

## RECENT DEVELOPMENTS IN FINITE ELEMENT ANALYSIS FOR TRANSONIC AIRFOILS\*

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## INTRODUCTION

The prediction of aerodynamic forces in the transonic regime generally requires a flow field calculation to solve the governing non-linear mixed elliptic-hyperbolic partial differential equations. Finite difference techniques have been developed to the point that design and analysis applications are routine, and continual improvements are being made by various research groups. The principal limitation in extending finite difference methods to complex three-dimensional geometries is the construction of a suitable mesh system. Finite element techniques are attractive since their application to other problems have permitted irregular mesh elements to be employed. The purpose of this paper is to review the recent developments in the application of finite element methods to transonic flow problems and to report some recent results of our own study. In most cases, the reader is referred to the original paper for the details.

Finite element methods have been quite successful when applied to elliptic problems, particularly in elasticity and structures. A straightforward application of these techniques to the transonic problem involving mixed elliptic-hyperbolic equations with embedded shock waves has not proven successful. In general, either the finite element method must be modified to treat the transonic flow equations or the transonic flow equations must be modified to fit the finite element method. For the latter approach, additional terms (with hopefully small coefficients) are added to the equations representing artificial viscosity, artificial density or artificial time. The general mesh shapes used for elliptic problems result in matrices which are not well ordered. Thus either a direct matrix inversion or an explicit iterative method is required. Both of these approaches are computationally inefficient for realistic problems compared to the implicit methods which have been developed. Some work reported later in this paper may relax this conclusion. Finally, the higher order shape functions used in elliptic problems do not appear as attractive for mixed problems with shocks. In this paper, we review the subject by generally grouping together methods following the same general formulation.

## SYMBOLS

All symbols are dimensionals.

A matrix operator in equation  $Aw = f$   
 $C_p$  pressure coefficient  
 c coefficient in Tricomi equation  
 f right hand side in equation  $Aw = f$

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F	function in equation (5)
G	matrix in equations (6), (7)
I	functional
L	Laplace operator
M <sub>∞</sub>	freestream Mach number
P	pressure
Q	matrix operator
R	residual operator
s	streamwise coordinate
t	time-like variable
u,v	velocity component in x,y directions
w	dummy dependent variable
x,y	independent variables
α,β	coefficients in equation (1)
γ	ratio of specific heats
δ	incremental difference operator ( $\delta a = a^{n+1} - a^n$ )
Δ	finite difference operator ( $\Delta x = x_2 - x_1$ )
ε	coefficient in equation (1)
λ	parameter in equation (4)
μ	switching factor, equation (34)
ν	coefficient in equation (1)
ρ	density
ρ̄	artificial density
φ	potential function
*	transpose operator

#### TIME-DEPENDENT METHODS

The Euler equations as well as the unsteady full potential equation are hyperbolic in time, whether or not the flow is locally subsonic or supersonic in space. With this in mind, Wellford and Hafez (ref. 1) solved the following augmented system for the transonic small-disturbance equation:

$$\alpha u_t = \left( (1 - M_\infty^2)u - \frac{\gamma + 1}{2} M_\infty^2 u^2 \right)_x + \nu_y + \nu_1 u_{xx} - \epsilon_1 u \quad (1)$$

$$\beta v_t = u_y - \nu_x + \nu_2 v_{yy} - \epsilon_2 v$$

where regularization terms  $\beta v_t$ ,  $\nu_1 u_{xx}$ ,  $\nu_2 v_{yy}$ ,  $\epsilon_1 u$ ,  $\epsilon_2 v$  are added explicitly. The coefficients  $\epsilon$ ,  $\nu$ ,  $\beta$  are assumed to be small. An implicit Crank-Nicholson finite difference scheme was used for the time derivatives and a standard Galerkin finite element approach was used for the space derivatives. Stability and convergence were rigorously analyzed. It was shown that a minimum amount of viscosity  $\nu_1$ , related to the magnitude of the drag, is needed. Results shown in fig. 1 are generally in agreement with finite difference results, but the accuracy is not acceptable for the element density used. The method converged slowly, but it may be useful for unsteady problems.

