USE OF ASYMPTOTIC METHODS IN VIBRATION ANALYSIS*

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ABSTRACT

Two subjects are discussed, which are believed relevant to the structural analysis of vertical-axis wind turbines. The first involves the derivation of dynamic differential equations, suitable for studying the vibrations of rotating, curved, slender structures. The Hamiltonian procedure is advocated for this purpose. Various reductions of the full system are displayed, which govern the vibrating Troposkien when various order-of-magnitude restrictions are placed on important parameters.

The final section discusses the possible advantages of the WKB asymptotic method for solving these classes of problems. A special case of this method is used illustratively to calculate eigenvalues and eigenfunctions for a "flat" turbine blade with small flexural stiffness.

PRINCIPAL SYMBOLS

k	Argument of elliptic integrals & functions
ℓ	Length of blade
$\bar{m}$	(Constant) mass per unit length of blade
s	Coordinate measuring distance along blade
t	Time
t_1, t_2	Limits in Hamiltonian integral
$x = x_o + \xi$, $y = y_o + \eta$, $z = \zeta$	Instantaneous coordinates of point on vibrating blade
$x_o(s)$, $y_o(s)$	Coordinates of blade in rotational equilibrium
x_m	Coordinate of blade end
y_m	Maximum value of y_o
EI_s	Flexural rigidity for bending in x-y-plane

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EI_z	Flexural rigidity for bending normal to x-y-plane
H_o	(Constant) x-component of equilibrium tensile force
K, K_o	Curvatures defined by Eqs. (4)
L	Kinetic minus potential energy
$T = T_o + \tau$	Instantaneous tensile force in blade
$T_o(s)$	Equilibrium value of T
$U(s)$	Vector of unknowns in WKB method
$\delta(\cdots)$	Variation of a quantity
λ	Eigenvalue in WKB method
μ	Parameter defined by Eq. (9)
ω	Circular frequency of simple harmonic motion
Ω	Angular velocity about support axis of blade

Superscripts, Etc.

$(\dot{\ })$	Partial derivative with respect to time.
$(\ )'$	Total derivative with respect to s

INTRODUCTION

The title of this paper is somewhat misleading, since it actually addresses two subjects relevant to the structural analysis of vertical-axis wind turbines. The first involves the derivation of differential equations needed for the vibration and aeroelastic analysis of a rotating, curved, slender structure like a beam in the Troposkien shape. It is emphasized that a systematic procedure, such as that furnished by Hamilton's principle, can be useful both for ensuring that the results are correct and for developing a hierarchy of simplifications when various approximations are made.

The second subject concerns the possible advantages of what Steele (Ref. 1) chooses to call the WKB (Wentzel-Kramers-Brillouin) method for solving certain of these problems. An elementary illustrative example is offered.

Admittedly, detailed structural design studies of Darrieus-type wind turbines are best carried out by means of finite-element approximations. This approach is unavoidable when the designer is faced with such complications as interactions with the support structure, reinforcing struts, or variable properties along the blades. Calculations of operating stresses, vibration modes, etc. by finite elements appear in several published papers. Examples are Weingarten and Nickell (Ref. 2); Weingarten and Lobitz (Ref. 3); and Biffle (Ref. 4). Many other related citations are given in the survey paper by Blackwell et al. (Ref. 5).

Parametric studies and the goal of fundamental understanding are not so well served, on the other hand, by the purely numerical investigations of point designs. It is believed that Troposkien vibrations offer some interesting problems in applied mechanics, and the discovery of analytical or semi-analytical solutions to appropriate differential equations can certainly not be ruled out. The aim of this paper is to encourage that search.

Figure 1 depicts a Darrieus machine, with a typical blade both in its unperturbed equilibrium shape of $y_o(x_o)$ and slightly deformed. The instantaneous time-dependent position of mass element $\bar{m}\, ds$ is specified by coordinates x,y,z, measured in a frame rotating with the constant angular velocity Ω. Various simplifications are here adopted from the outset. Thus, attention is focused on a single, uniform blade by assuming its ends to be fully restrained at the points $(\pm x_m, o, o)$. Gravitational force is neglected (cf. comparative studies cited by Weingarten and Lobitz Ref. 3).

