LINEARIZED UNSTEADY JET ANALYSIS

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1. Introduction

The introduction of a time dependency into a jet flow to change the rate at which it mixes with a coflowing stream or ambient condition has been investigated by several researchers[1-7]. The advantage of the unsteady flow is an increase in the mixing rate as compared to a "steady" jet. The disadvantage, in the case of a jet nozzle, is the fact that the nozzle efficiency suffers significantly[6].

Examples of the types of jets which may be treated by the present analysis are shown in Figures 2-4. The jet exit position of Figure 2 oscillates from side to side and produces a relatively constant magnitude streamwise wave. In Figure 3 the velocity vector at the jet exit oscillates in direction and produces a growing streamwise wave. The unsteadiness of Figure 4 consists of a sinusoidal change with time of the mass flow at the jet exit and thus produces a nominally constant amplitude wave pattern.

The mathematical complexity of time dependent flows is such that one usually is forced to resort to one of three possible attacks:

(a) A fully numerical solution of the governing equations

(b) A phenomenological model

(c) The use of limiting assumptions which simplify the governing equations.

A fully numerical solution is possible but requires the dedication of very

significant amounts of time and effort. A recent interesting example of a phenomenological model is presented by Simmons, Platzer and Smith[8], who assume that the unsteady jet may be decomposed into a number of steady jets which exist during short intervals at the nozzle exit and produce short bursts along the steady jet trajectories.

Much of the unsteady flow work employing limiting assumptions is based in principle on the celebrated work of Lin[9], who analysed the boundary layer beneath a time dependent external flow. Lin's technique is limited to high frequencies and benefits from the fact that the particular problem allows the specification of an unsteady static pressure distribution within the boundary layer. The extension of Lin's analysis to the jet case is hampered by the fact that the time dependent pressure distribution is unknown and there is no comparable technique to Lin's use of the unsteady Bernoulli equation.

The application of Lin's technique to unsteady jets was carried out by McCormack, Cochran and Crane[4] for the case of a jet vibrated from side to side at high frequency (see Figure 2). However, the strict application of the Lin analysis leads to the conclusion that the convective term in the momentum equation is negligibly small. This result is not acceptable to McCormack, et al., since it is clear that the convective term is of importance as in the steady flow case. Thus, the authors present a phenomenological argument that the convective term must be included (in spite of the fact that the analysis excludes the term) and proceed to assume a linear form for it which then results in a linear equation. The analysis, therefore, is a mixture of types b and c above.

The present analysis follows the spirit of the linearized jet analyses due to Pai[10]. The linearization of the equations is achieved by an order of magnitude analysis which is rigorous and removes the need for a phenomenological argument. The requirement of high frequency is also removed and a technique described for including a time dependent pressure distribution which is produced

by the motion of the jet.

2. RELATIONSHIP BETWEEN UNSTEADY AND TIME AVERAGED FLOWS

The objective of this study is to determine the effect of the unsteady flow components on the time averaged flow. That is, what advantages does the unsteady flow hold in terms of steady state mass and momentum transfer? The answer should appear in a steady flow relation including time averaged effects of the unsteady components.

The two-dimensional boundary layer equations are

$$\frac{\delta u}{\delta t} + u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y} = -\frac{1}{\rho}\frac{\delta p}{\delta x} + \frac{1}{R}\frac{\delta^2 u}{\delta y^2} \qquad (1)$$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0 \qquad (2)$$

where each term is non-dimensionalized so R = Reynolds Number.

Consider the velocity profiles shown in Figure 1. The initial jet velocity is a function of time while the coflowing stream velocity is steady. The independent variables can then be separated into time averaged ($^-$) and time dependent ( )$^\prime$ quantities with U = coflowing stream velocity.

$$u(x, y, t) = U + \bar{u}(x, y) + u'(x, y, t)$$
$$v(x, y, t) = \bar{v}(x, y) + v'(x, y, t)$$
$$p(x, y, t) = \bar{p}(x, y) + p'(x, y, t) \qquad (3)$$

Substituting the expansions of the independent variables into the momentum equation and taking the time average of each term results in

$$(U+\bar{u})\frac{\delta\bar{u}}{\delta x} + \bar{v}\frac{\delta\bar{u}}{\delta y} = \frac{\delta\bar{p}}{\delta x} - \left[\overline{u'\frac{\delta u'}{\delta x}} + \overline{v'\frac{\delta u'}{\delta y}} \right] + \frac{1}{R}\frac{\delta^2\bar{u}}{\delta y^2} \qquad (4)$$

It may be seen that the effect of the unsteady terms on the average velocity is the same as an additional (or artificial) pressure gradient. Thus, if the time dependent velocities are known, the bracketed term can be evaluated and the effect of the unsteadiness on the mean flow determined. The following

section will discuss the evaluation of the unsteady flow conditions.

3. DETERMINATION OF TIME DEPENDENT VELOCITIES

The objective is, therefore, to determine the unsteady velocity components u´ and v´, to evaluate the bracketed term in Eqn. 4 and thereby to find the average velocity distribution $U + \bar{u}$, $\bar{v}$. Approximate solutions for u´ and v´ can be found by the following order of magnitude analysis.

Consider the case where the steady state jet velocity deviates only slightly from the coflowing stream velocity and, as well, the unsteady velocity components are small compared to the coflowing stream velocity.

