THE LIFT FORCE ON A DROP IN UNBOUNDED

PLANE POISEUILLE FLOW

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SUMMARY

The lift force on a deformable liquid sphere moving in steady, plane Poiseuille-Stokes flow and subjected to an external body force is calculated. The results are obtained by seeking a solution to Stokes' equations for the motion of the liquids inside and outside the slightly perturbed sphere surface, as expansions valid for small values of the ratio of the Weber number to the Reynolds number. When the ratio of the drop and external fluid viscosities is small, the lift exerted on a neutrally buoyant drop is found to be approximately one-tenth of the magnitude of the force reported by Wohl and Rubinow (ref. 1) acting on the same drop in unbounded Poiseuille flow in a tube. The resultant trajectory of the drop is calculated and displayed as a function of the external body force.

INTRODUCTION

Understanding the dynamics of a single particle and suspensions of particles in slow viscous flow is of fundamental importance in many branches of science and technology, such as air pollution, raindrop formation, fluidization in the chemical process industry, blood flow, the flow of fiber suspensions in paper making, et al. (refs. 2, 3).

The migration of a single spherical particle across the streamlines of a nonuniform creeping flow cannot be explained on the basis of Stokes' equations, even in the presence of bounding walls; i.e., a sphere experiences no transverse force at zero Reynolds number. A transverse force does exist theoretically if inertial forces are taken into account (refs. 4, 5, 6). However, the situation is different for flexible particles. Experimental observations (ref. 7) reveal that at low Reynolds numbers, even when a rigid sphere experiences a negligible transverse force, neutrally buoyant deforming drops (and

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flexible solid particles) migrate rapidly across streamlines. This suggests that the lift force which produces migration arises from the interaction between the particle deformation and the surrounding flow field, rather than from an inertial effect.

Chaffey et al.(ref. 8) considered the problem of a deformable liquid sphere in Couette-Stokes flow (linear shear). Assuming the drop to be 'close' to the plane wall bounding the flow, they found that the effect on the deformed drop was to produce a force tending to push the drop away from the wall. This force has two failings when used alone as a basis for explaining the migration of drops in plane Poiseuille flow. First, it neglects the force due to the interaction of the parabolic profile with the resultant deformation of the drop. Secondly, it cannot be expected to be valid when the drop is not close to the wall. In fact, Karnis & Mason (ref. 9) have shown experimentally that the migration rates calculated by Chaffey et al. are significantly larger than those which are observed. The experimental observations were recorded at a considerable distance from the walls relative to the particle size.

In this paper, the hydrodynamic force arising out of the interaction between the incident plane parabolic flow and the sphere deformation is calculated. The interaction between the drop and the boundary walls is neglected. The analysis herein follows that of reference 1 wherein the case of Poiseuille flow in a tube is considered (see also refs. 10, 11, 12).

FORMULATION

The drop surface is defined by
$$r = 1 + f(\theta, \phi), \tag{1}$$
where (r, θ, ϕ) are spherical polar coordinates with pole fixed at the center of the undistorted drop and the axis $\theta = 0$ along the direction of the undisturbed velocity $\vec{U}$. We denote the velocity of the fluid exterior to this surface by $\vec{v}$ and the pressure by p. All external lengths, velocities and stresses have been non-dimensionalized by a, U_0 and $\mu U_0 a^{-1}$, respectively, where a is the radius of the undeformed drop, U_0 is a reference velocity to be specified later, and μ is the viscosity of the fluid outside the drop. Quantities characterizing the interior of the drop are distinguished with a prime. It is assumed that $\{\vec{v},p\}$ and $\{\vec{v}',p'\}$ satisfy the Stokes' equations and conditions:

$$\Delta \vec{v} - \nabla p = 0, \quad \nabla \cdot \vec{v} = 0, \quad \vec{v} = \vec{U} \text{ at } r = \infty, \left.\begin{array}{c} \\ \\ \end{array}\right\}$$
$$\Delta \vec{v}' - \nabla p' = 0, \quad \nabla \cdot \vec{v}' = 0, \quad \vec{v}' \text{ bounded,} \qquad (2)$$

