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Bubble in a corner flow

Jean-Marc Vanden-Broeck

Mathematics Department and Mathematics Research Center, University of Wisconsin-Madison
Madison, Wisconsin 53706

Abstract

The distortion of a two-dimensional bubble (or drop) in a corner of angle δ, due to the flow of an inviscid incompressible fluid around it, is examined theoretically. The flow and the bubble shape are determined as functions of the angle δ, the contact angle β and the cavitation number γ. The problem is formulated as an integrodifferential equation for the bubble surface. This equation generalizes the integrodifferential equations derived by Vanden-Broeck and Keller[1,2]. The shape of the bubble is found approximately by using the slender body theory for bubbles presented by Vanden-Broeck and Keller[2]. When γ reaches a critical value $\gamma_0(\beta,\delta)$, opposite sides of the bubble touch each other. Two different families of solution for $\gamma < \gamma_0$ are obtained. In the first family opposite sides touch at one point. In the second family contact is allowed along a segment. The methods used to calculate these two families are similar to the ones used by Vanden-Broeck and Keller[3] and Vanden-Broeck[4].

1. Introduction and formulation

We consider the steady potential flow around a gas bubble or liquid drop in a corner of angle δ. The contact angle is denoted by β (see Figure 1). We shall write "bubble" to mean either bubble or drop. We take into account the surface tension σ at the interface, but we ignore the flow inside the bubble, assuming that the pressure is a constant p_b throughout it.

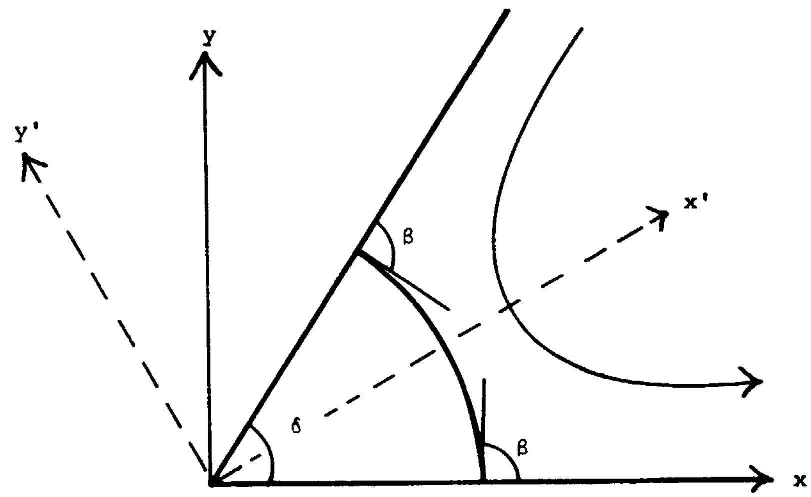

Figure 1. Sketch of the bubble and the coordinates

In order to formulate this problem we assume that the complex potential without the bubble is $\frac{\delta\alpha}{\pi}(x+iy)^{\pi/\delta}$, where α is a constant and x and y are Cartesian coordinates.

We introduce dimensionless variables by choosing $\left(\frac{2\sigma}{\rho\alpha^2}\right)^{\frac{\delta}{2\pi-\delta}}$ as the unit length and $\alpha\left(\frac{2\sigma}{\rho\alpha^2}\right)^{\frac{\pi-\delta}{2\pi-\delta}}$ as the unit velocity. We also introduce the dimensionless potential $b\phi$ and stream function $b\psi$. Here, $b > 0$ is a dimensionless constant to be chosen so that $\phi = \frac{1}{2}$ and $\phi = -\frac{1}{2}$ at the stagnation points on the walls $y = 0$ and $y = x\tan\delta$, respectively. We denote the streamline along the two walls and along the bubble boundary by $\psi = 0$. In these variables $b(\phi + i\psi) \sim \frac{\delta}{\pi}(x + iy)^{\frac{\pi}{\delta}}$ at infinity or, equivalently

$$x + iy \sim \left(\frac{\pi b}{\delta}\right)^{\frac{\delta}{\pi}} (\phi + i\psi)^{\frac{\delta}{\pi}} \tag{1}$$

at infinity.