Mathematically -

$$\bar{u}, \bar{v}, u', v', \ll U \qquad (5)$$

Expanding the momentum equation (1) by the steady and unsteady velocity components (Eqn. (3)) results in

$$\frac{\delta u'}{\delta t} + (U + \bar{u} + u') \frac{\delta}{\delta x} (U + \bar{u} + u') + (\bar{U} + v') \frac{\delta}{\delta y} (U + \bar{u} + u')$$

$$= \frac{\delta}{\delta x} (\bar{p} + p') + \frac{1}{R} \frac{\delta^2}{\delta y^2} (U + \bar{u} + u') \qquad (6)$$

In view of the assumptions (5), all products of small variables in Eqn. (6) are neglected and only terms up to first order in the small variables are retained.

(Note also that U = constant)

$$\frac{\delta u'}{\delta t} + U \frac{\delta \bar{u}}{\delta x} + U \frac{\delta u'}{\delta x} = - \frac{\delta p}{\delta x} - \frac{\delta p'}{\delta x} + \frac{1}{R} \left(\frac{\delta^2 \bar{u}}{\delta y^2} + \frac{\delta^2 u'}{\delta y^2} \right) \qquad (7)$$

The steady and unsteady portions of Eqn. (7) may be separated by taking the time average of (7):

$$U \frac{\delta \bar{u}}{\delta x} = - \frac{\delta p}{\delta x} + \frac{1}{R} \frac{\delta^2 \bar{u}}{\delta y^2} \qquad (8)$$

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and subtracting this from (7) to yield

$$\frac{\delta u'}{\delta t} + U \frac{\delta u'}{\delta x} = -\frac{\delta p'}{\delta x} + \frac{1}{R} \frac{\delta^2 u'}{\delta y^2} \qquad (9)$$

This, then, is the governing equation for the unsteady velocity distribution for the jet in Figure 1 with the small perturbation assumptions of equation (5). The initial condition may be seen from Figure 1 to be a top hat velocity profile whose magnitude is a function of time. The lateral boundary conditions are that

$$\lim_{y \to \pm \infty} u' = 0 \qquad (10)$$

The unsteady pressure gradient may, in general, be a function of position and time,

$$\frac{\delta p'}{\delta x} = f(x,y,t), \qquad (11)$$

but is not known directly in the jet case. In a later section a technique is described which allows an approximation to the unsteady pressure distribution. For the present, the pressure will be neglected and thus the governing equation reduces to

$$\frac{\delta u'}{\delta t} + U \frac{\delta u'}{\delta x} = \frac{1}{R} \frac{\delta^2 u'}{\delta y^2} \qquad (12)$$

This equation is similar to that developed in Reference 4, but in this case it is not limited to high frequencies and requires no phenomenological arguments. The initial and boundary conditions are the same as those for Equation (9). The solution to the linear Equation (12) is

$$u' = \frac{\alpha U}{2} e^{i(wT - wX)} \left[erf\left(\frac{1-y}{2\sqrt{X}}\right) + erf\left(\frac{1+y}{2\sqrt{X}}\right) \right] \qquad (13)$$

where
$$T = t/R, \quad X = x/RU, \quad \alpha = \left. \frac{u'}{U} \right|_{x=0}$$
and the error function is defined as

$$erf \, \eta = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta \qquad (14)$$

The actual form of the exponential depends upon the initial condition on the jet. If the initial condition is that the unsteady flow varies about some mean as a cosine function, then the solution is -

$$u' = \frac{\alpha U}{2} \cos(wT - wX) \left[erf\left(\frac{1-y}{2\sqrt{X}}\right) + erf\left(\frac{1+y}{2\sqrt{X}}\right) \right] \qquad (15)$$

4. DETERMINATION OF THE STEADY ARTIFICIAL PRESSURE GRADIENT

The overall continuity Equation (2) can be expanded by the velocity definitions, Eqn. (3), and then split into steady and unsteady forms,

$$\frac{\delta\bar{u}}{\delta x} + \frac{\delta\bar{v}}{\delta y} = 0 \quad \text{and} \quad \frac{\delta u'}{\delta x} + \frac{\delta v'}{\delta y} = 0 \qquad (16)$$

from which it follows that

$$v' = -\int_0^y \frac{\delta u'}{\delta x} \, dy \qquad (17)$$

Then from Eqns. (15) and (17)

$$v' = -\frac{\alpha w}{2} \sin(wT - wX) \int_0^y \left[erf\left(\frac{1-y}{2\sqrt{X}}\right) + erf\left(\frac{1+y}{2\sqrt{X}}\right) \right] dy$$

$$\frac{\alpha \cos(wT - wX)}{4\sqrt{\pi} \, X^{3/2}} \int_0^y \left[\exp\left(-\left(\frac{1-y}{2\sqrt{X}}\right)^2\right)(1-y) \right.$$

$$\left. + \exp\left(-\left(\frac{1+y}{2\sqrt{X}}\right)^2\right)(1+y) \right] dy \qquad (18)$$