The current experimental Darrieus configurations deviate somewhat from the true Troposkien shape (Blackwell and Reis, Ref. 6), and it is known that these deviations can produce substantial equilibrium bending stresses (Ref. 3). Here $y_o(x_o)$ is chosen, however, to be the perfect "skiprope." This is a restriction that is easy to remove, as are the prescribed limitations to essentially infinite torsional and elongational rigidity of the blade. In fact, during free vibration, torsion is often nearly uncoupled from the other degrees of freedom when the blade elastic axis and line of C.G.'s coincide — a design feature that would seem desirable in practice.

VIBRATION EQUATIONS FOR SEVERAL IDEALIZED TROPOSKIENS*

It is well known (e.g., Ref. 6) that the zero bending moment shape is governed by

$$\frac{d^2 y_o}{dx_o^2} + \frac{\Omega^2 m}{H_o}\left[1 + \left(\frac{dy_o}{dx_o}\right)^2\right]^{\frac{1}{2}} y_o = 0 \tag{1}$$

With support conditions as in Fig. 1, the solution of (1) is conveniently expressed as

$$\frac{y_o}{y_m} = \mathrm{sn}\left[K(k)\left(1 + \frac{x_o}{x_m}\right);\ k\right] \tag{2}$$

Here sn is Jacobi's elliptic function, whereas K is the complete elliptic integral of the second kind with argument

$$k = \left[1 + \frac{4H_o}{\bar{m}\Omega^2 y_m^2}\right]^{-\frac{1}{2}} \tag{3}$$

Equation (3) introduces a dimensionless group $\bar{m}\,\Omega^2 y_m^2 / H_o$ which, with various slight modifications, is perhaps the key parameter for vibration and stability studies.

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*Principal symbols are defined in Fig. 1 and the list at the beginning.

As suggested by Mallett in his work on the catenary (Ref. 7), distance s along the blade seems to be an efficient choice for the spatial independent variable. In terms of s, the instantaneous curvatures of the undeformed and deformed shapes can be written, respectively,

$$K_o = \left[\left(\frac{d^2x_o}{ds^2}\right)^2 + \left(\frac{d^2y_o}{ds^2}\right)^2\right]^{\frac{1}{2}} \tag{4a}$$

$$K = \left[\left(\frac{\partial^2 x}{\partial s^2}\right)^2 + \left(\frac{\partial^2 y}{\partial s^2}\right)^2\right]^{\frac{1}{2}} \tag{4b}$$

For the inextensible blade with flexural stiffnesses EI_s and EI_z, an appropriate Hamiltonian is

$$\begin{aligned} H = \int_{t_1}^{t_2} \Big[L + (\text{Length constraint}) \Big] \, dt = \tfrac{1}{2} \int_{t_1}^{t_2} \int_o^{\ell} \Big\{ \bar{m} \Big[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ + 2\Omega y\dot{z} - 2\Omega z\dot{y} + \Omega^2 y^2 + \Omega^2 z^2 \Big] - EI_s \Big[K - K_o \Big]^2 - EI_z \left(\frac{\partial^2 z}{\partial s^2}\right)^2 \\ + T(s,t) \left[1 - \left(\frac{\partial x}{\partial s}\right)^2 - \left(\frac{\partial y}{\partial s}\right)^2 - \left(\frac{\partial z}{\partial s}\right)^2 \right] \Big\} \, ds\,dt \end{aligned} \tag{5}$$

It is easily seen that the Lagrange multiplier on the constant-length constraint is simply the tension $T(s,t)$.

The principle $\delta H = 0$ is enforced by taking variations of (5) on x,y,z and T. Suitable integrations by parts with respect to s or t then lead to the general nonlinear dynamical system:

$$\frac{\partial^2}{\partial s^2}\left[EI_s\left(1 - \frac{K_o}{K}\right)\frac{\partial^2 x}{\partial s^2}\right] - \frac{\partial}{\partial s}\left[T\frac{\partial x}{\partial s}\right] + \bar{m}\,\ddot{x} = 0 \tag{6a}$$

$$\frac{\partial^2}{\partial s^2}\left[EI_s\left(1 - \frac{K_o}{K}\right)\frac{\partial 2 y}{\partial s^2}\right] - \frac{\partial}{\partial s}\left[T\frac{\partial y}{\partial s}\right] + \bar{m}\left[\ddot{y} - \Omega^2 y - 2\Omega\dot{z}\right] = 0 \tag{6b}$$

$$\frac{\partial^2}{\partial s^2}\left[EI_z\frac{\partial^2 z}{\partial s^2}\right] - \frac{\partial}{\partial s}\left[T\frac{\partial z}{\partial s}\right] + \bar{m}\left[\ddot{z} + 2\Omega\dot{y} - \Omega^2 z\right] = 0 \tag{6c}$$