The flow occupies the region $\psi > 0$ of the ϕ, ψ plane, and the bubble boundary corresponds to the segment $-\frac{1}{2} < \phi < \frac{1}{2}$ of the axis $\psi = 0$. The problem of finding the flow consists of determining $x + iy$ as an analytic function of $\phi + i\psi$ in the half plane $\psi > 0$ satisfying Equation (1) at infinity. Then the bubble surface is given by setting $\psi = 0$ in $x(\phi + i\psi)$ and $y(\phi + i\psi)$ and letting ϕ range from $-\frac{1}{2}$ to $\frac{1}{2}$. The contact angle conditions require that the bubble surface meets the walls at the angle β, which yields

$$\frac{y_\phi}{x_\phi} \to \begin{cases} \tan\beta & \text{as } \phi \to \frac{1}{2} \\ -\tan(\beta - \delta) & \text{as } \phi \to -\frac{1}{2} \end{cases} \tag{2}$$

On the bubble surface the pressure in the fluid, which is given by the Bernoulli equation, must differ from p_b by σk, where k is the curvature of the interface. This leads to the boundary condition

$$p_s - \frac{\rho q^2}{2} = p_b - \sigma k \quad \text{on} \quad -\frac{1}{2} < \phi < \frac{1}{2}, \quad \psi = 0 . \tag{3}$$

Here, p_s, ρ and q are, respectively, the stagnation pressure, the density and the speed of the fluid outside the bubble. In dimensionless variables (3) becomes

$$q^2 = k - \gamma \quad \text{on} \quad -\frac{1}{2} < \phi < \frac{1}{2}, \quad \psi = 0 , \tag{4}$$

where γ is the cavitation number defined by

$$\gamma = \frac{p_b - p_s}{\sigma} \left(\frac{2\sigma}{\rho a^2}\right)^{\frac{\delta}{2\pi - \delta}} . \tag{5}$$

The problem can be further simplified by requiring the bubble to be symmetric about the line $y = x\tan\frac{\gamma}{2}$. This implies that

$$y_\phi(\phi,0) = y_\phi(-\phi,0)\cos\delta - x_\phi(-\phi,0)\sin\delta, \quad 0 < \phi < \frac{1}{2} . \tag{6}$$

By using Equation (6) we can restrict our analysis to the interval $0 < \phi < \frac{1}{2}$.

2. Reformulation as an integrodifferential equation

It is convenient to reformulate the boundary value problem as an integrodifferential equation by considering the function

$$(\phi + i\psi)^{1 - \frac{\delta}{\pi}}(x_\phi + iy_\phi) - \left(\frac{\pi b}{\delta}\right)^{\frac{\delta}{\pi}} \frac{\delta}{\pi} ,$$

which is analytic in the half plane $\psi > 0$ and vanishes at infinity as a consequence of Equation (1). Therefore, on $\psi = 0$, its real part is the Hilbert transform of its imaginary part. The imaginary part vanishes on $\psi = 0$, $|\phi| > \frac{1}{2}$ and therefore the Hilbert transform yields

$$\phi^{1-\frac{\delta}{\pi}} x_\phi(\phi,0) - \left(\frac{\pi b}{\delta}\right)^{\frac{\delta}{\pi}} \frac{\delta}{\pi} = \frac{1}{\pi}\int_0^{1/2} \frac{(\phi')^{1-\frac{\delta}{\pi}} y_\phi(\phi',0)}{\phi'-\phi}\, d\phi'$$

$$+ \frac{1}{\pi}\int_{-1/2}^{0} \frac{(-\phi')^{1-\frac{\delta}{\pi}}[-y_\phi(\phi',0)\cos\delta + x_\phi(\phi',0)\sin\delta]}{\phi'-\phi}\, d\phi' \ . \tag{7}$$