Hafez, Wellford and Murman (ref. 2) considered an extension of this method to the full potential equation in the conservative form

$$\rho_t = -(\rho u)_x - (\rho v)_y, \quad u_t = w_x, \quad v_t = w_y \quad (2)$$

where

$$w = \rho^{\gamma-1} - \left(1 - \frac{\gamma-1}{2} M_\infty^2 (u^2 + v^2 - 1)\right) = \phi_t \quad (3)$$

Again, artificial viscosity terms as well as damping terms ( $\alpha w$ ) are added to the system. If we are interested only in the steady state solution, the transient behavior is not important and a faster iterative procedure may be possible. The work is still in progress.

Phares and Kneile (ref. 3) have reported finite element solutions of the unsteady Euler equations using a time dependent method. The equations are in non-conservative form with the dependent variables being  $u, v, t$ . Isoparametric elements are used along with a Galerkin method for the spatial derivatives. A result from their study is shown in figure 2. The results look quite good, but the rate of convergence is about a hundred times slower than relaxation solutions of the full potential equation.

#### VARIATIONAL METHODS

For subsonic flows, the full potential equation is elliptic and there are many impressive finite element solutions. Most of this work is based on the Bateman variational principle. For transonic flows, however, the second variation ceases to be positive definite. Hafez, Wellford and Murman (ref. 2) introduced mixed variational formulations using two different functionals. The first given in terms of  $\phi, u$  and  $v$  is:

$$I(\phi, u, v) = \iint \left\{ \frac{\lambda}{2} \left[ (\phi_x - u)^2 + (\phi_y - v)^2 \right] + \rho \left[ (u^2 - u\phi_x) + (v^2 - v\phi_y) \right] + p \right\} dA \quad (4)$$

where  $p = \rho^\gamma / \gamma M^2$ ,  $\lambda$  is a free parameter and  $\rho = \rho(u, v)$ . Numerical results shown in figure 3 for the case of small-disturbance approximation indicate approximately the right solution but some apparent inaccuracies. The second functional given in terms of  $\phi$  and  $\rho$  is:

$$I(\phi, \rho) = \iint \left[ F(\rho) + \frac{1}{2} \rho (\phi_x^2 + \phi_y^2) \right] dA \quad (5)$$

where  $F(\rho) = -C_1 \frac{\rho^\gamma}{\gamma-1} + C_2 \rho$ . The two associated Euler equations are the continuity and the energy equations in  $\rho$  and  $\phi$ . Notice that the natural boundary condition associated with equation (5) admits the right jump conditions across the shock (i.e., mass is conserved). This is a consequence of the definition of weak solutions. The two Euler equations must be solved simultaneously. The method appears attractive, but no solutions are yet available. As an extension to the above ideas, we note that it is possible to construct a general gradient method

$$\begin{pmatrix} \delta\phi \\ \delta\rho \end{pmatrix} = -G \begin{pmatrix} \frac{\partial I}{\partial \phi} \\ \frac{\partial I}{\partial \rho} \end{pmatrix} \quad (6)$$

where  $G$  is  $2 \times 2$  matrix. Notice if we choose  $G = \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \end{pmatrix}$  where  $w_1 = \frac{\Delta t}{(\gamma-1)M_\infty^2}$  and  $w_2 = \Delta t$ , the system (equations (6) and (7)) is hyperbolic

in time for subsonic as well as supersonic regions. Different choices of  $G$  may lead to faster convergence. Dissipation terms will be needed for stability and to handle flows with shocks.

