

N82 23468

A variational approach to the study of capillarity phenomena

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Abstract

We consider the problem of determining the free surface of a liquid in a capillary tube, and of a liquid drop, sitting first on a horizontal plane and then on more general surfaces. With some modifications, the method applies to the study of pendent drops and of rotating drops as well.

Introduction

Several capillarity phenomena, such as the rise of water in tubes of narrow bore, and the formation of liquid drops or bubbles, can conveniently be studied from the general point of view of the Calculus of Variations. Such a possibility, which clearly originates in the energy-minimizing character of the observed equilibrium configurations, has the remarkable advantage of providing a unified treatment of the mathematical questions that arise in a variety of particular phenomena.

By using a well-known argument, based on the principle of virtual work, one is led to a variational formulation of the physical problem, in which a certain functional (representing the global energy of the system under consideration) has to be minimized, subject to some "natural" constraints, such as prescribed boundary conditions or fixed volume constraints. In general, the energy functional will consist of a "surface integral" plus a "volume integral": the latter corresponds to body forces, of which gravity is a typical representative, while the former results, for example, from the consideration of the forces acting on the surface of separation between the liquid and the gas surrounding it.

Now, the point is, that the classical definition of "surface area" is rather inadequate for treating this type of problem, mainly because it applies to smooth or Lipschitz-continuous surfaces only - a class which is not closed under the usual limit operations.

The difficulties arising from the presence of a surface integral become even more evident when compared with the relatively simple treatment of the corresponding volume integral, which is generally well-defined on measurable sets and enjoys (at least in the simplest cases) nice variational properties.

A satisfactory theory of surface area for a general class of surfaces of codimension one in \mathbb{R}^n , $n \geq 2$, has been developed by E. De Giorgi in the fifties.¹ It is a remarkable fact that some classical questions, concerning the existence and regularity of capillary surfaces, have been answered only quite recently, using the variational techniques introduced by De Giorgi, or even more general methods pertaining to the field of Geometric Measure Theory.²

The definition of "functions of Bounded Variation", together with the main results of the corresponding BV-functions theory, will be recalled in the next section. As a first application of the theory, we shall discuss in section 2 the "standard" capillary problem, i.e. the determination of the free surface of a liquid in a thin tube of general cross section, which results from the simultaneous action of surface tension, boundary adhesion and gravity. It turns out that in this case the existence of the solution surface depends heavily on the validity of a simple geometric condition about the mean curvature of the boundary curve of the cross section of the capillary tube. Some particular examples of physical interest will also be discussed.

Section 3 is devoted to the study of liquid drops, sitting on, or hanging from, a fixed horizontal plane. The symmetry of the solutions (which can actually be proved, as a consequence of a general symmetrization argument) now plays the chief role in deriving both the existence and the regularity of energy-minimizing configurations. When symmetry fails (this is the case, for example, when the "contact angle" between the drop and the plate is not constant, or when the supporting surface is not itself symmetric), then more sophisticated methods must be used. Extensions in this direction will be outlined in section 4.

We refer to the papers listed in the (fairly incomplete) bibliography at the end of the present paper for a deeper treatment of the subject, as well as for the discussion of related problems.

Functions of bounded variation

Given an open subset Ω of \mathbb{R}^n , $n \geq 2$, we denote by $BV(\Omega)$ the function space of Lebesgue integrable functions f over Ω , whose distributional gradient Df is a vector measure with finite total variation on Ω ; $\int_A |Df|$ will denote the total variation of the measure Df , evaluated at $A \subset \Omega$.

When A is open we obtain:

$$\int_A |Df| = \sup \left\{ \int f(x) \operatorname{div} G(x) dx : G \in C_0^1(A; \mathbb{R}^n), |G| \leq 1 \right\}$$

As a first result we can prove the lower semicontinuity of the map $f \rightarrow \int |Df|$, with respect to the local convergence on Ω ; that is, if $f_j \in BV(\Omega)$ for every j and

$$\lim_{j \rightarrow +\infty} \int_A |f_j - f| dx = 0$$

for every $A \subset\subset \Omega$ (i.e., A open and bounded with $\bar{A} \subset \Omega$), then:

$$\int_{\Omega} |Df| \leq \liminf_{j \rightarrow +\infty} \int_{\Omega} |Df_j|.$$

