N83 17372 D/3

A RECURSIVE ALGORITHM FOR ZERNIKE POLYNOMIALS

by

John W. Davenport
Associate Professor of Mathematics
and Computer Science
Georgia Southern College
Statesboro, Georgia

ABSTRACT

Many applications in optics, such as the diffraction theory of optical aberrations, involves the analysis of a function defined on a rotationally symmetric system, with either a circular or annular pupil. In order to numerically analyze such systems it is typical to expand the given function in terms of a class of orthogonal polynomials. Because of their particular properties, the Zernike polynomials are especially suited for numerical calculations. We develop a recursive algorithm that can be used to generate the Zernike polynomials up to a given order. The algorithm is recursively defined over J where R(J,N) is the Zernike polynomial of degree N obtained by orthogonalizing the sequence r^J, r^{J+2}, . . . , r^{J+2N} over $(\varepsilon, 1)$. The terms in the preceding row - the (J-1) row - up to the N+1 term is needed for generating the $(J,N)^{th}$ term. Thus, the algorith generates an upper left-triangular table. This algorithm has been placed in the computer with the necessary support program also included.

ORIGINAL PAGE IS OF POOR QUALITY

INTRODUCTION

An arbitrary function $W(r,\Theta)$, such as a wavefront error function over a circular or annular region, can be expanded in terms of an orthonormal series of orthogonal polynomials. If W is defined over a circular or annular region, it is convenient to expand W in terms of the Zernike polynomials, $Z_n^{\,l}$. This

$$W = \sum_{n,l = -\infty}^{\infty} A_{nl} Z_n^{\,l}.$$

It can be shown {3} that $Z_n^{\,l} = R_n^{\,l}(r)e^{il\Theta}$ where $R_n^{\,l}$ depends only on the radial coordinate and $e^{il\Theta}$ depends only on the angular coordinate. Also, l is the minimum exponent of the polynomials $Z_n^{\,l}$ and $R_n^{\,l}$ and the numbers n and l are either both even or both odd. The radial polynomials $R_n^{\,l}$ are of degree n and satisfy the relation

$$R_n^{\,l} = R_n^{-l} = R_n^{|l|}$$

If we write, using only the real part,

$$W\ (r,\Theta) = \sum_{j=o}^{\infty} \sum_{n=o}^{\infty} A_{nj} \ R^{2}{}_{jn+2} \begin{matrix} \text{Sin } j\ \Theta \\ \text{Cos } j\ \Theta \end{matrix}$$

the following properties are satisfied.

1. The Zernike polynomials are invarient in form with respect to rotations of axes about the origin {3}.

2. The Zernike polynomials are easily related to the classical aberrations {4}.

3. The function $W(r,\Theta)$ is usually found as a best least-squares fit to a collection of data points. Since the Zernike polynomials are orthogonal over an annular region, the well-known minimum-error property of Fourier expressions shows that each term

$$A_{nj} \, R^{J}_{jn+2} \quad \begin{matrix} \sin J\Theta \\ \cos J\Theta \end{matrix}$$

also represents individually a best least-squares fit to the data. Thus, the average amount of each term is given by the magnitude of that term, without the need to do a new least-squares fit.

Because the Zernike polynomials are being applied to an increasing number of physical problems {1,2,5,6}, there is an expressed interest in being able to generate the Zernike polynomials up to a given order. In this paper, we develop a numerical method due to Tatian {8} for generating the polynomials R^{J}_{2n+J} over an annular region. A discussion of the derivation of the algorithm is presented in section 2 and a general discussion of the computer program that was written to facilitate this algorithm is discussed in section 3.