$$v_n = v'_n = 0, \quad \vec{v}_t = \vec{v}'_t, \quad \vec{\tau}_t = \alpha \vec{\tau}'_t \text{ at } r = 1 + f(\theta, \phi), \qquad (3)$$

$$\varepsilon \tau_n = \varepsilon \alpha \tau'_n + R_1^{-1} + R_2^{-1} \text{ at } r = 1 + f(\theta, \phi), \qquad (4)$$

where v_n represents the normal velocity component, $\vec{v}_t$ the tangential velocity vector, $\vec{\tau}_t$ the tangential stress vector, τ_n the magnitude of the normal stress; $\alpha = \mu'/\mu$; R_1 and R_2 are the two principal radii of curvature of the drop surface; and the dimensionless parameter $\varepsilon = \mu U_0 T^{-1}$, where T is the constant surface tension associated with the interface between the two viscous media. The pressure p' includes a body force K per unit volume which is assumed to act on the drop in the positive-z direction. Equation (4) is Laplace's formula for the equilibrium between the normal stress across the surface and the tension and curvature of the surface.

We seek the solution of equations (2) - (4) as expansions valid for small values of ε which, upon neglecting terms of $O(\varepsilon)$, are of the form

$$\vec{v} = \vec{v}_0 + \varepsilon \vec{v}_1, \quad p = p_0 + \varepsilon p_1, \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\}$$
$$\vec{v}' = \vec{v}'_0 + \varepsilon \vec{v}'_1, \quad p' = \varepsilon^{-1} p'_{-1} + p'_0 + \varepsilon p'_1, \qquad (5)$$
$$f = \varepsilon f_1 .$$

The solution in general will depend on the parameter α. We shall see (eq. (6)) that the term $\varepsilon^{-1} p'_{-1}$, which gives the internal pressure of the drop in its spherical shape when $\vec{U} = 0$, is needed in order to satisfy condition (4). The deformation f is assumed to be $O(\varepsilon)$ so that the drop is spherical when $\vec{U} = 0$. Upon inserting equations (5) into equations (2) and equating coefficients of each power of ε in each equation, we find that $\{\vec{v}_i, p_i, \vec{v}'_i, p'_i\}$ for i = 0, 1 satisfy the Stokes' equations, $\nabla p'_{-1} = 0$, $\vec{v}_0 = \vec{U}$ at $r = \infty$, and $\vec{v}_1 = 0$ at $r = \infty$.

The boundary conditions (3) - (4) must be examined carefully in order to determine the contributions at the various orders of ε. It can be shown (ref. 1) that the normal and tangential components of the velocity and stress vectors can be expressed, through a small rotation, in terms of spherical coordinate components. This small rotation is defined with the aid of two Eulerian angles,

which are related to the deformation of the drop. A Taylor expansion about $r = 1$ of the quantities in equations (3) − (4) is then used to transform to an equivalent set of boundary conditions evaluated on the undeformed sphere surface. Finally, taking account of equations (5) yields:

$O(\varepsilon^0)$,

$$P'_{-1} = 2/\alpha, \tag{6}$$

$$\left.\begin{array}{l} v_{0r} = v_{0r}' = 0, \quad v_{0\theta} = v_{0\theta}', \quad v_{0\phi} = v_{0\phi}', \\[2mm] \tau_{0r\theta} = \alpha\tau_{0r\theta}', \quad \tau_{0r\phi} = \alpha\tau_{0r\phi}', \end{array}\right\} \text{ at } r = 1 \tag{7}$$