Eberle (ref. 4) has recently introduced a variational finite element method for the full potential equation. Artificial viscosity terms are added which may be interpreted as an artificial compressibility. The density is retarded upstream by a small ( $\Delta x$ ) amount. The equations are solved in a transformed plane using a relaxation method. A calculated result is shown in figure 4. Other results in Eberle's report indicate a noticeable dependence of the solution on element density, element shape and the magnitude of the artificial viscosity. We note that the work of Eberle motivated the artificial compressibility work discussed later.

#### LEAST SQUARES AND OPTIMAL CONTROL METHODS

Given the differential equation  $Aw = f$ , an associated functional whose second variation is positive definite can always be formulated using least squares, namely minimizing the residual

$$\begin{aligned} ||Aw - f||^2 &= (Aw, Aw) - 2(f, Aw) + (f, f) \\ &= (A^*Aw, w) - 2(A^*f, w) + (f, f) \end{aligned} \quad (8)$$

where the usual notation of inner products, norms, and adjoints are used. Notice that  $I = (A^*Aw, w) - 2(A^*f, w)$  is the Ritz variational functional for the problem  $A^*Au = A^*f$  which is automatically self-adjoint. However, the order of the equation has doubled, and therefore, a gradient method such as

$$\delta w = -I_w = -A^*(Aw - f) \quad (9)$$

will converge very slowly. In order to accelerate the convergence of the iterative process modified gradient methods can be used. We will discuss here three examples.

In the first gradient method, a modification is made such that a Poisson's equation is solved at each iteration; namely

$$L\delta w = I_w \quad \text{or} \quad \delta w = L^{-1}A^*(Aw - f) \quad (10)$$

Notice that higher order inter-element continuity of the shape functions is required for the least squares method compared to a Galerkin method. This problem may be avoided by writing the differential equation as a system of

lower order equations (see Lynn and Arya (refs. 5 and 6)). As an example, consider the Tricomi equation:

$$A\phi = c(x,y)\phi_{xx} + \phi_{yy} = 0$$

Minimizing

$$\iint (c\phi_{xx} + \phi_{yy})^2 dA \quad (11)$$

leads to:

$$\begin{aligned} A^*A\phi &= \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (c\phi_{xx} + \phi_{yy}) \\ &= (c^2\phi_{xx})_{xx} + \phi_{yyyy} + (c\phi_{yy})_{xx} + (c\phi_{xx})_{yy} = 0 \end{aligned} \quad (12)$$

Alternatively,

$$\text{let } u = \phi_x, \quad v = \phi_y; \quad \text{hence } cu_x + v_y = 0 \quad (13)$$

and minimize

$$\iint \left[ (u - \phi_x)^2 + (v - \phi_y)^2 + \lambda^2 (cu_x + v_y)^2 \right] dA \quad (14)$$

For simplicity,  $\lambda$  is taken to be unity. In general, the choice of  $\lambda$  may improve the rate of convergence with respect to the mesh size as well as with respect to iteration. The Euler equations then may be written as

$$\phi_{xx} + \phi_{yy} = u_x + v_y \quad (15)$$

$$(c^2 u_x)_x + (c v_y)_x - u = -\phi_x \quad (16)$$

$$(c u_x)_y + (v_y)_y - v = -\phi_y \quad (17)$$

A straightforward iterative procedure to solve equations (15) to (17) is as follows: Given  $\phi$ , find  $u$  and  $v$  from equations (16) and (17), then using the most recent values of  $u$  and  $v$  in equation (15), a new value of  $\phi$  is obtained. This procedure can be described by

$$(\delta u)_x + (\delta v)_y = (c^2 u_x)_{xx} + (c v_y)_{xx} + (v_y)_{yy} + (c u_x)_{xx} \quad (18)$$

In terms of  $\phi$ , equation (18) is essentially (except for compatibility terms)

$$L\delta\phi = A^*A\phi + \dots \quad (19)$$

It should be mentioned that the advantage of using equation (14) instead of equation (11) is that the elements used in the approximation of the latter must have continuous first derivatives, a fact which eliminates virtually all of the important practical elements. It is also worth mentioning here that a linear

element approximation leads to undesirable convergence characteristics with respect to mesh size as discussed by Lynn and Arya (refs. 5 and 6). It is obvious that if linear element trial functions are used for  $u$  and  $v$ , a bilinear approximation is needed for  $\phi$ .