Bhatia and Wolf {3} have shown that the polynomials $R^{J}_{2n+j}(r)$ are obtained by orthogonalizing the functions $r^{J}, r^{J+2}, \ldots, r^{J+2n}$ over the interval $\{\varepsilon,1\}$. The constant ε represents the inner radius of the annular region and 1 represents the outer radius. Thus $\varepsilon = o$ represents a circular region. We convert this into the associated problem of orthogonalizing $1, u, \ldots, u^n$ over $[\varepsilon^2, 1]$ with the weight function u^j by substituting $u = r^2$. This follows from

$$\langle r^{J+2l}, r^{J+2k} \rangle = \int_{\varepsilon}^{1} r^{J+2l}\, r^{J+2k} \quad rdr$$

$$= \int_{\varepsilon}^{1} r^{2J+2l+2K} \; rdr = \int_{\varepsilon}^{1} (r^2)^{J+l+k} \; rdr$$

$$= \frac{1}{2} \int_{\varepsilon^2}^{1} u^{j+l+k} \, du = \frac{1}{2} = \int_{\varepsilon}^{1} u^l \, u^k \, (u^j du)$$

First, let us consider the case $J = o$. Then $R_{2n}^{o}(r) = R_{n}^{o}(u)$ is obtained by orthogonalizing the sequence $1, u, \ldots, u^n$ over $[\varepsilon, 1]$ with the weight function 1. This however is a Jacobic problem over the shifted interval $[\varepsilon^2, 1]$. Hence,

(1) $R_{2n}^{o}(r) = R_{n}^{o}(u) = p_{n}^{(o,o)}$ (2) $\dfrac{r^2 - \varepsilon^2 - 1}{1 - \varepsilon^1}$

where $p_n^{(o,o)}$ is the Legendre polynomial of degree n.

ORIGINAL PAGE IS
OF POOR QUALITY

Now, let us consider the case when $J \neq o$. We will show that $R^J{}_{2n+2}$ can be obtained by a recursive method over the variable J. Suppose we have solved the problem of obtaining $Q_n^{J-1}(u)$ i.e. orthogonalize $1, u, \ldots, u^n$ over $\{\varepsilon^2, 1\}$ with weight function u^{J-1} and we want to obtain $Q^J{}_n(u)$; orthogonalize $1, u, \cdots u^n$ over $[\varepsilon^2, 1]$ with weight function u^J. We obtain a relationship between these 2 polynomials by making use of a special case of the following theorems {7}.

Theorem 1. Let $\{p_n(x)\}$ be the orthonormal polynomials associated with the distribution $d\alpha(x)$ on the interval $\{a,b\}$. Also, let

$$p(x) = c \cdot (x - x_1) \cdot (x - x_2) \ldots (x - x_1)$$

be non negative in this interval. Then the orthogonal polynomials $\{q_n(x)\}$ associated with the distribution $p(x)\, d\alpha(x)$ can be represented in terms of the polynomials $p_n(x)$ as follows:

$$p(x)\, q_n(x) = \begin{vmatrix} Pn(x) & Pn+1(x) & \cdots Pn+1(x) \\ Pn(x_1) & Pn+1(x_1) & \cdots Pn+1(x_1) \\ & \vdots & \\ Pn(x1) & Pn+1(x1) & \cdots Pn+1(x_e) \end{vmatrix}$$

Theorem 2. The following relation holds for any 3 consecutive orthogonal polynomials:

$$p_n(x) = (A_n X + B_n)\, p_{n-1}(x) = C_n \cdot p_{n-2}(x)$$

of the highest coefficient of $p_n(x)$ is denoted by k_n, we have

$$A_n = \frac{k_n}{k_{n-1}} \quad \text{and} \quad C_n = \frac{A_n}{A_{n-1}}$$

Theorem 3.

$$p_o(x)\, p_o(y) + p_1(x)\, p_1(y) + \cdots p_n(x)\, p_n(y)$$

$$= \frac{k_n}{k_{n+1}} \; \frac{p_{n+1}(x)\, p_n(y) - p_n(x)\, p_{n+1}(y)}{x-y}$$

using Theorem 1 with $p_n(u) = Q_n^{J-1}(u)$, $q_n(u) = Q_n^J(u)$ and $p(u) = u$, we have

$$2.1 \quad u_Q{}^J(u) = \frac{1}{Q_n^{J-1}(o)} \begin{vmatrix} Q_n^{J-1}(u) & Q_{n+1}^{J-1}(u) \\ Q_n^{J-1}(o) & Q_{n+1}^{J-1}(o) \end{vmatrix}$$