We now use the symmetry condition (6) to rewrite (7) in the form

$$x_\phi(\phi,0) = \left(\frac{\pi b}{\delta}\right)^{\frac{\delta}{\pi}} \frac{\delta}{\pi}\, \phi^{\frac{\delta}{\pi}-1} + \frac{\phi^{\frac{\delta}{\pi}-1}}{\pi}\int_0^{1/2} (\phi')^{1-\frac{\delta}{\pi}} y_\phi(\phi',0)\left(\frac{1}{\phi'-\phi} + \frac{1}{\phi'+\phi}\right) d\phi' \ . \tag{8}$$

Next we express the boundary condition (4) in terms of x_ϕ and y_ϕ noting that $q^2 = b^2(x_\phi^2 + y_\phi^2)^{-1}$. Then (4) becomes

$$\frac{b^2}{x_\phi^2 + y_\phi^2} = \frac{y_\phi x_{\phi\phi} - x_\phi y_{\phi\phi}}{(x_\phi^2 + y_\phi^2)^{3/2}} - \gamma, \qquad |\phi| < \frac{1}{2}, \qquad \psi = 0 \ . \tag{9}$$

Now (8) and (9) together constitute a nonlinear integrodifferential equation for $y_\phi(\phi)$ in the interval $0 < \phi < \frac{1}{2}$, $\psi = 0$. The contact angle conditions (2) complete the formulation of the problem for $y_\phi(\phi,0)$ and b.

For $\gamma = \pi$, the equation defined by (8) and (9) reduces to the integrodifferential equation derived by Vanden-Broeck and Keller[1]. The particular case $\beta = \frac{\pi}{2}$ represents half of a free bubble.

For $\gamma = \beta = \frac{\pi}{2}$, the equations (8) and (9) reduce to the integrodifferential equation derived by Vanden-Broeck and Keller[2]. This case represents a quarter of a free bubble in a straining flow.

The integrodifferential equation defined by (8) and (9) can be solved numerically for arbitrary values of β, γ and δ by using the numerical procedures described by Vanden-Broeck and Keller[1,2].

In the next section,, we shall find the shape of the bubble approximately by using the slender body theory for bubbles presented by Vanden-Broeck and Keller[1].

3. Slender body approximation

For γ large the bubble tends to an arc of a circle of radius γ^{-1}. As γ decreases numerical solutions show that the bubble elongates in the direction of the line which bisects the angle between the two walls. Then it develops a horn or spike which large curvature near its end. Finally when γ reaches a critical value $\gamma_0(\beta,\delta)$, opposite sides of the bubble touch each other. For $\beta < \frac{\delta}{2}$ the contact point is at $x = y = 0$. For $\beta > \frac{\delta}{2}$ the contact point is away from $x = y = 0$. Typical profiles for $\delta = \pi$ and $\delta = \beta = \frac{\pi}{2}$ can be found in Vanden-Broeck and Keller[1,2]. These profiles were obtained by solving numerically the integrodifferential equation of Section 2.

For $\gamma \sim \gamma_0(\beta,\delta)$ the bubble is slender. Therefore we shall use the slender body theory for bubbles presented by Vanden-Broeck and Keller[1] to get an approximate description of the flow around the bubble. In the lowest order, the flow about a symmetric slender bubble is approximated by the flow about a rigid plate lying along the center line of the bubble. In the present case the center line of the bubble consists of a straight segment of some length a lying along the line $y = x \tan\delta/2$. We introduce the coordinates x', y' (see Figure 1) and find the potential $b\phi(x',y')$ of the flow about these plates requiring that at infinity $b(\phi + i\psi) \sim \frac{\delta}{\pi}(x + iy)^{\frac{\pi}{\delta}}$. Evaluating the potential on the plate $y' = 0$, $x' > 0$ we obtain

$$b\phi(x',0) = \frac{\delta}{\pi}\left[a^{\frac{2\pi}{\delta}} - x'^{\frac{2\pi}{\delta}}\right]^{1/2} \quad . \tag{10}$$