The resulting artificial pressure gradient term is

$$\left[\overline{u' \frac{\delta u'}{\delta x}} + \overline{v' \frac{\delta u'}{\delta y}} \right]$$

$$= F_1 \, F_3 \, \overline{\cos(wT - wX) \, \sin(wT - wX)}$$

$$+ \, F_1 \, F_4 \, \overline{\cos^2(wt - wX)}$$

$$+ \, F_6 \, F_9 \, \overline{\sin(wt - wX) \, \cos(wT - wX)}$$

$$+ \, F_7 \, F_9 \, \overline{\cos^2(wT - wX)} \qquad\qquad (19)$$

where the F_i terms are independent of time. The averaging of the trigonometric terms over one cycle results in

$$\overline{\cos^2(wT - wX)} = \tfrac{1}{2}$$

$$\overline{\sin(wT - wX) \, \cos(wT - wX)} = 0 \qquad\qquad (20)$$

so-

$$\left[\overline{u' \frac{\delta u'}{\delta x}} + \overline{v' \frac{\delta u'}{\delta y}} \right] = \frac{1}{2} \, F_1 \, F_4 + \frac{1}{2} \, F_7 \, F_9 \qquad\qquad (21)$$

where

$$F_1 = \frac{\alpha U}{2}\left[\text{erf}\left(\frac{1-y}{2\sqrt{X}}\right) + \text{erf}\left(\frac{1+y}{2\sqrt{X}}\right)\right]$$

$$F_4 = \frac{-\alpha}{4\sqrt{\pi}}\, X^{3/2}\left\{\exp\left[-\left(\frac{1-y}{2\sqrt{X}}\right)^2\right]\left(1-y\right) + \exp\left[-\left(\frac{1+y}{2\sqrt{X}}\right)^2\right]\left(1+y\right)\right\}$$

$$F_7 = \frac{\alpha}{2\sqrt{\pi}}\, X^{1/2}\left\{\exp\left[-\left(\frac{1-y}{2\sqrt{X}}\right)^2\right] + \exp\left[-\left(\frac{1+y}{2\sqrt{X}}\right)^2\right]\right\}$$

$$F_9 = -U F_7 \tag{22}$$

With the artificial pressure gradient known, the steady state velocity distribution may be found numerically from Eqn. (4). The numerical results, however, are not yet available.

It should be noted that the final solution of Eqn. (4) cannot be a function of magnitude of the jet frequency since none of the terms in Eqn. (22) depend on frequency. The importance of this fact will become apparent in the next section.

5. EXPERIMENTAL OBSERVATIONS

The jet flow illustrated in Figure 3 has been investigated experimentally, as shown schematically in Figure 5, to determine the unsteady inputs into the time averaged jet behavior. The data are taken by a two channel hot wire anemometer probe, linearized, averaged, and read out on a set of digital volt-meters. The average is found by an electronic filter designed for this ex-periment by McCormack[11]. The jet nozzle is fluidically controlled, as shown in Figure 6, and is based on a design by Viets[5].

A set of time averaged velocity profiles showing the typical develop-ment of the jet in the streamwise direction is shown in Figure 7. The double

peaked profiles are caused by the time dependent flow inclination at the nozzle exit and disappear as the mixing progresses downstream. Although these profiles are for a jet streaming into an ambient condition, upcoming experiments will address the coflowing stream situation.

The most important data from the point of view of the present analysis is shown in Figure 8; the comparison of steady half width growth rates for the same jet at various oscillation frequencies. The half width is defined here as the distance from the jet centerline to the position on the profile where the velocity is half the _maximum_ velocity found on the profile. It may be seen that there is an appreciable effect of frequency on the jet half width growth, with the minimum growth at a frequency of zero, i.e. the steady two dimensional jet.

If one examines the time averaged term which reflects the effect of unsteadiness on the mean velocity distribution, Equation (21), it can be seen that this term is _not_ a function of frequency. This is true since none of its components [Eqn. (22)] depend on frequency. There are three strong possibilities for this discrepancy.

a. The analysis is linear while the jet is non-linear. ———
    This effect will be investigated in upcoming tests which will feature an experiment satisfying the linearizing assumptions.

b. The analysis is applicable but the eddy viscosity is not the same as the steady state (a very likely situation) and is, in particular, a function of frequency. ———
    This point may be clarified by comparison of the analysis to the experiments in a.

c. The time dependent pressure distribution in Equation (9) is not zero as was assumed earlier in the analysis but is really a function of frequency. ———
    This possibility is examined in the following section.

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6. EFFECT OF A TIME DEPENDENT PRESSURE VARIATION

The basis for an unsteady pressure distribution is the interaction between the unsteady jet and the coflowing stream. If one considers the simplest case of a jet which does not mix with the ambient fluid, then the jet surface appears as a traveling sinusoidal wave to the coflowing stream. The inclusion of mixing modifies the shape of this wave but near the jet exit the shape is still very nearly sinusoidal.

The simplest model for the pressure variation produced by a wave pattern is that produced by the inviscid flow over a wavy wall, shown in Figure 9. A linearized treatment of this problem is given by Liepmann and Roshko[12] and results in the pressure distribution

$$P = P_\infty \left(1 - B \, \epsilon \alpha \sin \alpha x\right) \qquad (23)$$

where B depends upon the freestream conditions, P_∞ is the freestream pressure and the other variables are defined in Figure 9. It may be seen that the pressure variation is in phase with the wall shape.

The real jet case is, of course, a viscous problem (as is the real wavy wall case). Thus, the pressure dependence is not as straight-foreward as indicated above. This has been demonstrated experimentally by Kendall[13], who studied a mechanical wave traveling relative to a free stream. Kendall's results indicate a phase shift between the wall shape and the pressure distribution. The magnitude of the phase shift depends upon the ratio of the wave speed to the coflowing stream velocity. For a wave speed approximately equal to the coflowing stream velocity, the phase shift is approximately 10° downstream.