Furthermore, we have the following compactness property: if $f_j \in BV(\Omega)$, and for every $A \subset\subset \Omega$ and for every j it holds

$$\int_A |Df_j| \leq c(A)$$

with $c(A)$ independent of j , then there exists a subsequence of $\{f_j\}$, locally converging in Ω to some limit function f .³

If $f \in BV(\Omega)$ and Ω has a Lipschitz-continuous boundary $\partial\Omega$, then we can define the "trace" of f on $\partial\Omega$ (still denoted by f), which is summable on $\partial\Omega$ and satisfies⁴

$$\int_{\partial\Omega} |f| \leq (1+L^2)^{1/2} \int_{\Omega} |Df| + c(\Omega) \int_{\Omega} |f| \quad (1)$$

where L (the Lipschitz constant of $\partial\Omega$) and $c(\Omega)$ depend only on the geometry of Ω but not on f .

By specializing the above definitions and properties to the case when f is the characteristic function ϕ_E of a measurable set $E \subset \Omega$, we get a parallel theory of sets of finite perimeter, where by definition:

$$\text{"perimeter of } E \text{ in } \Omega" = \int_{\Omega} |D\phi_E|$$

A straightforward application of the Gauss-Green theorem shows that this quantity coincides with the area of $\partial E \cap \Omega$, at least when ∂E is a smooth $(n-1)$ -dimensional surface in Ω . The connection between BV-functions and sets of finite perimeter is given by the coarea formula:

$$\int_{\Omega} |Df| = \int_{-\infty}^{+\infty} dt \int_{\Omega} |D\phi_{F_t}| \quad (2)$$

where $f \in BV(\Omega)$ and $F_t = \{x \in \Omega : f(x) < t\}$.³

Moreover⁵, if $E = \{(x, t) : x \in \Omega, t < f(x)\}$, then it holds

$$\int_{\Omega \times \mathbb{R}} |D\phi_E| = \int_{\Omega} \sqrt{1 + |Df|^2}$$

where the second integral represents the total variation on Ω of the vector measure, whose $n+1$ components are respectively the Lebesgue measure on \mathbb{R}^n , and the distribution derivatives $D_i f$, $i=1, \dots, n$. When f is Lipschitz-continuous on Ω , this yields of

course the area of the graph of f over Ω .

Capillary surfaces in cylindrical vertical tubes

Let us consider a capillary tube, open at both ends and partially immersed into a liquid; for simplicity, we may assume that the liquid rises in the tube, so that - in view of the preceding discussion - the energy corresponding to a certain configuration of the liquid within the tube, described by the graph a function $f \geq 0$, can be expressed as

$$\mathcal{J}(f) = \int_{\Omega} \sqrt{1+|Df|^2} - \nu \int_{\partial\Omega} f + \frac{\kappa}{2} \int_{\Omega} f^2 \quad (3)$$

Here, Ω (open and bounded in \mathbb{R}^2 , with Lipschitz boundary $\partial\Omega$) denotes the cross section of the tube, $f \in BV(\Omega)$, and κ, ν are physical constants, with $\kappa > 0$ and $0 < \nu \leq 1$. It is easily seen that $f \equiv 0$ is the trivial solution of $\mathcal{J}(\cdot) \rightarrow \min$, when $\nu \leq 0$, while $\inf \mathcal{J}(\cdot) = -\infty$ when $\nu > 1$.

We can immediately check that a configuration of minimal energy (i.e., a solution to the problem $\mathcal{J}(\cdot) \rightarrow \min$) satisfies the equations

$$\begin{cases} \operatorname{div} Tf(x,y) = \kappa f(x,y) & \text{in } \Omega \\ Tf(x,y) \cdot n(x,y) = \nu & \text{on } \partial\Omega \end{cases}$$

where $Tf = \frac{Df}{\sqrt{1+|Df|^2}}$, and $n(x,y)$ denotes the outward unit normal to $\partial\Omega$ at (x,y) .

This is true if, for example, $\partial\Omega$ is of class C^1 and $f \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

This way, we realize that the mean curvature of a capillary surface is, at any point $(x,y,f(x,y)) \in \Omega \times \mathbb{R}$, proportional (with constant κ) to its height above the reference plane $z = 0$, and that ν corresponds to the cosine of the contact angle between the surface and the walls of the tube.