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 - 1 = 0 \tag{6d}$$

Small perturbations ξ, η, ζ and τ are introduced on the dependent variables, products of small quantities are neglected, and the equilibrium relations corresponding to (1) are subtracted out. There results the most general system, governing the free dynamics of a uniform, inextensible Troposkien:

$$EI_s \frac{\partial^2}{\partial s^2}\left[\frac{x_o''}{K_o^2}\left(x_o''\frac{\partial^2\xi}{\partial s^2} + y_o''\frac{\partial^2\eta}{\partial s^2}\right)\right] - \frac{\partial}{\partial s}\left[T_o(s)\frac{\partial\xi}{\partial s} + x_o'\tau\right] + \bar{m}\ddot{\xi} = 0 \tag{7a}$$

$$EI_s \frac{\partial^2}{\partial s^2}\left[\frac{y_o''}{K_o^2}\left(x_o''\frac{\partial^2\xi}{\partial s^2} + y_o''\frac{\partial^2\eta}{\partial s^2}\right)\right] - \frac{\partial}{\partial s}\left[T_o(s)\frac{\partial\eta}{\partial s} + y_o'\tau\right] + \bar{m}\left[\ddot{\eta} - 2\Omega\dot{\zeta} - \Omega^2\eta\right] = 0 \tag{7b}$$

$$EI_z \frac{\partial^4\zeta}{\partial s^4} - \frac{\partial}{\partial s}\left[T_o(s)\frac{\partial\zeta}{\partial s}\right] + \bar{m}\left[\ddot{\zeta} + 2\Omega\dot{\eta} - \Omega^2\zeta\right] = 0 \tag{7c}$$

$$x_o'\frac{\partial\xi}{\partial s} + y_o'\frac{\partial\eta}{\partial s} = 0 \tag{7d}$$

In principle, the assumption of simple harmonic time dependence at frequency ω converts Eqs. (7) into an ordinary system, whose eigenvalues and eigenfunctions describe the free vibration. It is also not difficult to add forcing terms, such as aerodynamic loads.

Equations (7), as written, offer little attraction to the analyst. Accordingly, several simpler versions will be discussed, along with the approximations that lead to them. For reasons of space, little attention is paid to boundary conditions, but their formulation is not a difficult matter.

1. The Rotating Rope or Chain

When $EI_s = EI_z = 0$, a catenary-like structure is left. Its linearized differential equations are

$$\bar{m}\ddot{\xi} - \frac{\partial}{\partial s}\left[T_o\frac{\partial\xi}{\partial s} + x_o'\tau\right] = 0 \tag{8a}$$

$$\bar{m}\left[\ddot{\eta} - 2\Omega\dot{\zeta} - \Omega^2\eta\right] - \frac{\partial}{\partial s}\left[T_o\frac{\partial\eta}{\partial s} + y_o'\tau\right] = 0 \tag{8b}$$

$$\bar{m}\left[\ddot{\zeta} + 2\Omega\dot{\eta} - \Omega^2\zeta\right] - \frac{\partial}{\partial s}\left[T_o\frac{\partial\zeta}{\partial s}\right] = 0\ , \tag{8c}$$

with (7d) unchanged. Equations (8) are not relevant to wind turbines. The reason for displaying them is to make the point that, even in this case, the four variables remain coupled. In addition to the familiar centrifugal term

$-\Omega^2 \eta$, Coriolis effects couple the in- and out-of-plane degrees of freedom. These latter gain in importance relative to the centrifugal as the vibration frequency increases.