Different least square formulations of the full potential equation are given in Table I. Chan and Brashears (ref. 7) solved the transonic small-disturbance equation by least squares similar to equation (11). Fix and Gunzburger (ref. 8) as well as Glowinski et al. (refs. 9 and 10) used a formulation similar to equation (14).

For the second method, we note that  $A$  is a second order operator. The "closest" positive operator to  $A^*A$  is the Biharmonic  $L^*L$ ; hence an iterative procedure based on

$$L^*L\delta w = -I_w = -A^*(Aw - f) \quad (20)$$

is expected to be fast provided the inverse of  $(L^*L)^{-1}$  is easily obtained. Equation (20) is also equivalent to

$$R = Aw - f, \quad L^*Z = A^*R, \quad L\delta w = Z. \quad (21)$$

The third method results from a more elegant decomposition of equation (20) obtained by modifying the inner product, namely

$$\begin{aligned} \tilde{I} &= ((Aw - f), Q(Aw - f)) \\ &= (A^*QAw, w) - 2(A^*Qf, w) + (f, Qf) \end{aligned} \quad (22)$$

hence,

$$\tilde{I}_w = A^*QAw - A^*Qf \quad (23)$$

where  $Q^* = Q$  the choice of  $Q = L^{-1}$  makes  $A^*L^{-1}A$  effectively second order. The gradient method gives

$$\delta w = A^*L^{-1}(Aw - f) \quad (24)$$

and the modified gradient method gives

$$L\delta w = -A^*L^{-1}(Aw - f) \quad (25)$$

which may be decomposed into

$$LZ = Aw - f, \quad L\delta w = -A^*Z \quad (26)$$

Effectively, each step has an operator of zero order. Notice that

$$\tilde{I} = -((Lz), L^{-1}(Lz)), = -(Lz, z). \quad (27)$$

Upon integration by parts,  $\bar{I}$  becomes

$$\bar{I} = (\nabla Z, \nabla Z) = \|\nabla Z\|^2 = \iint |\nabla Z|^2. \quad (28)$$

Recently Glowinski et al. (refs. 9 and 10) solved the full potential transonic flow equation using an equivalent procedure, namely

$$\text{minimize} \quad \iint |\nabla Z|^2 \, dA \quad \text{w.r.t. } w \quad (29)$$

under the constraint

$$LZ = Aw - f \quad \text{where } Z = \phi - w. \quad (30)$$

The discussion with Professor Antony Jameson of Courant Institute helped the author in the preparation of this section of this paper.

#### TREATMENT OF SHOCKS

In the above formulations, expansion as well as compression shocks are admitted since the potential flow is reversible. In order to exclude expansion shocks, Glowinski et al. (refs. 9 and 10) introduced an entropy constraint\* to the least squares formulation, namely

$$\nabla^2 \phi = M^2 \phi_{ss} < +\infty \quad (31)$$

A result shown in figure 5 indicates quite good agreement between the results and an exact solution for a shock-free airfoil. An alternative procedure has been introduced by Bristeau (ref. 12) in which artificial viscosity terms are explicitly added to the continuity equation in addition to the above constraint. A result is shown in figure 6. We notice here that the special assembly procedure used by Chan and Brashears (ref. 7) may effectively produce some sort of artificial viscosity in an obscure way without which the solution cannot be obtained.