The classical approach consisted in solving the above system of equations in the special case when Ω was a disc of radius R , and $f = f(r)$, $r = (x^2 + y^2)^{1/2}$ (axially symmetric solutions). In this case, one is led to the ordinary differential equation

$$\left(\frac{rf_r}{\sqrt{1+f_r^2}} \right)_r = \kappa rf(r) \quad \text{for } 0 < r < R$$

with the boundary conditions: $f_r(0) = 0$, $f_r(R) = \nu \cdot (1-\nu^2)^{-1/2}$.

In order to prove the existence of a solution f to the problem $\mathcal{J}(\cdot) \rightarrow \min$, we make the following assumption on the domain Ω :

$$\int_{\partial\Omega} \phi_E \leq \alpha \int_{\Omega} |D\phi_E| + \beta \int_{\Omega} \phi_E \quad (4)$$

for every $E \subset \Omega$, with $\alpha \geq 1$ and $\beta > 0$. From (4) and the coarea formula (2) we get

$$\int_{\partial\Omega} f \leq \alpha \int_{\Omega} |Df| + \beta \int_{\Omega} f$$

for every $f \in BV(\Omega)$ with $f \geq 0$. By introducing this last inequality in (3) we find:

$$\mathcal{J}(f) \geq (1-\nu\alpha) \int_{\Omega} |Df| + \frac{\kappa}{4} \int_{\Omega} f^2 - \frac{\nu^2 \beta^2}{\kappa} \cdot |\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω . In particular, if $\nu \leq 1/\alpha$ we obtain $\inf \mathcal{J}(\cdot) \geq -\nu^2 \beta^2 |\Omega| / \kappa$, while if $\nu > 1/\alpha$, then for any minimizing sequence f_j , satisfying $\mathcal{J}(f_j) \rightarrow \inf \mathcal{J}(\cdot)$, we obtain:

$$\int_{\Omega} |Df_j| \leq \frac{\text{const.}}{1-\nu\alpha}, \quad \int_{\Omega} f_j^2 \geq \text{const.}$$

so that, in view of the results in section 1, we conclude that a subsequence of $\{f_j\}$ converges to a function f , which clearly solves our problem; as a consequence of the strict convexity of the energy functional, such a solution is in fact unique.

The hypothesis (4) now comes into discussion: clearly, it is implied by the trace estimate (1), so that capillary surfaces always exist for v in the range $0 < v < (1+L^2)^{-1/2}$, with L = Lipschitz constant of $\partial\Omega$; that this is in fact an "almost necessary" condition can easily be seen with the aid of some simple examples.⁷ For a domain Ω in the form of a circular sector, no solution with bounded energy can exist when $v > (1+L^2)^{-1/2}$ (that is, when $\theta + 2\gamma < \pi$, where θ is the angle of the sector and γ is the contact angle). The discontinuity at $\theta + 2\gamma = \pi$ is also confirmed by physical experiments.⁷

On the other hand,⁶ interior corners do not affect the solution: in the special case when Ω satisfies an Internal Sphere Condition of radius $R > 0$ (i.e., when each point in Ω belongs to some ball $B_R \subset \Omega$), then (4) holds with $\alpha = 1$, and it can be shown, by using a result of Concus and Finn,⁶ that the solution exists for every v with $0 < v \leq 1$.

In conclusion, we remark that the variational method does not work when $\kappa = 0$, i.e. in the absence of gravity; one can actually show⁸ that the solutions f_κ , corresponding to values $\kappa > 0$, go uniformly to $+\infty$ in Ω , as $\kappa \rightarrow 0^+$.

Sessile drops and pendent drops

The energy of a liquid drop, sitting on the horizontal plane $\{z=0\}$ in \mathbb{R}^3 , can be written in the following way:

$$\mathcal{F}_\kappa(E) = \int_{z=0} |D\phi_E| + v \int_{z=0} \phi_E + \kappa \int_{z=0} z\phi_E \quad (5)$$

where E denotes the region of the half-space $\{z > 0\}$ occupied by the liquid. The first integral in (5) represents the area of the free boundary of the drop, the second integral gives the area of the region of contact, and the third integral corresponds to gravity. As usual, we assume $\kappa \geq 0$ and $v \in (-1, 1]$, since for $v = -1$ no solution can occur. The same functional, with $\kappa < 0$, represents the energy of a pendent drop; in both cases, a volume constraint has to be imposed, namely $|E| = v > 0$.