When (8) are made dimensionless, a two-parameter system is found, involving

$$\mu \equiv \frac{\bar{m}\,\omega^2\, x_m^2}{H_o} \tag{9}$$

and the frequency ratio Ω/ω. It appears that $\mu\,\Omega^2/\omega^2$ is of O(1) for all interesting designs. Ω/ω is less than unity, but it remains O(1) for the lower vibration modes.

2. Small In-Plane and Large Out-of-Plane Bending Stiffnesses

It is obvious that Darrieus turbine blades with typical airfoil shapes will have

$$\frac{EI_z}{EI_s} \gg 1 \tag{10}$$

Indeed, Ref. 3 quotes a value over 40 for this ratio on the SANDIA 17-meter design. An interesting reduction of system (7) is found when both inequality (10) is satisfied and when

$$\frac{EI_s}{H_o x_m^2} \ll 1 \tag{11}$$

(this ratio is estimated to be 0.015 for the 17-meter). It can then be reasoned that $\zeta \ll \eta$, so that the Coriolis term in (7b) is negligible, and that the flexural terms may be dropped from (7a,b). The resulting equations are

$$\bar{m}\,\ddot{\xi} - \frac{\partial}{\partial s}\left[T_o \frac{\partial \xi}{\partial s} + x_o'\,\tau\right] = 0 \tag{12a}$$

$$\bar{m}\left[\ddot{\eta} - \Omega^2 \eta\right] - \frac{\partial}{\partial s}\left[T_o \frac{\partial \eta}{\partial s} + y_o'\,\tau\right] = 0 \tag{12b}$$

$$\frac{\partial \xi}{\partial s} + \frac{dy_o}{dx_o}\frac{\partial \eta}{\partial s} = 0 \tag{12c}$$

Three coupled dependent variables are involved in (12). Nevertheless, they constitute a considerable simplification and might form the basis for useful parametric studies. At considerable labor, one can eliminate $\xi(s,t)$ among (12 a,b,c). The result does not seem of great value, however, and efforts to achieve a single equation in one unknown where unsuccessful.

3. The "Flat" Blade with Small Bending Stiffness

If in addition to the other assumptions behind system (12) the approximation is made that

$$\frac{dy_o}{dx_o} << 1 \quad , \tag{13}$$

then dropping the τ term in (12b) is justified. A further replacement of $T_o(s)$ by H_o/x_o', where H_o is its large, constant x-component leads to

$$\frac{\partial^2 \eta}{\partial x_o^2} - \frac{\bar{m}}{H_o} \frac{ds}{dx_o} \left[\ddot{\eta} - \Omega^2 \eta \right] = 0 \tag{14}$$

This is essentially the in-plane-deformation equation employed by Ham (Ref. 8), in his investigations on free vibration and flutter of the rotating blade. In view of the apparent limitations on its validity, further study would seem justified into the parameter ranges within which it can be used in practice.

DISCUSSION OF ASYMPTOTIC METHODS AND AN APPLICATION

The excellent review by Steele (Ref. 1) removes the need for any recapitulation here of how these methods are applied to ordinary differential equations arising in solid mechanics. The particular form useful in vibration problems with spatially-varying coefficients is known by such names as "phase integral," WKB and WKBJ. In the form most broadly applicable here, it starts with a homogeneous problem formulation in terms of a "state vector" U(s):

$$\frac{dU(s)}{ds} = A(s;\lambda)\, U(s) \tag{15}$$

Here $A(s,\lambda)$ is a prescribed coefficient matrix, containing an eigenvalue λ which, in some sense, is a large quantity. A is expanded according to

$$A\,(s;\lambda) = \lambda A_o(s) + A_1(s) + \frac{1}{\lambda} A_2(s) + \cdots \tag{16}$$

The series in (16) may be convergent or asymptotic. In either event, clever changes of variable are employed to construct a corresponding succession of approximate solutions.

After the assumption of simple harmonic motion, it is worth noting that all the systems presented in the preceding section can be recast in state vector form. The same is true of equations for flutter stability or forced sinusoidal response. The principal recommendation of this paper is that the WKB theory should be put to work in order to determine what contribution it may make to the subject at hand.