ORIGINAL PAGE IS
OF POOR QUALITY

We note that this formula becomes undeterminate for $u = r^2 = o$. But we need the values of Q_n^J at $u = o$ to calculate the next set of polynomials, so expand equation 2.1 and use Theorem 3 to get

$$(2) \quad Q_n^J(u) = \frac{h_n^{J-1}}{(1-\varepsilon^2)\; Q_n^{J-1}(o)} \sum_{i=o}^{n} \frac{Q_i^{J-1}(o)\; Q_i^{J-1}(u)}{h_i^{J-1}}$$

where

$$(3) \quad h_n^J = \frac{1}{2} \int_{\varepsilon^2}^{1} \{Q_n^J(u)\}^2 u^J \, du$$

is the normalization constant for $Q_n^J(u)$. By substituting the expression in equation 1 for $Q_n^J(u)$ into equation 3 and using Theorem 3, we obtain

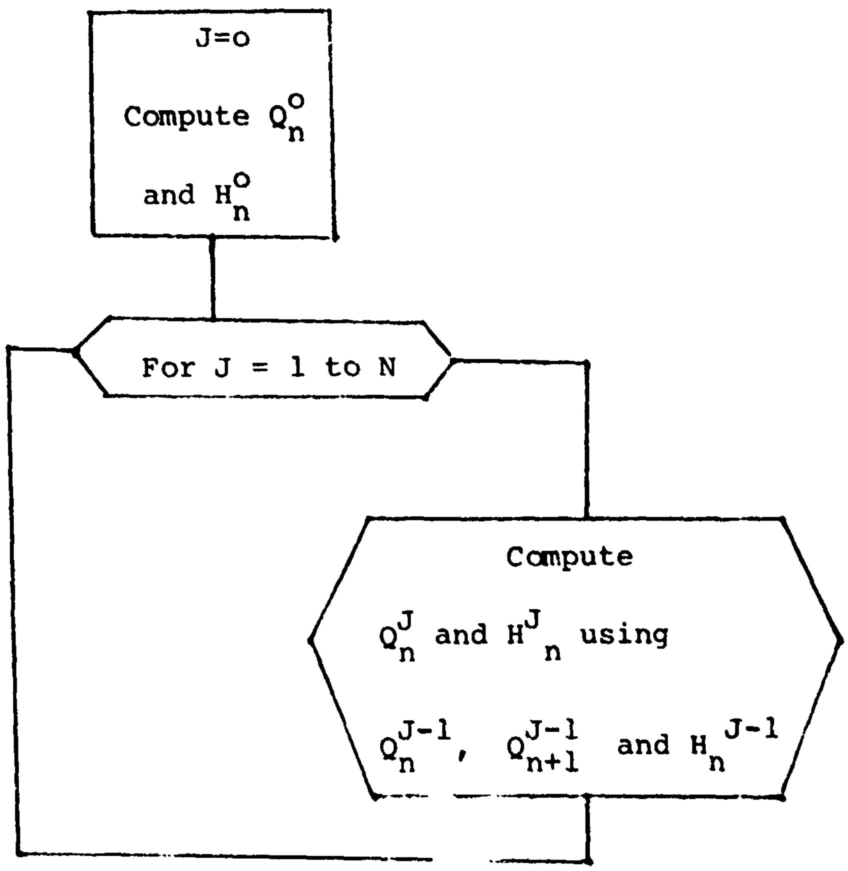
$$(4) \quad h_i^J = \frac{-1}{1-\varepsilon^2} \cdot \frac{Q_{i+1}^{J-1}(0)}{Q_i^{J-1}(0)} h_i^{J-1}$$

converting back to the variable r, we have

$$(5) \quad R_{2n+J}^J(r) = r^J \; Q_n^J(r^2).$$

3. Some comments about the computer program.

The outline for this computer program is:



For a given value of N, the following table shows the values of Q_n^J and H_n^J that are generated.

TABLE 1

Q_n^J or H_n^J	0	1	2	3	·	·	·	N-2	N-1	N
0	*	*	*	*				*	*	*
1	*	*	*	*				*	*	
2	*	*	*	*				*		
3										
·										
·										
·										
N-2	*	*	*							
N-1	*	*								
N	*									

NOTE: Since H_n^J is dependent upon knowing Q_{n+1}^{J-1}, each of the above rows are shorter by 1 entry.