#### FINITE VOLUMES

Finite volume methods use general nonorthogonal coordinates and consider the governing integral equations as balances of mass, momentum, and energy fluxes for each finite volume defined by the intersection of the coordinate surfaces. Rizzi (ref. 13) applied the finite volume method to the Euler equation for transonic flows and calculated the time-accurate solution until it converged to a steady state. Factored explicit and implicit difference operators were used to accelerate the calculations. Several computed examples are given in the reference.

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\*A different procedure was used by Chattot (ref. 11) in his least squares formulation of Euler Equations to exclude the expansion shock by imposing  $US_x + VS_y \geq 0$ .

Jameson and Caughey (ref. 14) have recently applied the finite volume method to the steady full-potential equation in conservation form by using mixed-type flux operators. Centered difference operators are used with artificial viscosity added in supersonic cells. Isoparametric mesh system is used and the equations are solved by relaxation. The method combines the advantage of finite elements for handling complicated geometries and the advantage of using simple finite difference schemes in the transformed coordinates. A typical calculation is shown in figure 7.

The same concept has been recently used by Lucchi (ref. 15) with different higher order elements and with the velocities and the density as variables. He used direct inversion to solve the matrix equation.

#### ARTIFICIAL COMPRESSIBILITY METHODS

Following the work of Eberle (ref. 4), we have explored the application of an artificial compressibility approach for both finite element and finite difference methods for mixed equations. The idea behind these new methods is to modify the density (and/or the speed of sound) in the supersonic region slightly (within the same order of the truncation error) and solve the resulting problem iteratively with standard methods used for the solution of elliptic problems. The density modification can be interpreted as an artificial viscosity effect. The modified equation reads

$$(\tilde{\rho}\phi_x)_x = (\tilde{\rho}\phi_y)_y = 0 . \quad (32)$$

For example, Jameson's fully conservative schemes can be approximated by this form where

$$\tilde{\rho} = \rho + \mu \frac{\rho}{a^2} (uu_x \Delta x + vv_y \Delta y) \quad (33)$$

and 
$$\mu = \max (0, 1 - 1/M^2) . \quad (34)$$

We have tested an alternative form

$$\tilde{\rho} = \rho - \mu \rho_s \Delta s \quad (35)$$

where 
$$\Delta s \rho_s = \frac{u}{q} \rho_x \Delta x + \frac{v}{q} \rho_y \Delta y \quad (36)$$

Other possible forms are under study including modification of the speed of sound.

Standard central difference formulas are used for equation (32) and  $\tilde{\rho}$  evaluated at the previous iteration. In the formula for  $\tilde{\rho}$ , upwind differencing is used in the supersonic region. Various iterative methods have been tested successfully including SOR (vertical-horizontal-SSOR), ADI, a second order explicit method, fast solver and multigrid method. A numerical result is shown in figure 8 using a three level explicit scheme. The rate of convergence of this calculation is comparable to the relaxation method. The method is still under development and details will be reported in a separate paper.

#### CONCLUDING REMARKS

A considerable amount of work has been reported during the last few years exploring the application of finite element methods to transonic flow problems. In general, it is found that classical finite element methods are not directly applicable. Modifications must be made either in the finite element method or the governing equation. For the latter approach, artificial viscosity terms must be added. Elliptic type solvers are required for the spatial derivatives with a suitable iterative scheme required in the time-like direction. Some of the methods appear attractive and are being applied to realistic airfoil geometries.

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TABLE I.- LEAST-SQUARE FORMULATIONS FOR FULL POTENTIAL EQUATION

$$S = (\rho u)_x + (\rho v)_y$$

$$\Omega = u_y - v_x$$

$$h = \frac{1}{(\gamma-1)} (\rho^{\gamma-1} - a^2 M_\infty^2)$$

Functional

$$I(u,v) = \iint s^2 \Omega^2$$

where  $\rho = 1 - \frac{\gamma-1}{1} M_\infty^2 (u^2 + v^2 - 1)^{\frac{1}{\gamma-1}}$