$O(\varepsilon)$,

$$v_{1r} + f_1 \frac{\partial v_{0r}}{\partial r} = [\text{primes}] = \frac{\partial f_1}{\partial \theta} v_{0\theta} + \text{cosec}\theta \frac{\partial f_1}{\partial \phi} v_{0\phi},$$

$$v_{1\theta} + f_1 (\partial v_{0\theta}/\partial r) = [\text{primes}],$$

$$v_{1\phi} + f_1 (\partial v_{0\phi}/\partial r) = [\text{primes}], \qquad\qquad \text{at } r = 1$$

$$\tag{8}$$

$$\tau_{1r\theta} + f_1 \frac{\partial \tau_{0r\theta}}{\partial r} + \frac{\partial f_1}{\partial \theta} (\tau_{0rr} - \tau_{0\theta\theta}) - \text{cosec}\theta \frac{\partial f_1}{\partial \phi} \tau_{0\theta\phi} = \alpha [\text{primes}],$$

$$\tau_{1r\phi} + f_1 \frac{\partial \tau_{0r\phi}}{\partial r} - \frac{\partial f_1}{\partial \theta} \tau_{0\theta\phi} + \text{cosec}\theta \frac{\partial f_1}{\partial \phi} (\tau_{0rr} - \tau_{0\phi\phi})$$
$$= \alpha [\text{primes}]$$

$$\tau_{0rr} = \alpha\tau_{0rr}' - (2 + \text{cosec}\theta \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial}{\partial \theta}) + \text{cosec}^2\theta \frac{\partial^2}{\partial \phi^2}) f_1 \qquad \text{at } r = 1. \tag{9}$$

Equations (7) involve only the $O(\varepsilon^0)$ terms of the expansion, and equation (9) gives the deformation f_1 in terms of the quantities determined from the $O(\varepsilon^0)$ solution. The function f_1 is determined as a particular solution of equation (9) subject to the auxiliary conditions that the volume of the drop remains constant, and that the centroid of the drop is chosen to coincide with the origin of the coordinate system.

PLANE PARABOLIC FLOW

We now consider U to be a plane parabolic flow represented by

$$U = \beta + \delta x + \gamma x^2, \tag{10}$$

with respect to a coordinate system fixed at the centroid of the drop, where β, δ and γ are constants. In order to relate this to plane Poiseuille flow, let the drop be located at an orthogonal distance $X = b$ measured from the mid-plane between two parallel walls, one stationary, with the positive X-axis pointing

in the direction of the other wall moving in the positive-z direction with velocity U_w. Suppose that the drop is moving with dimensionless velocity $c = dZ/dt$, with respect to a coordinate system fixed in the stationary wall, and in the same direction as the Poiseuille flow field. Then the velocity of the flow with respect to a coordinate system fixed at the centroid of the drop is

$$\left. \begin{aligned} U = 1 - (\frac{X}{b_0})^2 + \frac{U_w}{2U_0}(1 + \frac{X}{b_0}) - c \\ X = b + ax. \end{aligned} \right\} \tag{11}$$

where

Here the reference velocity $U_0 = -b_0^2 G/2\mu$ where b_0 is one-half the distance between the walls bounding the flow, and $G \neq 0$ is the constant applied pressure gradient. By comparing equation (10) with equation (11) it follows that

$$\left. \begin{aligned} \beta \equiv 1 - (\frac{b}{b_0})^2 + \frac{U_w}{2U_0}(1 + \frac{b}{b_0}) - c, \\ \delta \equiv \frac{aU_w}{2U_0 b_0} - \frac{2ab}{b_0^2}, \quad \gamma \equiv -(a/b_0)^2. \end{aligned} \right\} \tag{12}$$

