83N19368

Reports of the Department of Geodetic Science and Surveying

Report No. 337

THE USE OF BASELINE MEASUREMENTS AND GEOPHYSICAL MODELS FOR THE ESTIMATION OF CRUSTAL DEFORMATIONS AND THE TERRESTRIAL REFERENCE SYSTEM

by

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Prepared for

National Aeronautics and Space Administration
Goddard Space Flight Center
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Grant No. NSG 5265
OSURF Project No. 711055

The Ohio State University
Research Foundation
Columbus, Ohio 43212

December, 1982

DEDICATION

To those who preceded me - Philip and Sara Bock

To the one who walks with me - Lydia

and

To those who will follow me - Jonathan and Elinor

PREFACE

This project is under the supervision of Professor Ivan I. Mueller, Department of Geodetic Science and Surveying, The Ohio State University. The Science Advisor is Dr. David E. Smith, Code 921, Geodynamics Branch, and the Technical Officer is Mr. Jean Welker, Code 903, Technology Applications Center, both at Goddard Space Flight Center, Greenbelt, Maryland 20771.

This report was originally presented as a dissertation to the Graduate School of The Ohio State University in partial fulfillment of the requirements for the PhD degree.

ABSTRACT

Four possible estimators are investigated for the monitoring of crustal deformations from a combination of repeated baseline length measurements and adopted geophysical models, particularly an absolute motion plate model. The first estimator is an extension of the familiar free adjustment. The next two are Bayesian type estimators, one weak and one strong. Finally, a weighted constraint estimator is presented. The properties of these four estimators are outlined and their physical interpretations discussed.

A series of simulations are performed to test the four estimators and to determine whether or not to incorporate a plate model for the monitoring of deformations. It is concluded that it is preferable to adopt even a weak but realistic model than none at all. In this case, the weak Bayesian estimator (Best Linear Estimator--BLE) is preferred. It filters the signal (deformations) from the measurement noise in an optimal manner and, furthermore, is not overly sensitive to the errors in the adopted geophysical deformation model.

The application of these estimations to the maintenance of a new conventional terrestrial reference system is discussed. The functions of the system are twofold. The first is to monitor the global rotations and translations of the earth with respect to an inertial frame. The second is to monitor the nonglobal motions or deformations of the earth. The relationship between these two functions is outlined.

ACKNOWLEDGMENTS

The time has finally come when I can sit down and reflect for a moment on the conclusion of a most rewarding period of my life, that spent at the Department of Geodetic Science and Surveying. It is rare to find an environment so conducive to study, scientific inquiry and exchange of ideas. It is therefore appropriate that I acknowledge first this institution as a major factor for the successful completion of this report. Of course, the whole is the sum of its parts and that means people. However, it is impossible to appropriately thank everyone within and outside the department so therefore I will have to limit myself to just a few.

First and foremost, I would like to thank my lovely wife Lydia for her constant support and encouragement, and for putting up with a part-time husband for the last five years. I plan to make it up to her. What makes our stay in Columbus even more rewarding was being able during this time to bring two beautiful children, Jonathan and Elinor, into this world.

Next, I would like to express my gratitude to my supervisor Prof. Ivan I. Mueller, for directing me towards the topic of this investigation and for providing useful suggestions. His guidance, support and patience are greatly appreciated. It was a pleasure to be associated with him. I would like to also thank him for the use of his excellent and well organized library. The other members of the review committee, Prof. Urho A. Uotila and Prof. Richard H. Rapp are to be thanked for their constructive criticisms. I would also like to express my appreciation to Dr. Burkhard Schaffrin for reading through the original manuscript very carefully and suggesting improvements.

I owe my colleagues at the Department of Geodetic Science and Surveying a special thanks for making my stay here a stimulating and rewarding one. First and foremost is Sheng Yuan Zhu who has since returned to Shanghai. The many hours that we spent talking about the subject of reference frames and other matters have greatly enriched me. Dr. Erricos C. Pavlis was always available for useful and long discussions as well as Dr. Lenny A. Krieg who also aided in making the computational effort less burdensome. I would also like to express my thanks to Dr. John Hannah, Dr. Christopher Jekeli, Mr. Edward M. Baker, and Mr. Brent A. Archinal for their help along the way as well as to many others too numerous to mention.

I would like to thank Prof. Armin Gruen for useful comments on estimation and to Prof. Joseph C. Loon, particularly for discussions during the last summer. I enjoyed talking with Dr. W. McWorter of the Dept. of Mathematics about polyhedra and with Dr. Angela M. Dean of the Dept. of Statistics about optimal design. I appreciate the useful comments of Dr. Erik W. Grafarend (Stuttgart University), Dr. Peter L. Bender (NBS and University of Colorado) and Dr. Jean B. Minster (S-Cubed, La Jolla, California).

I am very grateful for the financial support received as a Graduate Research Associate and Teaching Assistant of The Ohio State

University, and to the extensive computer support provided by the Instruction and Research Computer Center.