By differentiating (10) we find that the flow speed q on the plate is

$$q(x',0) = x'^{\frac{2\pi}{\delta}-1}\left(a^{\frac{2\pi}{\delta}} - x'^{\frac{2\pi}{\delta}}\right)^{-1/2} \quad , \quad x' > 0 \; . \tag{11}$$

Before using q to get the bubble shape, we shall determine the length a. We do so by requiring the suction force F, exerted by the flow on the end of the spike, to balance the surface tension 2σ. As we see in Ref. 5 [p. 412, Eq (6.5.4)], $F = \pi\rho A^2/4$. Here A, is the coefficient in the expansion $b\phi \sim Ar^{1/2}\cos\frac{\theta}{2}$ in terms of polar coordinates with their origin at the end of the plate. Upon setting $F = 2\sigma$ and introducing dimensionless variables we obtain

$$A^2 = \frac{4}{\pi} \; . \tag{12}$$

From (10) we find $A^2 = \frac{2\delta}{\pi}a^{\frac{2\pi}{\delta}-1}$, so (12) yields

$$a = \left(\frac{2}{\delta}\right)^{\frac{\delta}{2\pi-\delta}} \; . \tag{13}$$

We next use (11) for q in (4) and approximate the curvature k by $-\eta_{x'x'}(x')$. Here the equation of the bubble is $y' = \eta(x')$. Then (4) becomes

$$\eta_{x'x'} = -x'^{\frac{4\pi}{\delta}-2}\left(a^{\frac{2\pi}{\delta}} - x'^{\frac{2\pi}{\delta}}\right)^{-1} - \gamma \; . \tag{14}$$

At the end of the spike we require

$$\eta(a) = 0 \; . \tag{15}$$

In addition the contact angle condition yields

$$\eta'(\alpha) = -\tan\left(\beta - \frac{\delta}{2}\right) \; . \tag{16}$$

Here α is defined by the equation

$$\eta(\alpha) = \alpha\tan\frac{\delta}{2} \; . \tag{17}$$

The function $\eta(x')$ is easily obtained by integrating (14) twice with the auxiliary conditions (15) and (16). In the particular case $\delta = \pi$, the result of the integration is

$$\eta(x') = (a^2 - x'^2)(\gamma - 1)/2 - \frac{1}{2}a(a + x')\log(a + x')$$
$$- \frac{1}{2}a(a - x')\log(a - x') + a^2\log 2a + (a - x')\tan\left(\beta - \frac{\pi}{2}\right) \; . \tag{18}$$

For $\delta = \pi$, (13) becomes

$$a = \frac{2}{\pi} \; . \tag{19}$$

Vanden-Broeck and Keller[1] have shown that the approximate solution (17), (18) is in fair agreement with the exact numerical solution of (8) and (9) for $\beta \sim \frac{1}{2}\pi$ and $\gamma \sim \gamma_0(\beta,\pi)$.

For $\gamma < \gamma_0(\beta,\delta)$ (14)-(16) yield unphysical profiles in which opposite sides of the bubble cross over. In the next two sections we construct physically acceptable families of solutions for $\gamma < \gamma_0(\beta,\delta)$. We shall present these results in the important particular case $\delta = \pi$.

4. Solution with one point of contact

To obtain solutions for $\gamma < \gamma_0(\pi,\beta)$ we require the free surface to be in contact with itself at one point. Then the bubble contains a small sub-bubble near its tip (see Figure 2). We denote by c the x' coordinate of the contact point.

