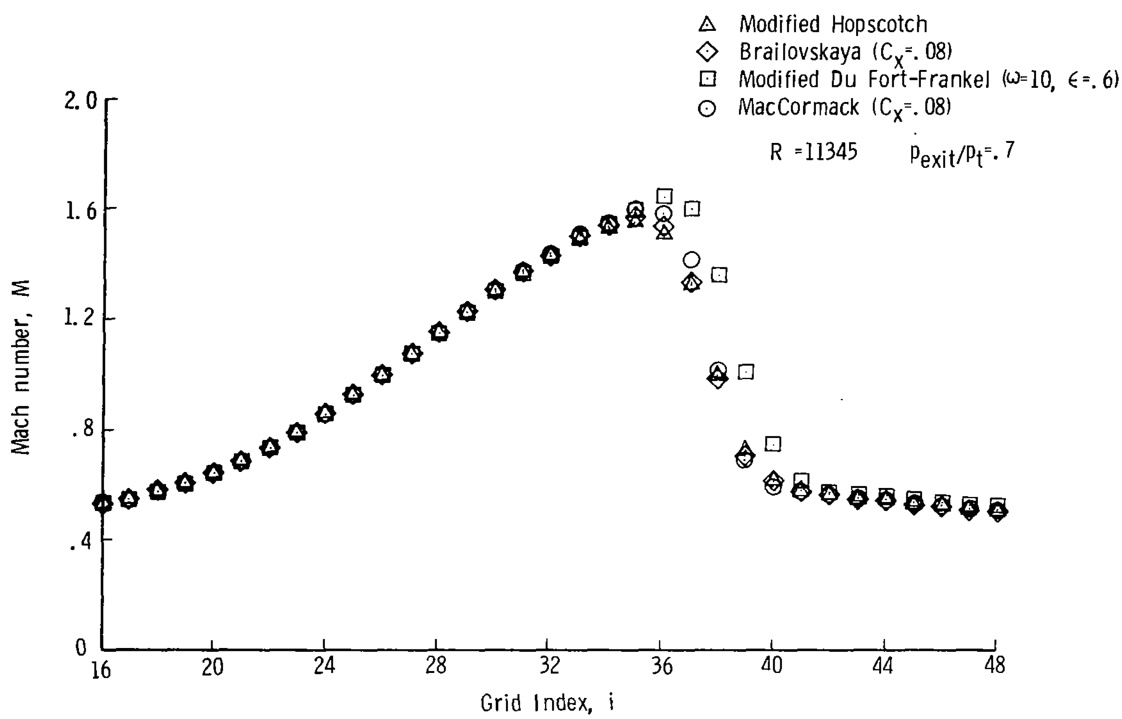

Figure 5.- Mach number distribution (case 3).