With the velocity varying as a cosine function as in Eqn. (15), the static pressure should vary as a sine function. In addition there must be a phase shift and the boundary conditions require that the pressure approach

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the limit of the free stream pressure as the distance from the jet increases. The pressure dependence satisfying these conditions as well as the requirement that the pressure be proportional to the square of the velocity difference between the coflowing stream and wave speed is

$$P' = -\frac{C_p \rho (U - c)^2}{2} e^{-\sqrt{R}|y|} \sin\left[w(T - T_0) - wX\right] \qquad (24)$$

where C_p may be obtained from Kendall[13]. Including this term in the governing differential Equation (9) gives rise to another term in the solution for u´ in order to balance the $\frac{dp}{dx}$ term.
Then

$$u' = u'_{p'=0} - \frac{C_p \rho (U - C)^2}{U/w} e^{-\sqrt{R}|y|} \cos\left[w(T - T_0) - wX\right] \qquad (25)$$

The main point here is that u´ now is a function of frequency and therefore the bracketed term in Eqn. (4) is also a function of frequency. Thus the inclusion of the time dependent pressure allows the prediction of an average velocity $\bar{u}$ which depends upon frequency. The numerical results for this case are not yet available.

7. CONCLUSION

The foregoing analysis shows that the unsteady flowfield generated by a time dependent jet can be treated by a linearized attack which is not limited by frequency constraints and evolves through a rigorous simplification of the equations of motion. The numerical integration of the full non-linear equations to produce the time averaged solution is currently underway.

8. REFERENCES

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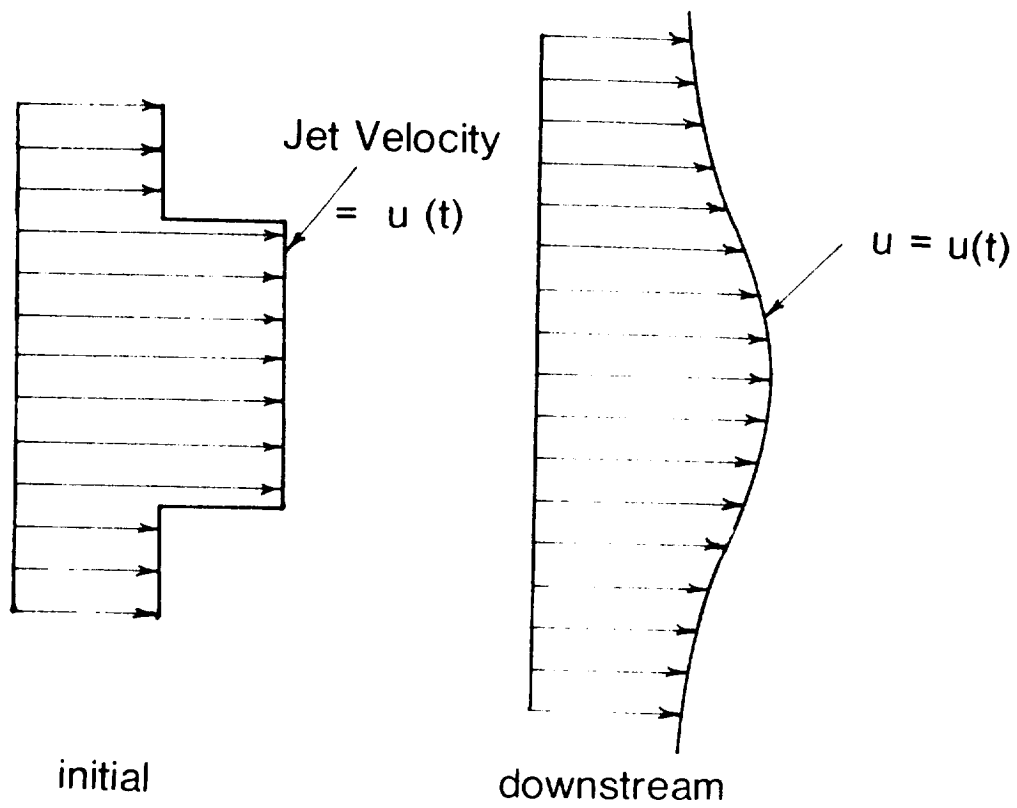
Figure 1. Initial and downstream velocity profiles.