We describe the profile of the bubble by the equations $y' = \eta_1(x')$ $0 < x' < c$ and $y' = \eta_2(x')$ $c < x' < a$. Then by symmetry we have

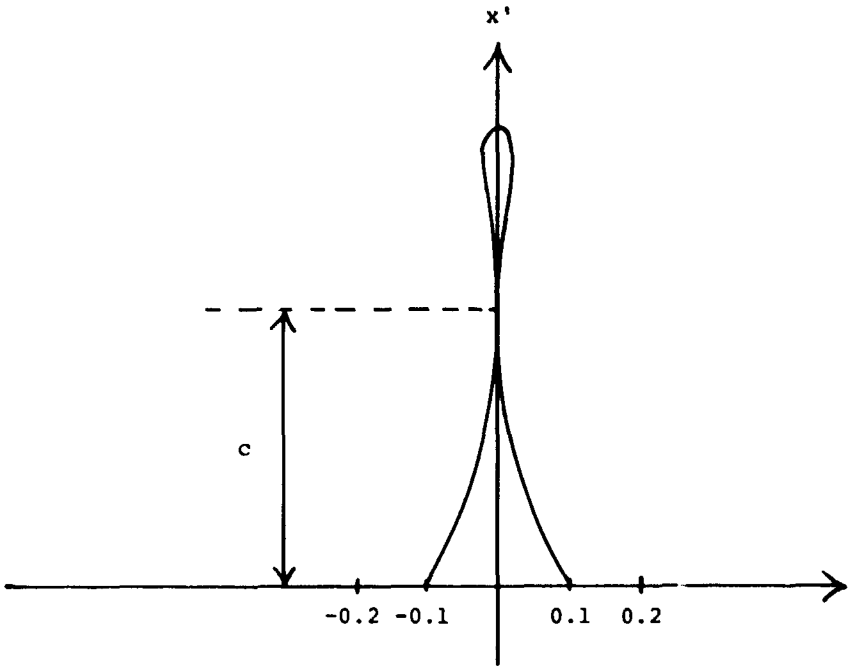

Figure 2. Profile of the bubble with one point of contact for $\gamma = \gamma_0 \sim -1.7$ and $\beta = 2\pi/3$. The vertical scale is the same as the horizontal scale. The cavitation number in the sub-bubble is equal to γ_0.

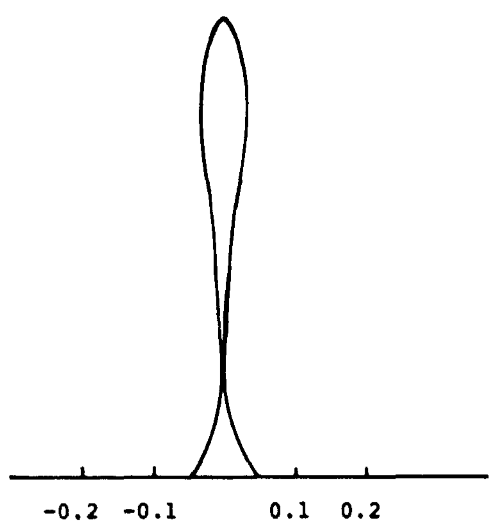

Figure 3. Profile of the bubble with one point of contact for $\gamma = -4.5$ and $\beta = 2\pi/3$. The vertical scale is the same as the horizontal scale. The cavitation number in the sub-bubble is $\mu = -0.6$.

$$\eta_1(c) = \eta_2(c) = 0 \tag{20}$$

$$\eta_1'(c) = \eta_2'(c) = 0 \ . \tag{21}$$

The conditions (15) and (16) yield

$$\eta_2(a) = 0 \tag{22}$$

$$\eta_1'(0) = -\tan(\beta - \frac{\pi}{2}) \ . \tag{23}$$

The functions $\eta_1(x')$ and $\eta_2(x')$ are obtained by integrating (14) twice. The four constant of integrations and the value of c have to be evaluated to satisfy the six conditions (20)-(23). This yields a system of six equations with five unknowns. Therefore we cannot expect this system of equations to have a solution for any value of γ other than $\gamma_0(\beta,\pi)$.