$$I(u,v,\rho) = \iint s^2 + \Omega^2 + h^2$$

where  $a^2 = \frac{1}{M_\infty^2} - \frac{\gamma-1}{2} (u^2 + v^2 - 1)$

$$I(\phi,u,v) = \iint s^2 + (u - \phi_x)^2 + (v - \phi_y)^2$$

where  $\rho = 1 - \frac{\gamma-1}{2} M_\infty^2 (u^2 + v^2 - 1)^{\frac{1}{\gamma-1}}$

$$I(\phi,u,v,\rho) = \iint s^2 + h^2 + (u - \phi_x)^2 + (v - \phi_y)^2$$

where  $a^2 = \frac{1}{M_\infty^2} - \frac{\gamma-1}{2} (u^2 + v^2 - 1)$

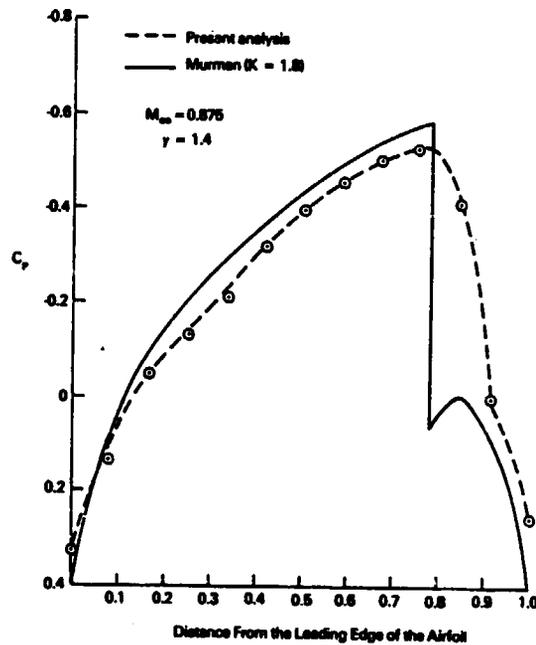


Figure 1.- Solution of transonic small-disturbance equations for parabolic-arc airfoil using implicit time-dependent finite-element method of Hafez, Wellford, and Murman (fig. from ref. 2).

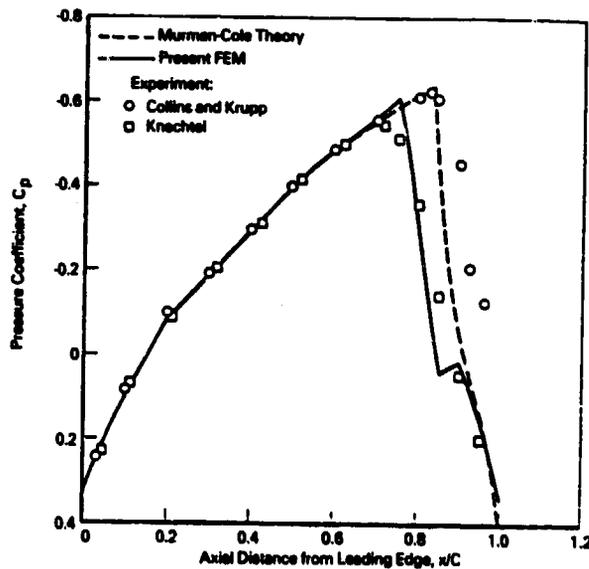


Figure 2.- Solution of Phares and Kneile using time-dependent finite-element method to solve Euler equations for flow past parabolic arc (fig. from ref. 3).