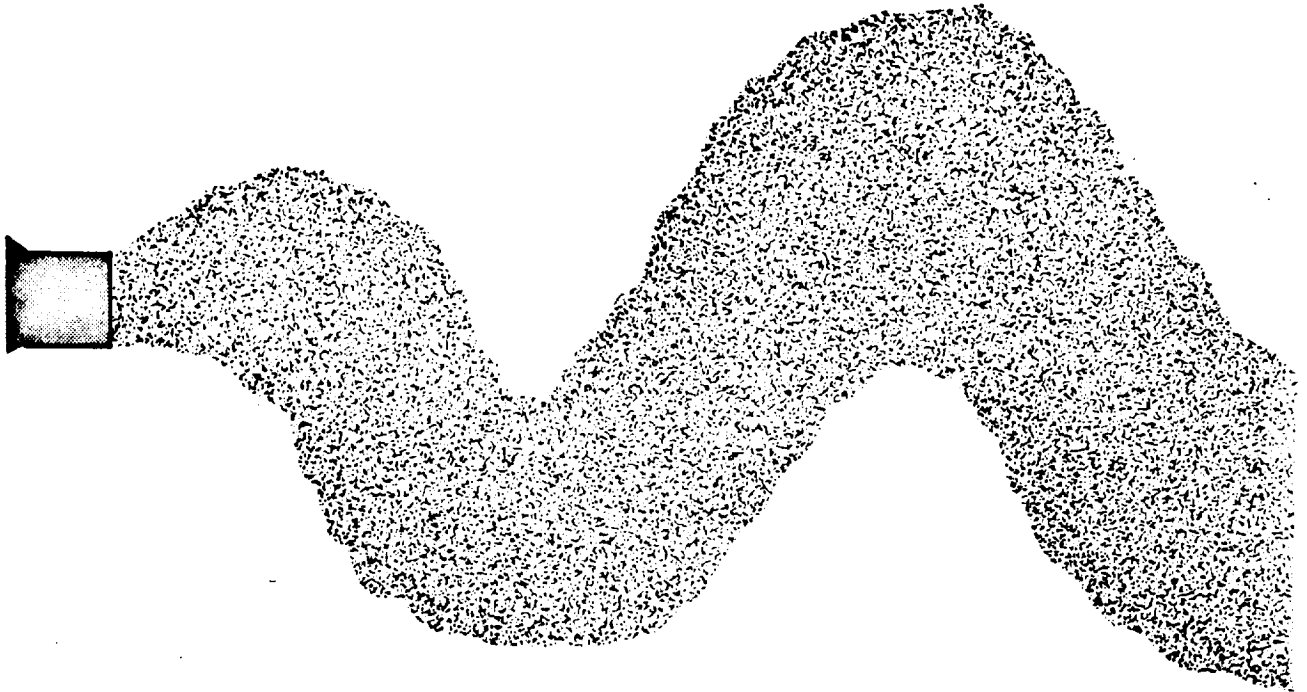Figure 2. Schematic of the vibrated jet studied by McCormack,
Cochran and Crane[4].

Figure 3. Flapping jet with angular time variation at the exit studied by Viets et al.[5-6] and Platzer et al.[7-8]
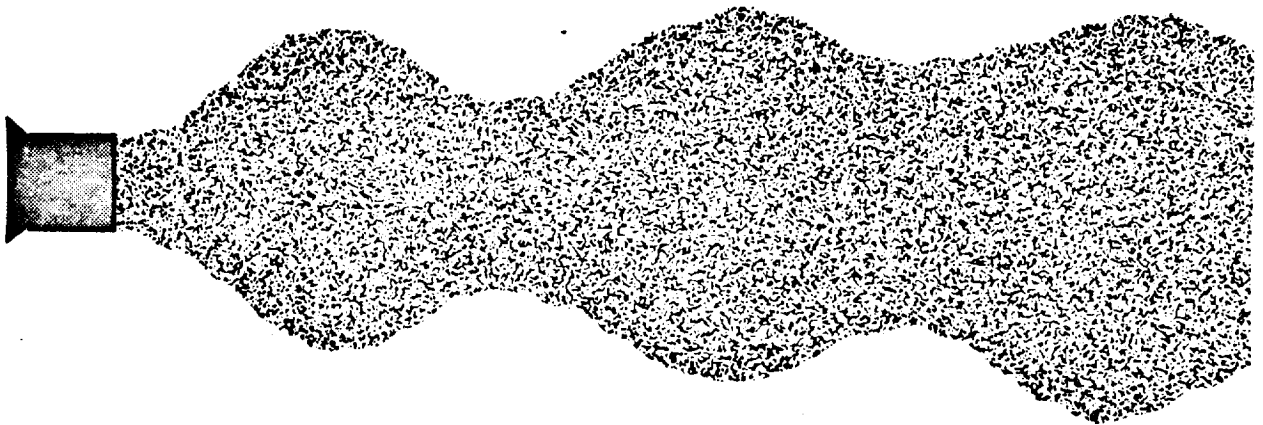