All that can be offered at present is a simple example to illustrate the possibilities. For harmonic oscillations at frequency ω, with mode shape $\bar{\eta}(x_o)$, it is an easy matter to recast the "flat - Troposkien" eq. (14) as follows:

$$\frac{d^2\bar{\eta}}{dx_o^2} + \frac{\Lambda^2\bar{m}}{H_o}\left[1 + \frac{m\Omega^2 y_m^2}{H_o}\left(1 - \frac{y_o^2(x_o)}{y_m^2}\right)\right]\bar{\eta} = 0 , \tag{17}$$

where

$$\Lambda \equiv \sqrt{\Omega^2 + \omega^2} \tag{18}$$

In view of (2), the quantity in brackets of (17) may be rewritten

$$\left[\cdots\right] = 1 + \frac{\bar{m}\,\Omega^2 y_m^2}{H_o}\,\mathrm{cn}^2\left(K(k)\left(1 + \frac{x_o}{x_m}\right);\ k\right) \tag{19}$$

Now (17), with the associated boundary conditions $\bar{\eta}(\pm x_m) = 0$, is a special case of the problem solved by Steele (Ref. 1, Sect. 2), using the Green-Liouville transformation. In more general terms, Steele transforms the differential equation

$$\frac{d}{dx}\left[p(x)\ \frac{dy}{dx}\right] + \left[\lambda^2 q(x) - r(x)\right]\ y = 0 \tag{20}$$

by introducing

$$\zeta = \zeta(x) , \quad y = \psi(x)\ \eta(\zeta) \tag{21a,b}$$

With the following optimal choice of these functions:

$$(\zeta')^2 = p/q \tag{21c}$$

and $\psi = \left[pq\right]^{-1/4}$, (21d)

(20) is reduced to

$$\frac{d^2\eta}{ds^2} + \left[\lambda^2 - \frac{\nu}{q}\right]\eta = 0 , \tag{22}$$

where

$$\nu(x) = r - \frac{(p\psi')'}{\psi} \tag{23}$$

When λ is large, the limiting form of the solution can be re-transformed to

$$y(x) = \psi(x) \exp\left[\pm i\lambda\,\zeta(x)\right], \tag{24}$$

where

$$\zeta(x) = \int_{x_b}^{x} \left[\frac{q}{p}\right]^{\frac{1}{2}} dx \tag{25}$$

has the name "phase integral." x_b is a point associated with one of the boundary conditions. Eigenvalues are readily estimated by combining (24), (25) and the second B.C.

In the present example, one associates Steele's x variable with x_o/x_m and y with $\bar{\eta}$. Here also

$$p(x) = 1 \tag{26a}$$

$$q(x) = 1 + \frac{\bar{m}\Omega^2 y_m^{\ 2}}{H_o} \quad cn^2\left(K(k)\left(1 + \frac{x_o}{x_m}\right); k\right) \tag{26b}$$

$$r(x) = 0 \tag{26c}$$

$$\lambda^2 = \frac{\bar{m}\Lambda^2 x_m^{\ 2}}{H_o} \tag{26d}$$

After specialization of the general solution, it is found that a first approximation to the free-vibration eigenvalues is

$$\tilde{\lambda}_n \cong \frac{n\pi}{\zeta_L}, \quad (n = 1,2, \cdots), \tag{27a}$$

where [cf. (26b)]

$$\zeta_L = \int_{-1}^{1} \sqrt{1 + \frac{\bar{m}\Omega^2 y_m^{\ 2}}{H_o} \quad cn^2\, d\left(\frac{x_o}{x_m}\right)} \tag{27b}$$

The mode shape corresponding to $\tilde{\lambda}_n$ is

$$\bar{\eta}_n \cong \psi \sin\left[\tilde{\lambda}_n \ \zeta\left(\frac{x_o}{x_m}\right)\right], \tag{28a}$$

where

$$\zeta\left(\frac{x_o}{x_m}\right) = \int_{-1}^{x_o/x_m} \sqrt{1 + \frac{\bar{m}\Omega^2 y_m^2}{H_o} \, cn^2} \; d\left(\frac{x_o}{x_m}\right) \tag{28b}$$

It is even possible, by one additional integration, to obtain a second approximation to the eigenvalues $\tilde{\lambda}_n$ in accordance with a scheme developed in Sect. 3.3, Ref. 1. Details are not given here, since corrections are generally less than 5%.