Now, it can be shown¹⁰ that by replacing each horizontal section of a given configuration E by a disc of the same area, centered on the z -axis, a new configuration E' results, which is of less energy than E (in fact, $\mathcal{F}_\kappa(E) = \mathcal{F}_\kappa(E')$ iff E is already axially symmetric).

From this fact, by using the obvious estimate

$$\int_{z=0} \phi_E \leq \int_{z=0} |D\phi_E|$$

which implies

$$\mathcal{F}_\kappa(E) \geq \frac{1+v}{2} \int_{z=0} |D\phi_E|$$

for every E , if $\kappa \geq 0$, one gets easily the existence of a minimum of $\mathcal{F}_\kappa(\cdot)$, when $\kappa \geq 0$. On the other hand, when $\kappa < 0$ the situation is completely different, and we can look only for local solutions of $\mathcal{F}_\kappa(\cdot) \rightarrow \min$, since clearly $\inf \mathcal{F}_\kappa(\cdot) = -\infty$ in this case.

To this aim, we introduce the following definition: E is a local minimum of the energy functional (a pendent drop) if $|E| = v$, and there exist $T > 0$ and $\alpha \in (0, 1)$ such that E is contained in the region $\{0 < z < \alpha T\}$ and, for every $F \subset \{0 < z < T\}$ with $|F| = v$, there holds $\mathcal{F}_\kappa(E) \leq \mathcal{F}_\kappa(F)$.

In order to prove the existence of pendent drops, we start by observing that, when $\kappa = 0$ (that is, in the absence of gravity), the minimum E_0 of $\mathcal{F}_0(\cdot)$ is a portion of a ball, completely determined from the data v and v . With this in mind, we choose T greater than the maximum height of E_0 and find the solution E_κ of the problem $\mathcal{F}_\kappa(\cdot) \rightarrow \min$, restricted to the configurations $E \subset \{0 < z < T\}$.

An easy calculation shows that $E_\kappa \rightarrow E_0$ as $\kappa \rightarrow 0$ as a consequence of general results concerning the convergence of surfaces of prescribed mean curvature, we get that for $|\kappa|$ small enough, the solution E_κ is actually contained in $\{0 < z < \alpha T\}$ for a suitable $\alpha \in (0, 1)$,¹¹ thus concluding the proof of the existence of pendent drops of given volume, in a weak gravitational field. Estimates on the effective smallness of $|\kappa|$ are also explicitly known.

For example, Giusti¹² showed that pendent drops exist, if the product $|\kappa|v^{2/3}$ does not exceed a constant, which can be written down explicitly, and which depends only on the value of the contact angle between the free surface of the drop and the plane $\{z=0\}$.

Concluding comments

The regularity (analyticity) of the equilibrium surface of a liquid in a capillary tube can be derived from general regularity results for hypersurfaces of least area or, more generally, of prescribed mean curvature in \mathbb{R}^n .¹³ The boundary behaviour of the solutions has also been studied, and several results in this direction are presently known.¹⁴

As far as the regularity of liquid drops is concerned, we note firstly that the free surface of a drop sitting on a horizontal plane, being rotationally symmetric, can also be described (locally) as the graph of a suitable function f , defined on a 2-dimensional domain. It turns out that f minimizes a functional of the type:

$$\int \sqrt{1 + |Df|^2} + \text{gravity} + \lambda f$$

where the Lagrange multiplier λ takes into account the volume constraint. From this, the interior analyticity of the free surface of the drop follows at once.¹⁵

Secondly, it is not difficult to prove that the configuration representing a sessile drop is a convex set.¹⁶ We then conclude that its free surface is in fact smooth up to the plane $\{z=0\}$, and that the cosine of the contact angle coincides with ψ .