The physical reason why the problem does not have a solution for $\gamma \neq \gamma_0$ is that it requires the cavitation number in the sub-bubble to be the same as in the main bubble. It is to be expected that the cavitation number within the sub-bubble will have some value other than γ, which we cannot prescribe. Following the general philosophy of the method used by Vanden-Broeck and Keller[3] we shall introduce the unknown cavitation number μ in the interval $c < x' < a$.

Integrating (14) twice we obtain

$$\eta_1(x') = (a^2 - x'^2)\frac{\gamma - 1}{2} - \frac{1}{2}a(a + x')\log(a + x')$$
$$- \frac{1}{2}a(a - x')\log(a - x') + A + Bx' \ , \tag{24}$$

$$\eta_2(x') = (a^2 - x'^2)\frac{\mu - 1}{2} - \frac{1}{2}a(a + x')\log(a + x')$$
$$- \frac{1}{2}a(a - x')\log(a - x') + E + Dx' \ . \tag{25}$$

Here A, B, E and D are the four constants of integration. Using the six conditions (20)-(23) we obtain a system of six algebraic equations for the six unknowns A, B, E, D, μ and c. This system can easily be solved and yields a unique solution for any γ in the interval $-\infty < \gamma < \gamma_0(\beta,\pi)$. Typical profiles for $\beta = \frac{2\pi}{3}$ are shown in Figures 2 and 3. The value of γ_0 is approximately equal to -1.7. As γ decreases the size of the sub-bubble increases and the size of the main bubble decreases. For $\gamma = -\infty$, $\mu = -0.39$ and the main bubble vanishes. It is interesting to note that the present solution also exists in the interval $\gamma_0 < \gamma < \gamma^*$. Here γ^* is the value of γ for which $\eta_1''(c) = 0$. A similar result was found by Vanden-Broeck and Keller[3].

The results are summarized in Figure 4. The solution before contact described in Sections 2 and 3 correspond to the interval $\gamma_0 < \gamma < \infty$. It is represented by the straight line $\mu = \gamma$ in Figure 4. The other curve in Figure 4 corresponds to the present solution. It exists in the interval $-\infty < \gamma < \gamma^*$. Therefore there are two possible solutions in the interval $\gamma_0 < \gamma < \gamma^*$.

5. Solution with an interval of contact

In this section we derive another solution for $\gamma < \gamma_0(\beta,\pi)$ by requiring the bubble to be collapsed between $x' = f$ and $x' = g$ (see Figure 5). We describe the profile of the bubble by the equations $y' = \eta_1(x')$, $0 < x' < f$ and $y' = \eta_2(x')$, $g < x' < a$. The functions $\eta_1(x')$ and $\eta_2(x')$ must satisfy the following conditions