Since the $O(\epsilon^0)$ velocity field satisfies the boundary condition $\vec{v}_0 \to U\vec{k}$ as $r \to \infty$, where $\vec{k}$ is the unit vector in the z direction, it follows from the linearity of Stokes' equations that the $O(\epsilon^0)$ solution is easily obtained as the sum of the known flows past a liquid sphere in uniform stream, linear shear and plane quadratic shear. From these solutions the $O(\epsilon^0)$ hydrodynamic force $\vec{F}_0$ on the drop is calculated to be

$$\vec{F}_0 = 6\pi\mu a \ U_0 \left[\frac{\alpha + \frac{2}{3}}{\alpha + 1} \beta + \frac{1}{3} \frac{\alpha}{\alpha + 1} \gamma \right] \vec{k}. \tag{13}$$

This is the drag force acting on the undeformed drop. The value of β (or equivalently the velocity c of the drop via eq. (12)) is determined so that the body force $\frac{4}{3}\pi a^3 K\vec{k}$ balances $\vec{F}_0$, viz.

$$\beta = - \frac{\alpha}{3\alpha + 2} \gamma - \frac{2}{3} k (\frac{\alpha + 1}{3\alpha + 2}), \tag{14}$$

where k is the non-dimensional body force given by $k = K a^2 / \mu' U_0$.

The deformation f_1 is found from equation (9) to be

$$f_1 = f_{11} + f_{12},$$

$$f_{11} = \frac{19}{24} \delta \frac{\alpha + \frac{16}{19}}{\alpha + 1} \cos\phi \; P_2^1 (\cos\theta),$$

$$f_{12} = \frac{-11}{40} \gamma \frac{\alpha + \frac{10}{11}}{\alpha + 1} [P_3^0(\cos\theta) - \frac{1}{6} \cos 2\phi \; P_3^2 (\cos\theta)],$$

(15)

where P_2^1, P_3^0 and P_3^2 are associated Legendre polynomials. It is seen that the deformation is independent of the body force.

To determine the $O(\varepsilon)$ fields $\{\vec{v}_1, p_1, \vec{v}_1', p_1'\}$ we must solve Stokes' equations subject to the interface boundary conditions (8) at $r = 1$, in addition to having $\vec{v}_1$ vanish at infinity. It is found that the drop does not experience any hydrodynamic force owing to incident linear shear (δx) and quadratic shear (γx^2), considered independently. However, their combination gives rise to an additional $O(\varepsilon)$ flow needed to satisfy additional interaction terms in the boundary conditions (8). It is this interaction field which produces a transverse force on the drop. In particular, define the interaction contributions w_{1r} and $\sigma_{1r\theta}$ by

$$v_{1r} = v_{1r}^{(1)} + v_{1r}^{(2)} + w_{1r}, \quad \tau_{1r\theta} = \tau_{1r\theta}^{(1)} + \tau_{1r\theta}^{(2)} + \sigma_{1r\theta},$$

(16)

where $v_{1r}^{(1)}$ and $v_{1r}^{(2)}$ represent the linear shear and quadratic shear flows, considered independently. There are no contributions arising from the uniform flow (β) in equations (16) because to $O(\varepsilon)$ uniform flow produces no deformation of the drop (eqs. (15)). Substituting equations (16) and analogous expressions for the other $O(\varepsilon)$ velocity and stress components into conditions (8) together with the deformations (15), we find that the interaction field must satisfy, at $r = 1$:

$$w_{1r} + \frac{\partial}{\partial r} (f_1 v_{0r}^{(0)} + f_{12} v_{0r}^{(1)} + f_{11} v_{0r}^{(2)}) = [\text{primes}]$$

$$= \frac{\partial f_1}{\partial \theta} v_{0\theta}^{(0)} + \frac{\partial f_{12}}{\partial \theta} v_{0\theta}^{(1)} + \frac{\partial f_{11}}{\partial \theta} v_{0\theta}^{(2)}$$

$$+ \text{cosec}\theta \; (\frac{\partial f_{11}}{\partial \phi} v_{0\phi}^{(2)} + \frac{\partial f_{12}}{\partial \phi} v_{0\phi}^{(1)}),$$