Much more difficult is the study of liquid drops, when symmetry ceases to exist. However, some partial results have recently been obtained, such as the existence and regularity of solutions, corresponding to drops sitting on (or hanging from) a surface $z = \psi(x,y)$ in \mathbb{R}^3 , satisfying $\psi(r) \rightarrow +\infty$ for $r = (x^2 + y^2)^{1/2} \rightarrow +\infty$. From this "growth condition at infinity", which however is not satisfied in a number of interesting and still open situations, the necessary compactness results can easily be derived.¹⁷

A theorem, which extends to minima of area-like functionals, subject to a volume constraint, the regularity theory for minimal boundaries, is now available.^{17,18} A technique, developed in connection with its proof, can also be used to prove the existence of liquid drops, in rotation around an axis through the center of mass, and held together by surface tension.

In its simplest formulation, the problem asks for a local minimum of the functional

$$\mathcal{F}_\omega(E) = \int_{\mathbb{R}^3} |D\phi_E| - \omega \int_{\mathbb{R}^3} (x^2 + y^2) dx dy dz, \quad \omega > 0 \quad (6)$$

subject to a volume constraint ($|E| = 1$) and to a further constraint about the center of mass (which must coincide with the origin of the space).

The presence of the kinetic energy excludes, in general, the symmetry of the solutions. Anyway, following the treatment of the pendent drop problem, one can show¹⁹ the existence of relative minima of the energy functional, when ω is small enough, that is, when the rotation is sufficiently slow.

The proof of this result, which can only be outlined here, proceeds as follows: firstly, we define E to be a local minimum of the energy functional (6), if there exists $R > R_0 = (3/4\pi)^{1/3}$ such that $E \subset\subset C_R$ and, for every admissible $F \subset\subset C_R$, satisfying the above constraints, it holds $\mathcal{F}_\omega(E) \leq \mathcal{F}_\omega(F)$.

Here, C_R denotes the cylindrical container

$$C_R = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq R^2, |z| \leq R\}.$$

Then, we can easily prove that for every $\omega \geq 0$, and every $R > R_0$, there exists a solution E_ω to the problem $\mathcal{F}_\omega(E) \rightarrow \min$, restricted to the admissible configurations $E \subset C_R$. Such a solution, however, will not generally satisfy the condition $E_\omega \subset\subset C_R$ for a local minimum.

Next, we observe that for a fixed $R > R_0$, the solutions E_ω we have found in this way, converge, as $\omega \rightarrow 0^+$, to the ball $E_0 = \{(x,y,z) : x^2 + y^2 + z^2 < R_0^2\} \subset\subset C_R$. The local convergence $E_\omega \rightarrow E_0$ is unfortunately too weak to conclude that $E_\omega \subset\subset C_R$ for ω small enough. But we can prove, and this is actually the crucial point of the entire demonstration, that if ω is sufficiently small (and positive), then there exists a value $r \in (R_0, R)$ such that

$$\int_{\partial C_R} \phi_{E_\omega} = 0$$

From this result, the fact that $E_\omega \subset\subset C_R$ for ω small can be proved as follows. Define:

$$G = E_\omega \cap C_r \quad \text{and} \quad \alpha = |E_\omega - C_r|$$

and choose $F = (1-\alpha)^{-1/3}G$ (that is, F is constructed by expanding G , with a coefficient $(1-\alpha)^{-1/3} > 1$). Clearly, $|F| = 1$. Since $\alpha \rightarrow 0$ as $\omega \rightarrow 0$, it is clear that when ω is small, by means of a suitable translation $F \rightarrow \tilde{F}$ we obtain eventually an admissible configuration $\tilde{F} \subset C_R$, whence

$$\mathcal{F}_\omega(E_\omega) \leq \mathcal{F}_\omega(\tilde{F}) \quad (7)$$

On the other hand, the difference $\mathcal{F}_\omega(E_\omega) - \mathcal{F}_\omega(\tilde{F})$ can easily be estimated from below by the quantity:

$$\left[1 - (1-\alpha)^{-2/3} \right] \int_{\mathbb{R}^3} |D\phi_G| + \text{const.} \alpha + \text{const.} \alpha^{2/3}$$

Now, from the Taylor expansion of the coefficient between square brackets, we conclude that $\mathcal{F}_\omega(E_\omega) - \mathcal{F}_\omega(\tilde{F}) > 0$ if α is positive and small, which would contradict (7). Therefore, for ω small enough, α must be 0, so that the corresponding solution E_ω is contained in $C_R \subset C_R$.

In conclusion, we remark that the preceding argument, when used in connection with general regularization techniques,²⁰ allows the proof of the analyticity of the solution E_ω as well.

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