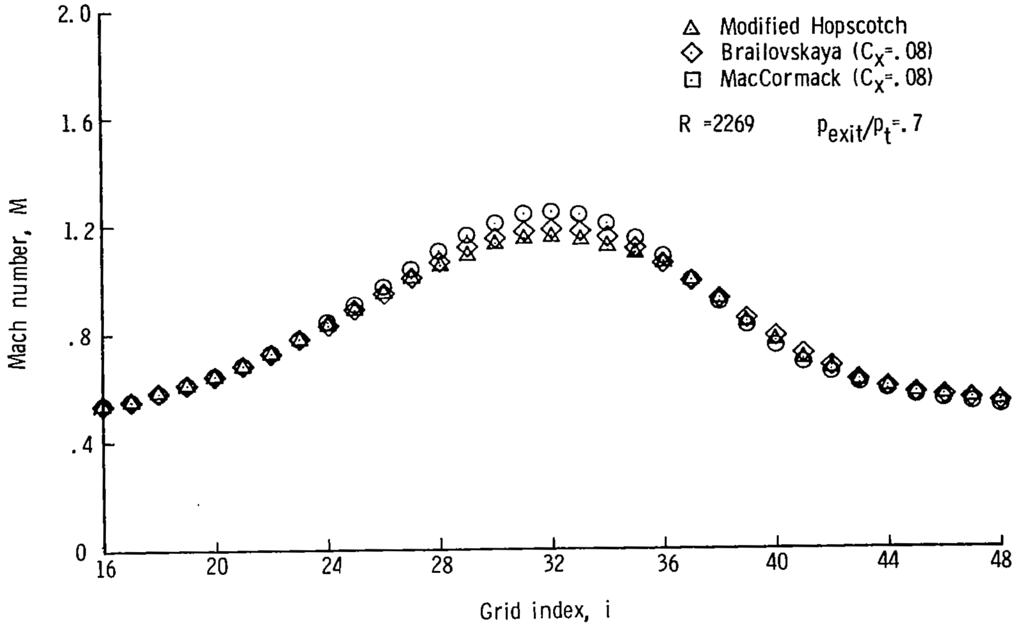

Figure 6.- Mach number distribution (case 4).

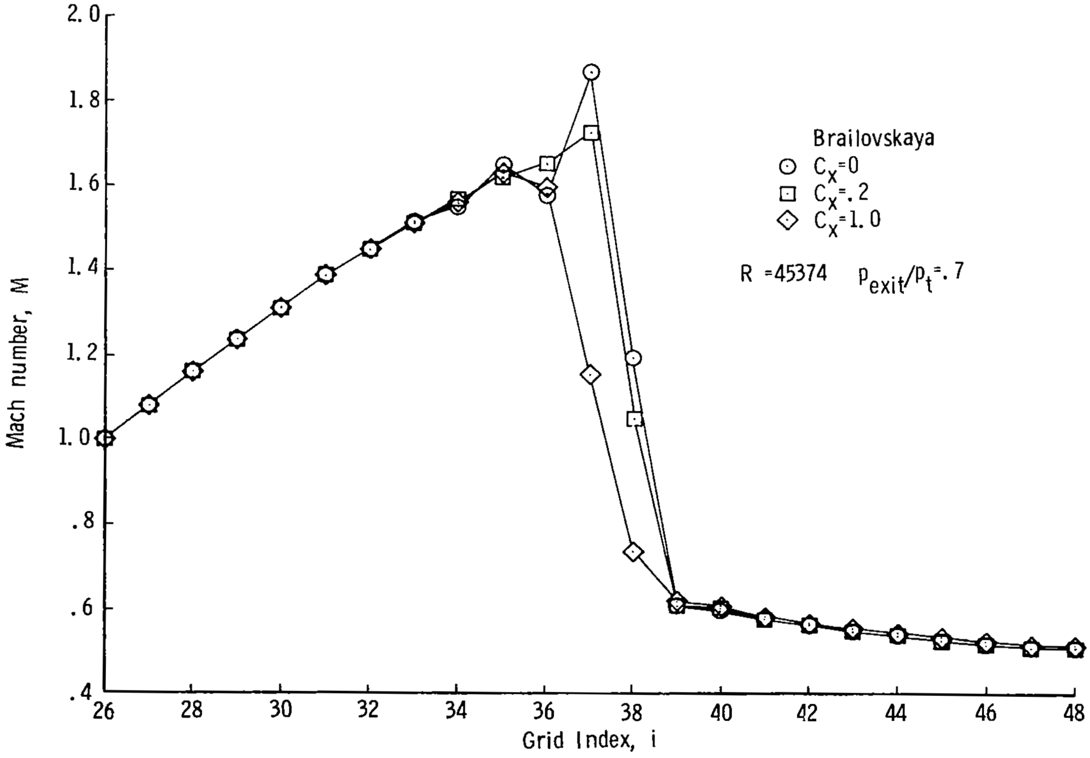

Figure 7.- Effect of artificial smoothing.