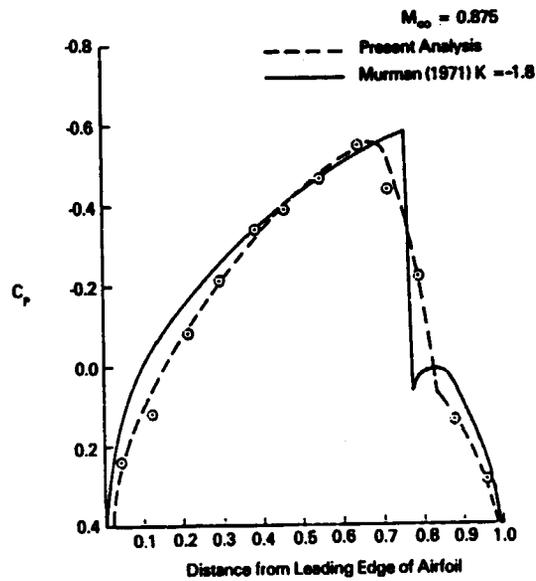


Figure 3.- Solution of transonic small-disturbance equation for flow past parabolic arc using mixed variational finite-element formulation of Hafez, Wellford, and Murman (fig. from ref. 2).

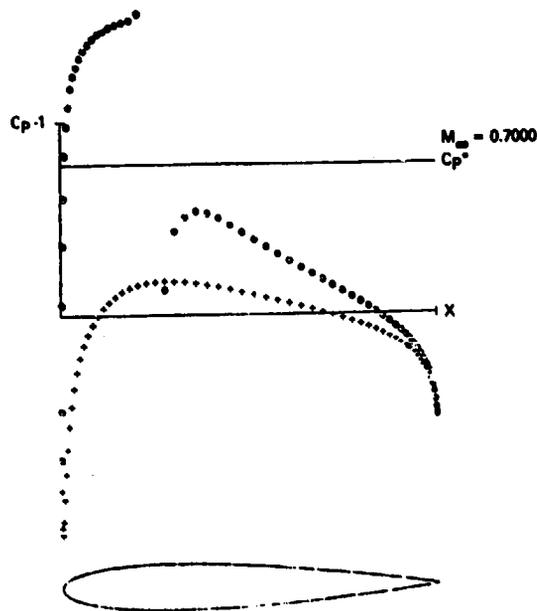


Figure 4.- Finite-element solution by Eberle for flow past NACA 0012 airfoil.

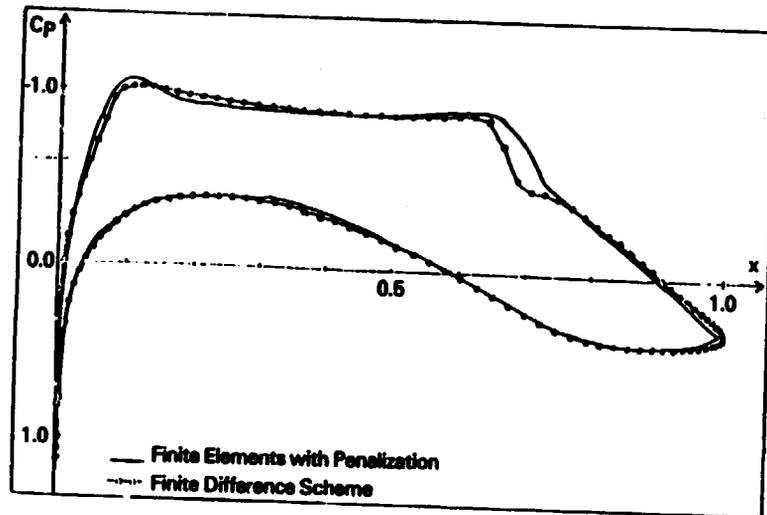


Figure 5.- Finite-element solution by Glowinski et al. for flow past Korn airfoil.