(17a)

$$w_{1\theta} + \frac{\partial}{\partial r} (f_1 v_{0\theta}^{(0)} + f_{12} v_{0\theta}^{(1)} + f_{11} v_{0\theta}^{(2)}) = [\text{primes}],$$

$$w_{1\phi} + \frac{\partial}{\partial r} (f_{11} v_{0\phi}^{(2)} + f_{12} v_{0\phi}^{(1)}) = [\text{primes}],$$

$$\sigma_{1r\theta} + \frac{\partial}{\partial r}(f_1\,\tau_{0r\theta}^{(0)} + f_{12}\,\tau_{0r\theta}^{(1)} + f_{11}\,\tau_{0r\theta}^{(2)}) - \frac{\partial f_1}{\partial \theta}\,\tau_{0\theta\theta}^{(0)}$$

$$+ \frac{\partial f_{12}}{\partial \theta}(\tau_{0rr}^{(1)} - \tau_{0\theta\theta}^{(1)}) + \frac{\partial f_{11}}{\partial \theta}(\tau_{0rr}^{(2)} - \tau_{0\theta\theta}^{(2)})$$

$$- \csc\theta\,\frac{\partial f_{11}}{\partial \phi}\,\tau_{0\theta\phi}^{(2)} + \frac{\partial f_{12}}{\partial \phi}\,\tau_{0\theta\phi}^{(1)} = \alpha\,[\text{primes}]$$

$$\sigma_{1r\phi} + \frac{\partial}{\partial r}(f_{11}\,\tau_{0r\phi}^{(2)} + f_{12}\,\tau_{0r\phi}^{(1)}) - \frac{\partial f_{11}}{\partial \theta}\,\tau_{0\theta\phi}^{(2)} - \frac{\partial f_{12}}{\partial \theta}\,\tau_{0\theta\phi}^{(1)}$$

$$+ \operatorname{cosec}\theta\,\frac{\partial f_{11}}{\partial \phi}(\tau_{0rr}^{(2)} - \tau_{0\phi\phi}^{(2)}) + \frac{\partial f_{12}}{\partial \phi}(\tau_{0rr}^{(1)} - \tau_{0\phi\phi}^{(1)})$$

$$= \alpha\,[\text{primes}]. \qquad (17b)$$

The quantity $v_{0r}^{(i)}$ in equations (17) denotes the $O(\varepsilon^0)$ radial velocity component corresponding to a uniform stream, linear shear flow and quadratic shear flow for $i = 0$, 1, and 2, respectively, and similarly for the other velocity and stress components.

THE LIFT FORCE AND TRAJECTORY

The $O(\varepsilon)$ interaction field can now be found by first inserting the known deformations and $O(\varepsilon^0)$ fields into boundary conditions (17). The solid spherical harmonics and their coefficients in Lamb's general solution (ref. 13, p. 595) to Stokes' equations for a sphere are then chosen so that conditions (17) are satisfied. However, the perturbation to the $O(\varepsilon^0)$ force experienced by the drop after deformation depends only upon one solid spherical harmonic of order -2 in Lamb's general solution (ref. 14). This $O(\varepsilon)$ force can be expressed by

where
$$\vec{F}_1 = -4\pi\mu\varepsilon U_0 a\,\nabla(r^3 P_{-2}),$$
$$P_{-2} = A_{-2}^1\,r^{-2}\,P_1^1(\cos\theta)\cos\phi \qquad (18)$$

in the case of a parabolic flow represented by equation (10). Upon substituting equation (14) into the computed value of the coefficient A_{-2}^1, we obtain the result

$$\vec{F}_1 = \frac{19}{20}\,\pi\mu\varepsilon U_0\,a\,\delta\,\gamma\sum_{n=0}^{2} F_{1n}\,\vec{i}, \qquad (19)$$