$$\eta_2(a) = 0 \tag{26}$$

$$\eta_1'(0) = -\tan(\beta - \frac{\pi}{2}) \ , \tag{27}$$

$$\eta_1(f) = \eta_2(q) = 0 \ , \tag{28}$$

$$\eta_1'(f) = \eta_2'(q) = 0 \ . \tag{29}$$

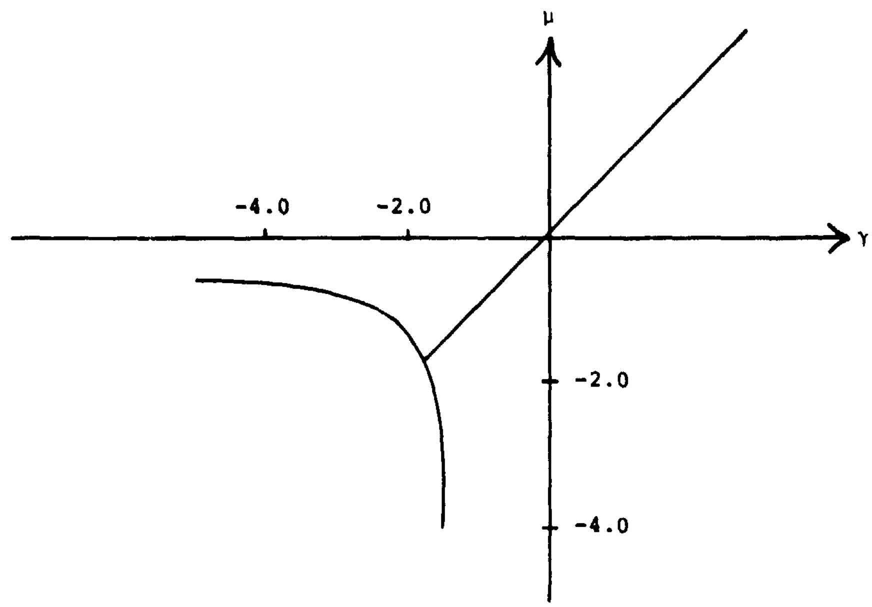

Figure 4. The cavitation number μ as a function of γ.

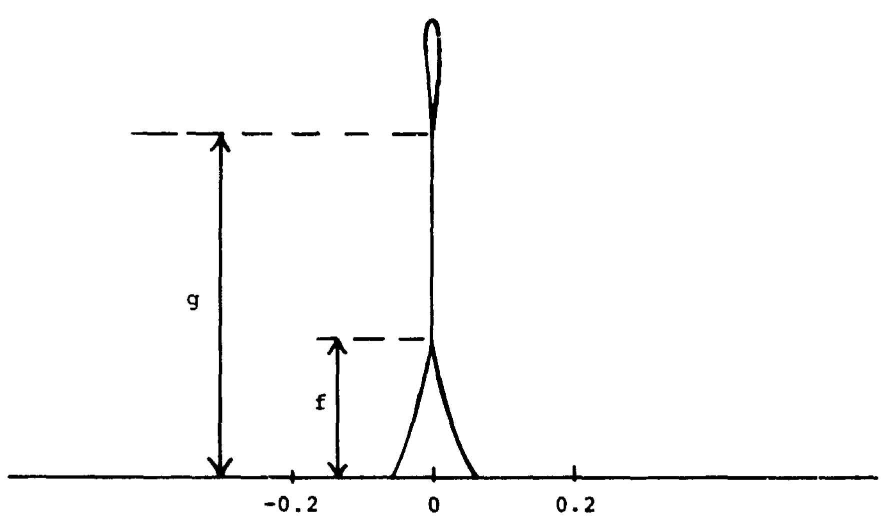

Figure 5. Profile of the bubble with one segment of contact for $\gamma = -3.0$ and $\beta = 2\pi/3$. The vertical scale is the same as the horizontal scale. The values of f and g are respectively 0.19 and 0.47.

The functions $\eta_1(x')$ and $\eta_2(x')$ are obtained by integrating (14) twice. They are therefore given by the relations (24) and (25). The six constants A, B, E, D, g and f are found by satisfying the six conditions (26)-(29). We note that the present solution can be found with the same cavitation number everywhere.

A typical profile for $\beta = \frac{2\pi}{3}$ is shown in Figure 5. As γ decreases the sizes of the main bubble and of the sub-bubble decrease. Furthermore the length of the contact segment increases as γ decreases. For $\gamma = -\infty$, the bubble reduces to a straight segment of length a lying on the x' axis.

Finally let us mention that the equilibrium of forces require the segment of contact to be a "film of impurities" characterized by a surface tension equal to 2σ. This is very unlikely to occur in reality. Therefore the bubble with a segment of contact is physically unrealistic. However, this mathematical solution is physically relevant to describe the deformation of an inflated membrane. For details see Vanden-Broeck[4].

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