In view of (18), (26d) and other knowledge of the properties of typical Darrieus turbines, one can conclude that λ is quite a large parameter, except possibly for the fundamental solution of (17). Accordingly, the WKB method was applied for values of the parameter k [eq. (3)] ranging from 0.2 to 0.8. ($k = 0.57$ characterizes a turbine with diameter equal to its height.) Table I furnishes some numerical results* for the first five eigenvalues and the corresponding dimensionless natural frequencies.

Figure 2 plots some representative vibration mode shapes for $k = 0.2$. This parameter was chosen because the condition (13), implied in (17), is most likely to be satisfied accurately in this rather "flat" case. The mode corresponding to $n = 1$ is not shown, because it involves a low-frequency, symmetrical stretching motion. Although not ruled out by (17), it obviously violates the condition (6d) of inextensibility and is regarded as a physical impossibility. For the same reason, $n = 1$ eigenvalues are enclosed by parentheses in Table I.

Time has not permitted extensive comparisons between the foregoing results and those of previous finite-element analyses. One possibility is furnished, however, by the data in Fig. 5 of Ref. 3. These relate to a blade like those of the SANDIA 17-meter turbine, but with the stiffening struts removed; the rotation rate is 75 RPM. From the modal symmetries, it is clear that their numbers are equivalent to $(\omega_2/\Omega) = 1.55$ and $(\omega_3/\Omega) = 2.95$ at $k = 0.57$. From the data in Table I, the corresponding numbers by the WKB approximation are 1.75 and 2.97, respectively.

The present computations were carried out on the digital computer only for convenience. Nothing more is involved than subroutines giving elliptic functions and the numerical evaluation of well-behaved integrals. It is believed unlikely that any much more efficient scheme exists for dealing with equations like (17). Further Troposkien dynamics investigations by means of the WKB method will hopefully be stimulated by this very elementary first attempt.

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DISCUSSION

Question:

Since torsion is structurally coupled with out-of-plane bending in a curved beam, isn't it true that whenever such bending occurs, torsion must occur?

Comment 1: Since coriolis forces dominate the coupling between in-plane and out-of-plane bending, blade chordwise elastic axis and center of gravity locations need not coincide, and blade design effort and manufacturing cost are greatly reduced.

Comment 2: In certain experimental cases of Darrieus blade flutter, in-plane and out-of-plane bending displacement perturbations are the same order of magnitude.

Answer: Agree with Comment 2. Feel we must gain more experience with various sizes of rotor and other structural details before Comment 1 can be accepted as a design principle for all Darrieus machines. Helicopter rotors also have important Coriolis couplings, yet practice with them is to massbalance so that the C.G. and quarter-chord axes coincide.

With regard to the question, author believes that the theory of his paper requires, as a strict condition, that the torsional rigidity GJ just be extremely large. He has found other possible circumstances, however, in which coupling with torsion may be negligible. This occurs when the C.G. is at the quarter chord and when a dimensionless parameter (involving GJ divided by the mass moment of inertia in torsion) is large compare to unity.

TABLE I. - SECOND-APPROXIMATE EIGENVALUES λ_n AND DIMENSIONLESS NATURAL FREQUENCIES ω_n/Ω FOR TROPOSKIEN BLADES OF SEVEN DIFFERENT ASPECT RATIOS

	k	0.2	0.3	0.4	0.5	0.6	0.7	0.8
n	y_m/x_m	0.2625	0.4100	0.5807	0.7908	1.071	1.486	2.226
1	$\tilde{\lambda}_1$	(1.511)	(1.525)	(1.489)	(1.440)	(1.376)	(1.057)	(1.181)
	ω_1/Ω	(0.7395)	(0.719)	(0.689)	(0.648)	(0.593)	(0.517)	(0.406)
2	$\tilde{\lambda}_2$	3.061	2.957	2.805	2.599	2.325	1.964	1.479
	ω_2/Ω	2.242	2.168	2.058	1.904	1.691	1.388	0.909
3	$\tilde{\lambda}_3$	4.593	4.444	4.235	3.966	3.638	3.251	2.801
	ω_3/Ω	3.545	3.446	3.306	3.126	2.907	2.649	2.356
4	$\tilde{\lambda}_4$	6.124	5.924	5.642	5.271	4.797	4.187	3.360
	ω_4/Ω	4.808	4.677	4.492	4.247	3.928	3.507	2.903
5	$\tilde{\lambda}_5$	7.655	7.405	7.053	6.593	6.016	5.304	4.433
	ω_5/Ω	6.057	5.895	5.665	5.364	4.983	4.510	3.926