where

$$F_{10} = \left(\frac{\alpha}{2} + \frac{\alpha+1}{3}\frac{k}{\gamma}\right)\left(\frac{1}{\alpha+1}\right)^3 \left(\frac{1}{\alpha+\frac{2}{3}}\right)\left(\alpha + \frac{16}{19}\right)\left(\alpha^2 - \frac{11}{3}\alpha + 6\right),$$

$$F_{11} = -\left(\frac{1}{\alpha+1}\right)^3 \left(\alpha + \frac{16}{19}\right)\left(\frac{1}{\alpha+4}\right)\left(\frac{43}{64}\alpha^3 - \frac{3119}{504}\alpha^2 - \frac{4233}{224}\alpha + \frac{321}{56}\right), \qquad (20)$$

$$F_{12} = -\left(\frac{1}{\alpha+1}\right)^3 \left(\alpha + \frac{10}{11}\right)\left(\frac{108867}{148960}\alpha^2 - \frac{153439}{223440}\alpha + \frac{28237}{15960}\right).$$

The direction of the force $\vec{F}_1$ is orthogonal to the direction of the undisturbed flow at infinity. We note that the factor γ appearing in equation (19) causes the contributions made by the coefficients F_{11} and F_{12} to vanish when the quadratic shear portion of the incident parabolic flow is set equal to zero, whereas the body force contribution appearing in the coefficient F_{10} remains finite and proportional to the body force parameter k. In this case $\vec{F}_1$ represents the transverse force exerted on a drop when the incident flow consists of a uniform stream plus a linear shear flow. It is observed from the detailed calculation which has been performed that the contributions made by the coefficients F_{11} and F_{12} to $\vec{F}_1$ arise, respectively, from interactions between (a) the $O(\epsilon^0)$ quadratic shear flow and the linear shear deformation f_{11}, and (b) the $O(\epsilon^0)$ linear shear flow and the quadratic shear deformation f_{12}. It is seen that the interaction between $O(\epsilon^0)$ uniform stream and f_{12} does not contribute to the $O(\epsilon)$ force.

The sign of the force $\vec{F}_1$ can be determined by inspection of equation (19) after inserting the value of b, given by equation (22). The coefficients (20) and their sum are plotted against α in figure 1 for a neutrally buoyant drop, when k is zero. For small α, the sum $F_{10} + F_{11} + F_{12}$ is negative. In this case the migration of the drop is always towards the point of zero velocity gradient, viz. $X = U_w b_0/4U_0$. It is observed from equations (20) that, in general, the direction of migration depends only on the parameters k/γ and α and not upon radial position (other than which side of the point where $dU/dX = 0$ the drop is).

We shall now calculate the trajectory of the drop. Its lateral velocity $d\vec{b}/dt$ is determined by equating the Hadamard & Rybczynski drag force for a liquid sphere to $\vec{F}_1$. Thus,

$$\frac{db}{dt} = \epsilon \left(\frac{a}{b_0}\right)^2 \left(\frac{aU_w}{4U_0 b_0} - \frac{ab}{b_0^2}\right) F,$$

$$F = -\frac{19}{60}\left(\frac{\alpha+1}{\alpha+\frac{2}{3}}\right)^2 \sum_{n=0}^{2} F_{1n}. \qquad (21)$$

Integration of equation (21) yields

$$b = \frac{U_w b_0}{4U_0} + (b_1 - \frac{U_w b_0}{4U_0})\, e^{-t/\tau} , \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}$$

$$\tau = b_0^4/\epsilon U_0 a^3 F, \tag{22}$$

where b_1 is the initial position of the drop. The trajectory of the drop is obtained by dividing db/dt, given by equation (21) into $dz/dt = c$, given by equation (14). Upon integration we obtain

$$\frac{z}{b_0} = \frac{-1}{\epsilon(a/b_0)^3 F}\left[(1 + \frac{2}{3}\frac{\alpha+1}{3\alpha+2}\, k - \frac{\alpha}{3\alpha+2}\frac{a^2}{b_0^2} + \frac{U_w}{2U_0} + \frac{U_w^2}{16U_0^2})\ln\left(\frac{\frac{b}{b_0} - \frac{U_w}{4U_0}}{\frac{b_1}{b_0} - \frac{U_w}{4U_0}} \right) \right.$$

$$\left. - \frac{1}{2}[(\frac{b}{b_0})^2 - (\frac{b_1}{b_0})^2] + \frac{U_w}{4U_0}(\frac{b}{b_0} - \frac{b_1}{b_0}) \right]. \tag{23}$$