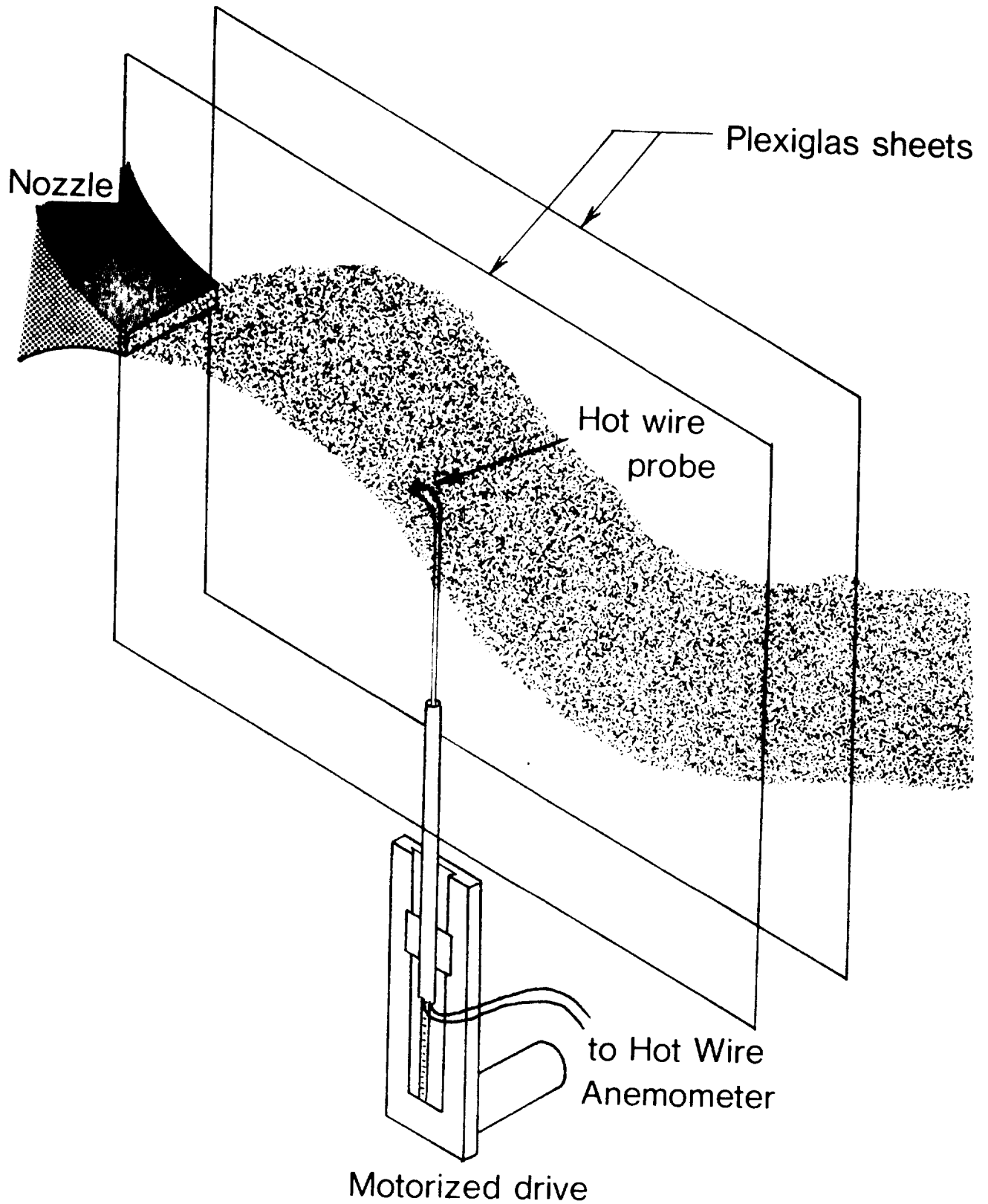Figure 4. Jet with time dependent mass flow studied by Binder and Favre-marinet[1] and Curtet and Girad[2].

Nozzle

Plexiglas sheets

Hot wire
probe

to Hot Wire
Anemometer

Motorized drive

Figure 5. Experimental Setup.