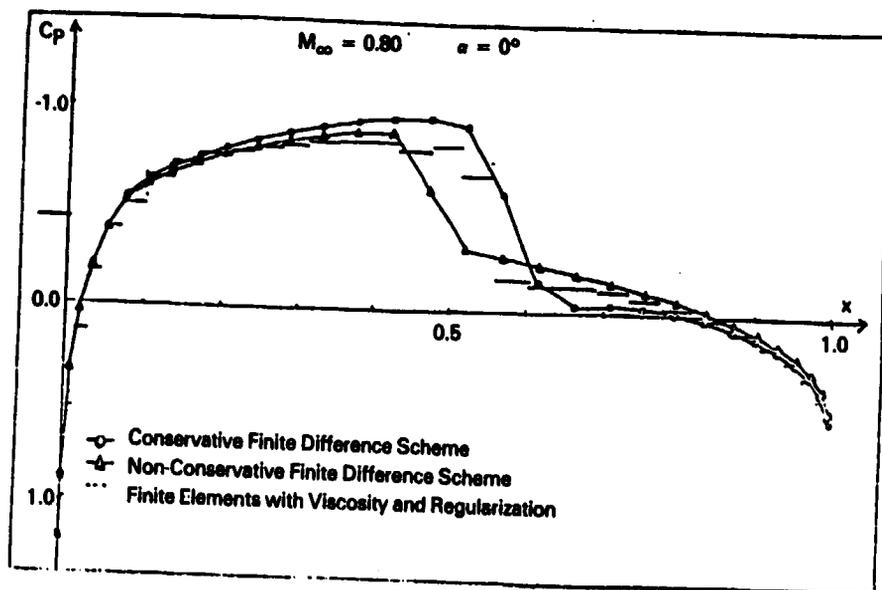


Figure 6.- Finite-element solution by Bristeau for flow past NACA 0012 airfoil.

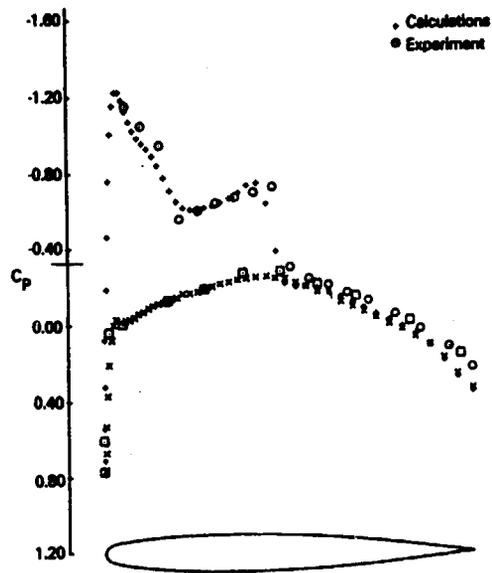


Figure 7.- Finite-volume solution of full-potential equation by Jameson and Caughey.

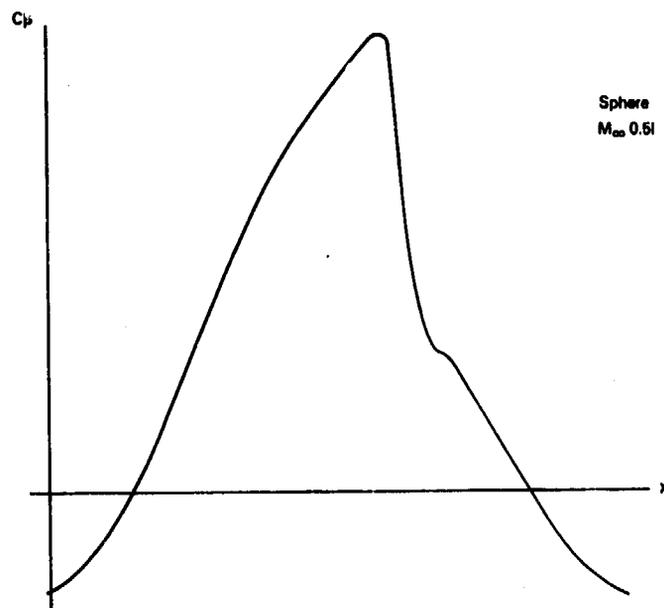


Figure 8.- Solution of full-potential equation using artificial compressibility method.