The ratio U_w/U_0 may be either positive or negative depending on the sign of the applied pressure gradient. In undisturbed flow some layers of the fluid will move in a direction opposite to U_w when U_0 is sufficiently negative. The critical pressure gradient occurs when $dU/dX = 0$ at $X = -b_0$, i.e., when $U_w/U_0 = -4$. Theoretical curves based on equation (23) are shown in figure 2(a) for $U_w = 0$, $U_0 > 0$ and in figure 2(b) for $U_w/U_0 = -4$, for the parameter values $\alpha = 0$, $\epsilon = 0.01$, $a/b_0 = 0.1$ and $b_1/b_0 = 0.5$, and for various values of the body force parameter k. These trajectories show that the drop may move inwards or outwards. The direction of migration as $t \to \infty$ for the various values of k can be determined by examination of equations (21) and (14).

Figure 2(a) (wherein $U_w = 0$) shows that the drop moves in the positive-z direction and approaches the axis midway between the walls ($b = 0$) asymptotically for $k > -0.011$ (to three decimal places). When $k = -0.011$, the drop travels along the line $b = b_1$ in the positive-z direction. For $-2.25 < k < -0.011$, the drop migrates radially outwards, initially in the positive-z direction. When $k < -2.25$, the trajectories are always in the negative-z direction and radially outwards. When the above trajectories are compared with those of a drop in Poiseuille flow in a tube, depicted in Figure 7 of Wohl and Rubinow (ref. 1), it is observed that there is negligible difference except in the case $|k| \ll 1$. This observation is to be expected because when $\alpha = 0$ and $a/b_0 = 0.1$, it is found that $F = 1.337 + 120k$ (to three decimal places) from equations (21) and

(20). The value of F for circular Poiseuille flow is 10.075+ 120k (to three decimal places). The body force parameter k dominates both expressions unless $|k| \ll 1$. Figure 2(b) (wherein $U_w/U_0 = -4$) displays the drop moving in the positive-z direction and approaching the stationary wall ($b = -b_0$) asymptotically for k < -0.011. For -0.011 < k < 6.75, the trajectories are always in the positive-z direction and radially outwards. When k > 6.75 the drop migrates radially outwards, initially in the negative-z direction.

SCOPE OF THE ANALYSIS

The incident parabolic flow is considered to be unbounded in the sense that the secondary effect of the wall on the perturbed flow produced by the drop is neglected, i.e., interaction between the drop and the boundary walls is neglected. Also, the results herein are not applicable unless the effects of $O(\varepsilon)$ and (nonlinear) inertial effects can be neglected.