The polynomials Q_n^o are directly computed from forumla (1) namely,

$$Q_n^o \quad (u) = p_n(o,o) \quad (\alpha u + \nu)$$

where $p_n^{(o,o)}$ is the Legendre polynomial of degree n, $\alpha=2/(1-\varepsilon^2)$ and $\nu = - (1 + \varepsilon^2)/(1 - \varepsilon^2)$. Likewise, the constants H_n^o are directly computed from (3); namely,

$$H_n^o = \frac{1}{2} \int_{\varepsilon^2}^{1} \left(Q_n^o \; (u)\right)^2 \; du.$$

NOTE: The above integral is computed in closed form. This is possible because Q_n^o (u) is a polynomial of degree n. This is done via a call to SQPOLY (square the polynomial) and a call to INTGRL (find the integral of a polynomial).

Once the first row is known (Q_n^J and H_n^J for J = o), the recursive algorithm can then be used to compute each succeeding row (Q_n^J and H_n^J for $J = J_o$). The results are then printed via a call to RJN (compute $R_{2n+J}^J = r^2 \; Q_n^J \; (r^2)$.

4. Example

A computer run for n = 3 is shown below.

```
ENTER N-LARGEST R(Ø,N) DESIRED
?3
R(Ø,Ø)
    1.ØØØØ
* * * * * * * * * * * * * * * *

R(Ø, 2)
NORM:           3.83767639
     -.5487 +     1.ØØØØ*R** 2
* * * * * * * * * * * * * * * *

R(Ø, 4)
NORM:          16.4661363Ø
     .2331  -    1.Ø973*R**2 +    1.ØØØØ*R** 4
* * * * * * * * * * * * * * * * * * * * * * * *

R(Ø, 6)
NORM:           71.94691818
      -.Ø981  =     .78Ø9*R** 2   - 1.646Ø*R** 4 +   1.ØØØØ*R** 6
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

R(1, 1)
NORM:                1.9Ø923153
     1.ØØØØ*R** 1 +
* * * * * * * * * * * * * * * *

R(1, 3)
NORM:          8.32586779
      -6724*R** 1 +  1.ØØØØ*R**3
* * * * * * * * * * * * * * * * *

R(1, 5)
NORM:          35,89625267
      .3191*R** 1 - 1.2252*R** 3 +  1.ØØØØ*R** 5
* * * * * * * * * * * * * * * * * * * * * * * * *

R(2, 2)
NORM:            2.32829Ø44
     1.ØØØØ*R** 2 +
* * * * * * * * * * * * * *
```

```
R(2, 4)
NORM:            11.844605 83
     -.7897*R** 2 +   1.ØØØØ*R** 4
* * * * * * * * * * * * * * * * * *

R(3,3)
NORM:               2.6873584
     1.ØØØØ*R** 3 +
* * * * * * * * * * * * * * *


* STOP * Ø
```

A COPY OF THE ABOVE PROGRAM CAN BE OBTAINED FROM THE AUTHOR.

ORIGINAL PAGE IS
OF POOR QUALITY

PEFERENCES

1. Baraket, Richard, Optimum Balanced Wavefront Aberrations for Radially Symmetric Amplitude Disturbances; Generalizations of Zernike Polynomials, J. Opt. Soc. Am. 70, (1980).

2. Bezdid'ko, S. N., The Use of Zernike Polynomials in Optics, Sov. J. Opt. Technol., 41, No. 9, Sept., (1974).

3. Bhatia, A. B. and Wolf, E., On the Circle Polynomials of Zernike and Related Orthogonal Sets, Proc. Camb. Phil. Soc., 50, (1954).

4. Nigboer, B. R. A., The Diffraction Theory of Aberrations, Physica (Utrecht) 10, (1947).

5. Noll, R. J., Zernike Polynomials and Atmospheric Turbulence, J. Opt. Soc. Am. 66, (1976).

6. Sumita, H., Orthonormal Expansion of the Aberration Difference Function and its Application to Image Evaluation, Japanese J. Applied Phys., 8, No. 8, Aug. (1969).

7. Szego, G., Orthogonal Polynomials, Am. Soc. Colloquium Pub., v. 23, (1939).

8. Tatain, B., Aberration Balancing in Rotationally Symmetric Lenses, J. Opt. Soc. Am. 64, (1974).