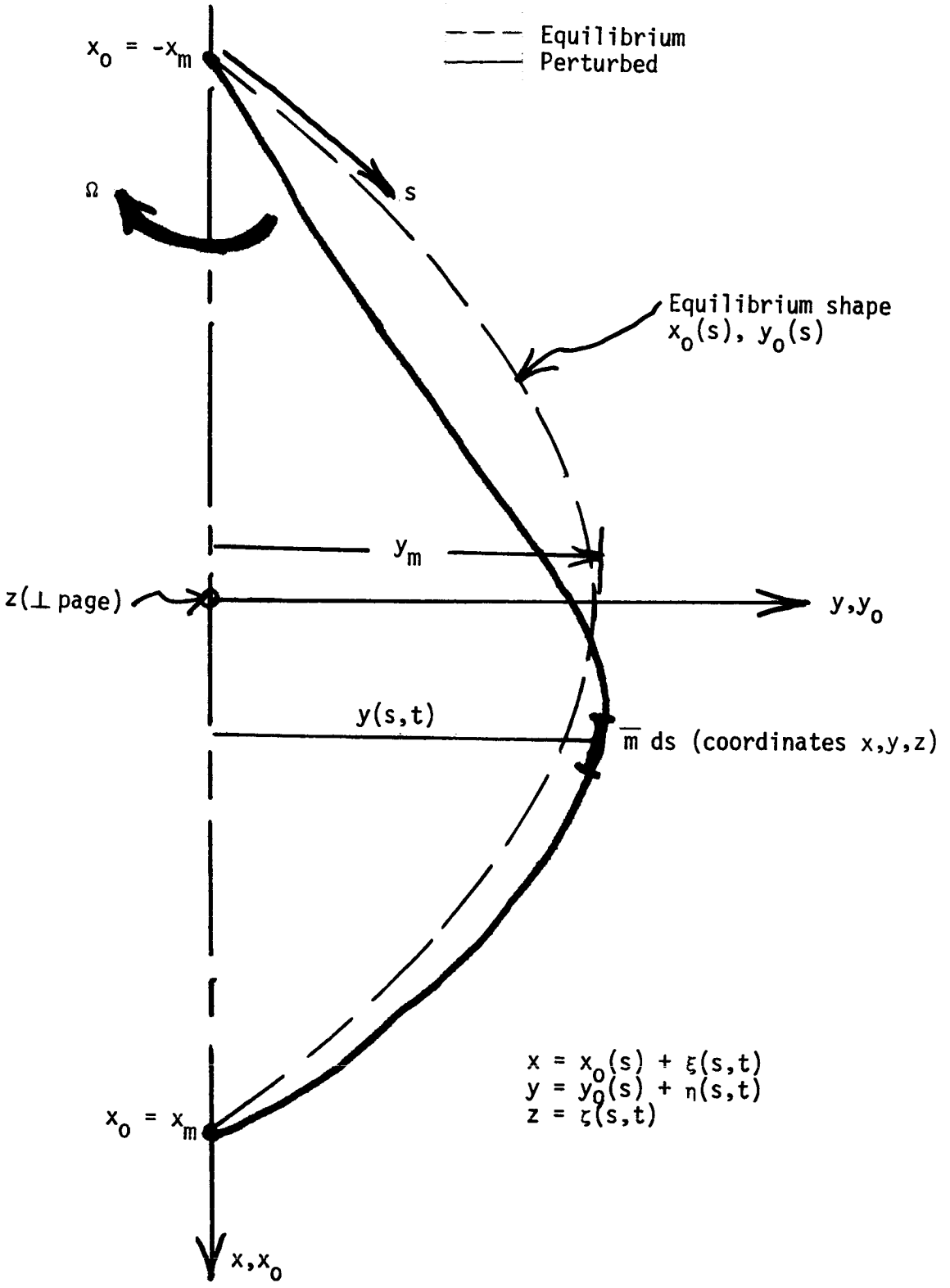

Figure 1. - Equilibrium and perturbed positions of a Troposkien shape which is vibrating while rotating about x_o-axis with constant angular velocity Ω.