FLOW
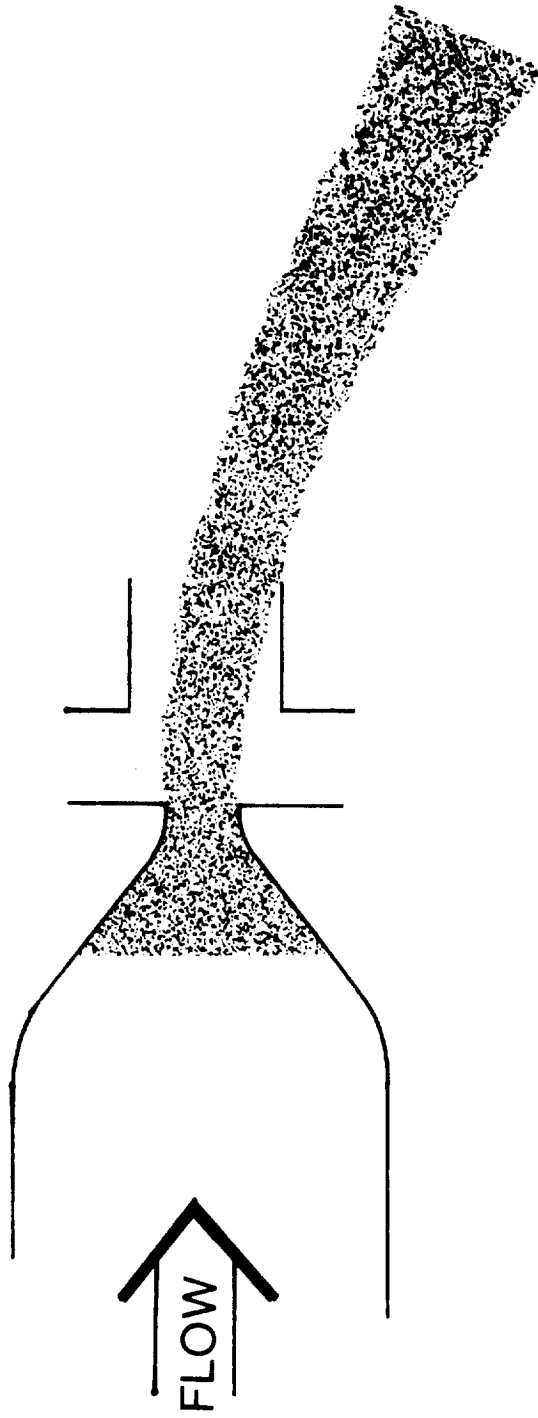
Figure 6.   Fluidically Controlled Jet Nozzle Design.
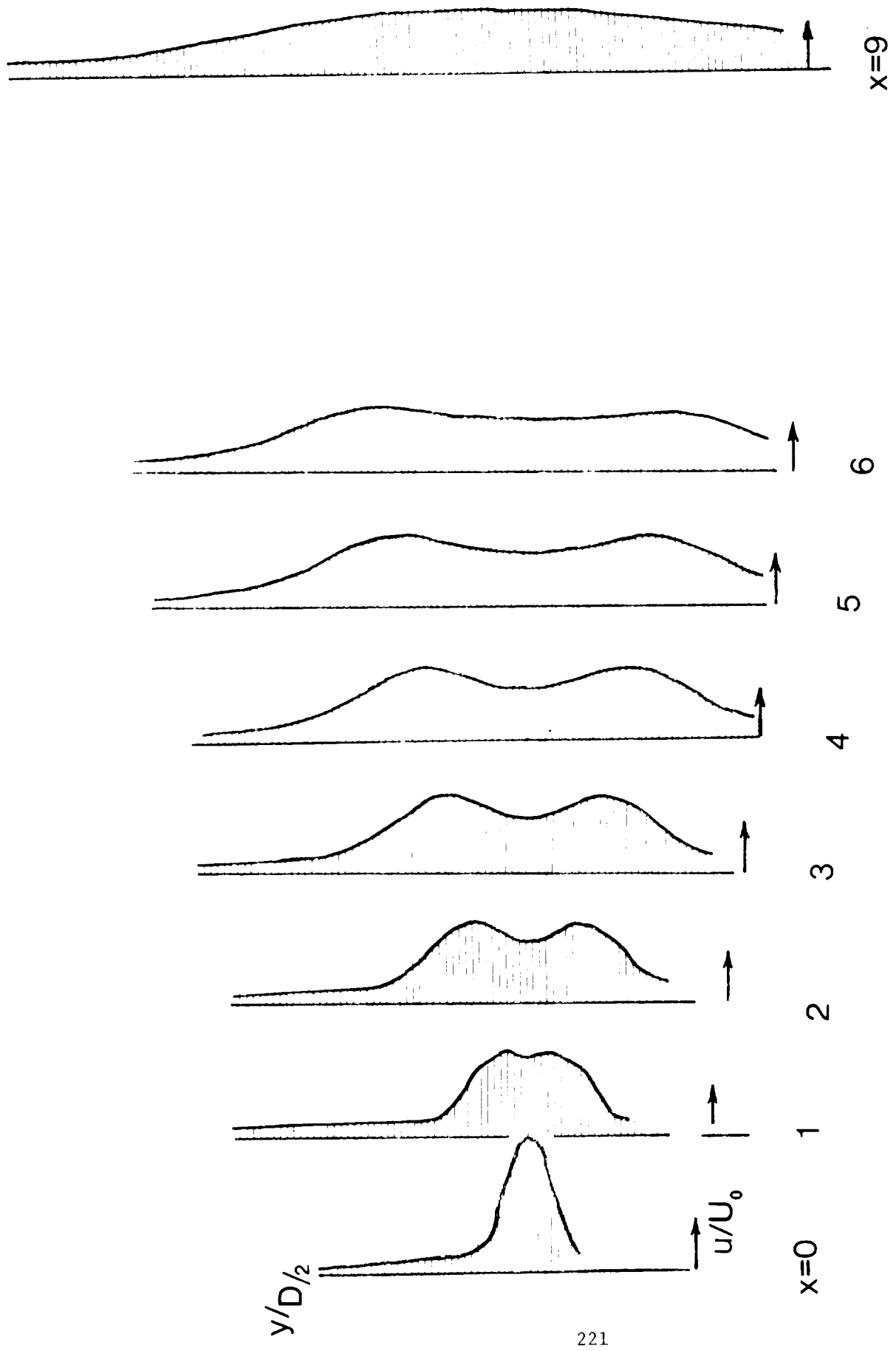
Figure 7. Development of the flapping jet in the streamwise direction.