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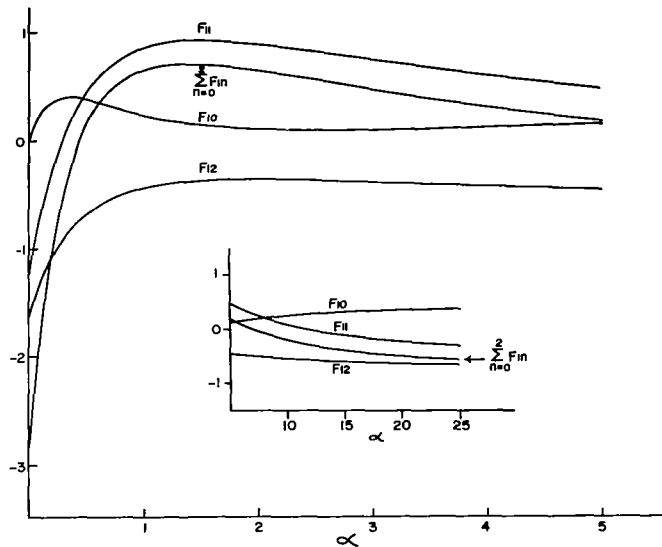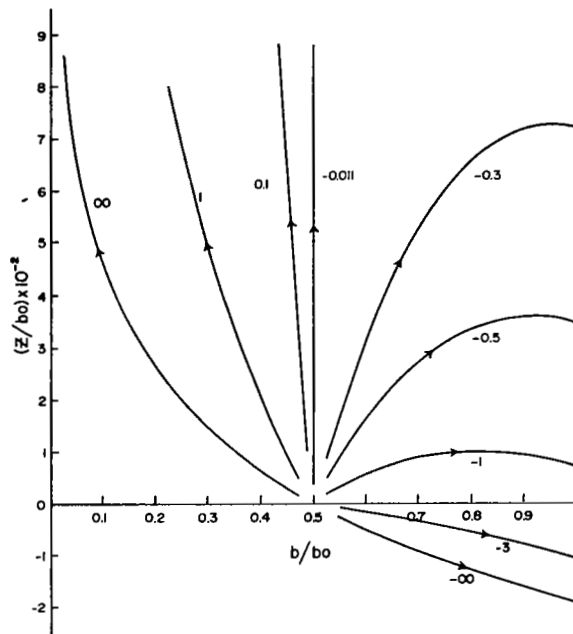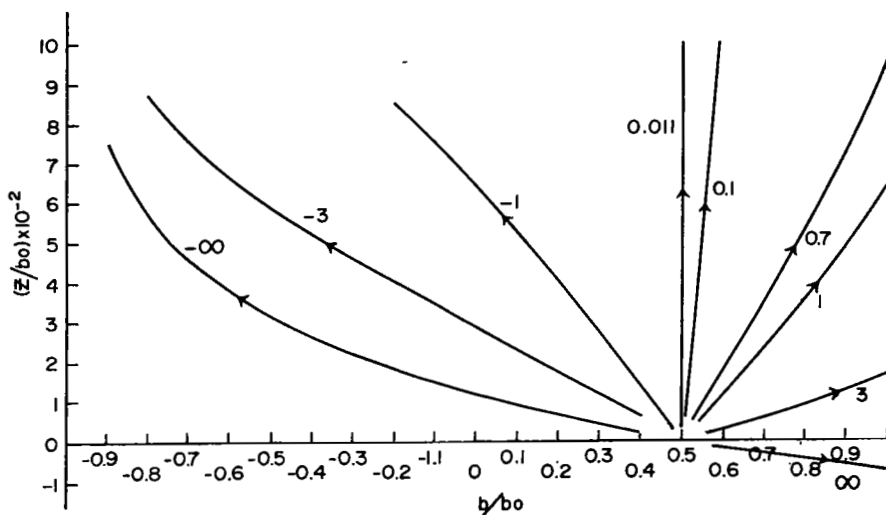14. Brenner, H.: 1964 Chem. Engng. Sci. 19, 519.

Figure 1.— Contributions to the force $\vec{F}_1$ defined by equations (20) with the body force parameter k = 0, as a function of the viscosity ratio α.

(a) $U_w = 0$; $U_0 > 0$.

(b) $U_w/U_0 = -4$.

Figure 2.- Trajectories of a drop in plane Poiseuille flow and subject to a body force, according to equation (23). It is assumed that $\alpha = 0$, $\varepsilon = 0.01$, $a/b_0 = 0.1$, and $b_1/b_0 = 0.5$. The labels denote various values of the body force parameter $k = Ka^2/\mu'U_0$. The line $b = 0$ is located midway between the two walls bounding the flow. The arrowheads indicate the direction of migration.