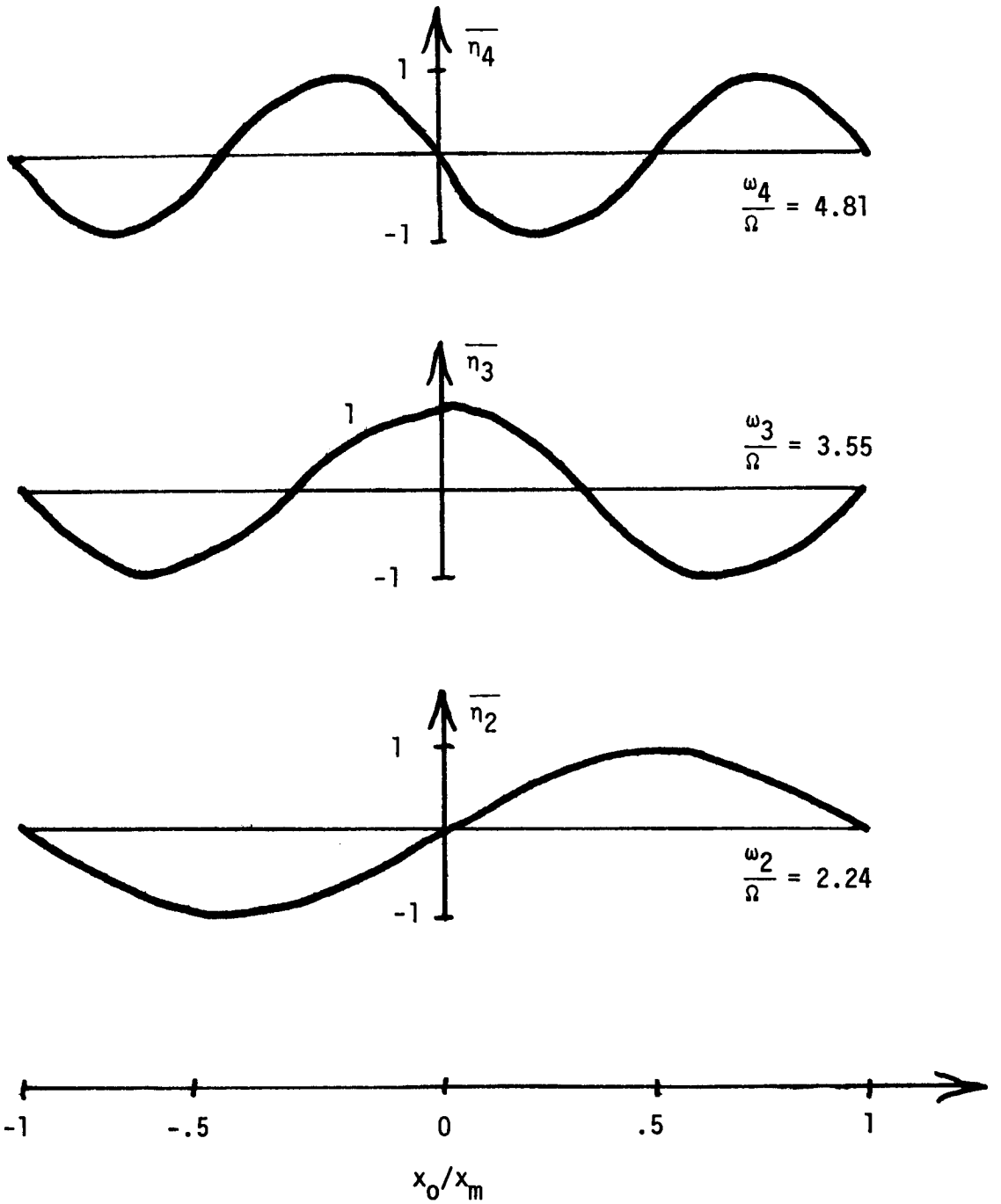

Figure 2. - First three physically possible natural modes of vibration for the Troposkien with $(\bar{m}x_m^2\Omega^2/H_o)' = 1.555$, corresponding to $k = 0.2$.