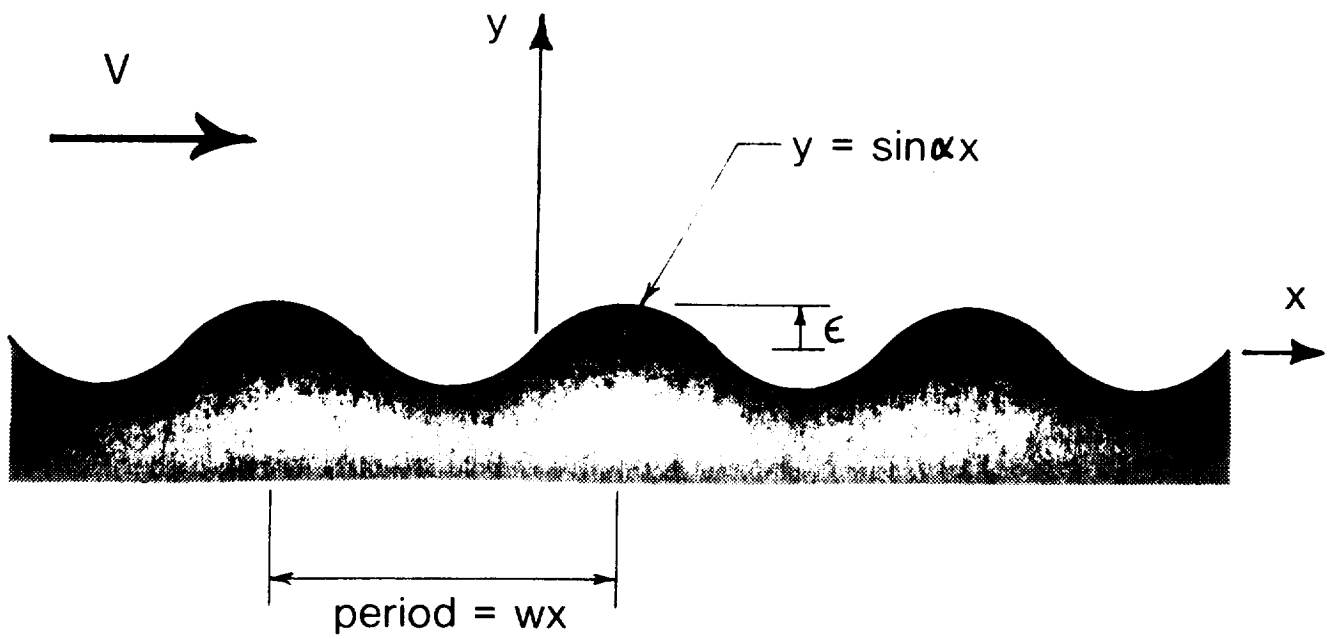
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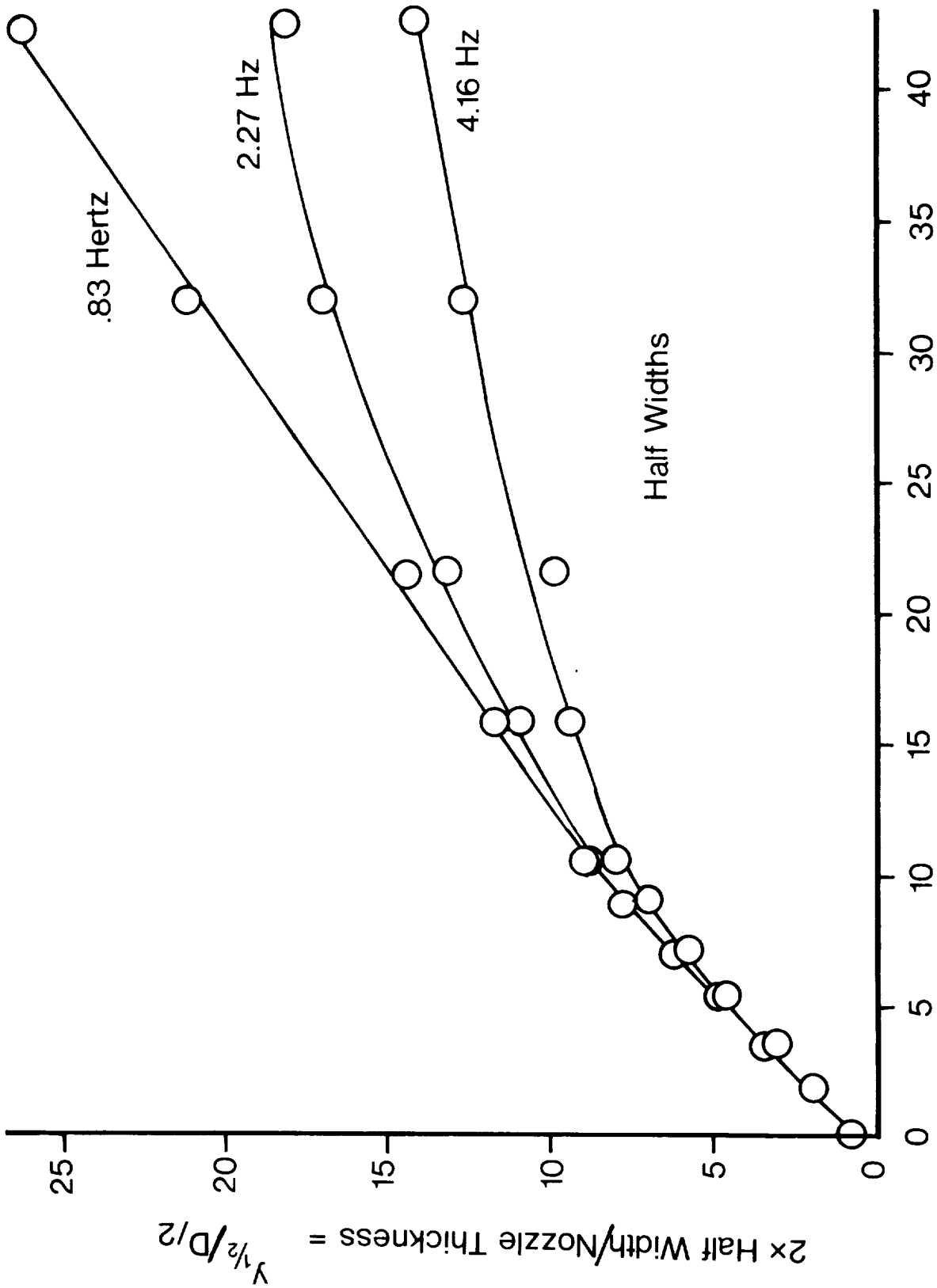
Figure 8. Half width growth dependent upon oscillation frequency.

Figure 9. Flow over a wavy wall.