A great deal of thanks are to be extended to Ms. Irene Tesfai, Ms. Susan Carroll and Ms. Roberta Canegali for typing assistance and particularly to Ms. Linda Wright for her excellent typing of the final manuscript under a tight schedule.

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1 INTRODUCTION

"Length, Breadth and Thickness take up the whole of Space. Nor can Fansie imagine how there should be a Fourth Local Dimension beyond these Three."

(John Wallis, 1685)

1.1 Background

The relatively new concept of a dynamic and deformable earth and the recent availability of highly accurate geodetic measurement systems lead us to address the problem of how to establish and maintain a suitable terrestrial reference system on such an earth. But first let us review briefly the history and status of the present system (see (Guinot, 1978) for a more detailed account) and how it came to be considered no longer satisfactory.

In the latter part of the 19th century, the existence of polar motion was recognized. In order to monitor and correct for this phenomena which as it turned out could vary station coordinates by several meters, the International Latitude (ILS) was established in 1899. The fundamental concept was to remove the systematic errors in the star positions from the latitude observations by choosing observatories on the same parallel (approximately 39°08') with identical instruments and procedures. Furthermore, it was assumed that the stations were motionless with respect to each other and without variations in their respective local plumblines. Thus, polar motion could be determined in a consistent and well defined manner by assigning conventional values for the initial latitudes of the five ILS stations. This convention was adopted in 1967 by the International Astronomical Union (IAU) and the International Union of Geodesy and Geophysics, defining the Conventional International Origin (CIO) for polar motion.

The CIO pole definition, therefore, excluded the incorporation of observations from other stations. However, the polar motion determinations from only the ILS stations were not accurate enough particularly in monitoring short-term variations. This became especially apparent with the advent of new optical instruments and methods. Therefore, in 1962 the ILS was reorganized into the International Polar Motion Service (IPMS) which today provides pole coordinates at 0.05 year intervals based on latitude measurements from about 80 stations with a precision of about 1 meter. Here again a consistent set of initial latitudes is required and furthermore weighting procedures are applied since the quality of observations differ from station to station. Therefore, the IPMS pole differs from the CIO pole.

The first axis of the present CTS is defined by the assigned longitudes of about 50 time observatories of the Bureau International de l'Heure (BIH). The initial task of the BIH, created in 1912, was to maintain a uniform time scale. This function evolved, in addition, to monitoring variations in the earth's rotation rate. Since this requires the pole position, the BIH began to compute its own polar motion values, hence a BIH pole. Furthermore, the BIH began applying corrections to its computations based on earth rotation variations estimated from satellite Doppler and lunar laser ranging (LLR) observations (Feissel, 1980). These observations, as well as those from the other new space methods discussed below, are no longer referenced to the directions of the local plumblines as are the optical instrument observations, but to terrestrial directions.

Thus, today one has the choice of several sets of earth orientation parameters. The CIO pole adopted by the IAU and IUGG can no longer be considered accessible and of practical use (Kovalevsky and Mueller, 1981). The BIH provides the most frequent (five-day averages) and precise estimates (approximately $0\overset{''}{.}01$ (30 cm) for the pole components and 1 ms (45 cm) for UT1). However, today's requirements include estimates of polar motion every two days and variations in the earth's rotation rate each day, both with an accuracy of 5 cm or better (National Research Council, 1981). There is a further reason why the present situation is unsatisfactory. By coincidence, in the same year (1967) that the CIO pole was adopted, several scientific papers appeared in two distinct areas that effectively render one of the fundamental assumptions of the present system invalid.

The theory of continental drift was first proposed by Wegener in 1912 although it was not accepted at the time by the scientific community (Wegener, 1924; Uyeda, 1978). This idea was reborn fifty years later as plate tectonic theory. Culminating a previous decade of geophysical investigations (McKenzie and Parker, 1967) and one year later (Morgan, 1968), advanced the hypothesis that the earth's lithosphere is composed of rigid plates that are in relative motion along their boundaries, over an underlying asthenosphere. According to this theory, observations on the earth's surface are expected to have relative secular motions on the order of 1-10 cm per year, depending on which plates the particular observatories are located. This, of course, invalidates the CIO-BIH assumption of no relative motions between stations.

This theory alone did not in itself cause the IAU and IUGG to defer their CIO decision because at the time such small secular motions were lost within the noise of the available optical instruments. However, in that same eventful year results of the first experiments with Very Long Baseline Interferometry (VLBI) were published (Broten, et al, 1967; Bare, et al, 1967). The opportunities presented by this measurement system cover nearly all aspects of reference frame requirements except the origin problem. First, recalling that radio interferometry was initially developed for astrometric applications, it has the potential for defining what is considered today, the most useful inertial reference system based on an adopted catalogue of extra-galactic radio source coordinates (Robertson, 1981). Second, it has the ability to measure intercontinental baselines with near centimeter repeatability (e.g., Herring, et al, 1981), and can be expected to monitor the relative motions between the observatories. Finally, it provides accurate estimates of short term variations in polar motion and earth rotation (e.g., Robertson, et al, 1978; Fanselow, et al, 1978). VLBI together with the other modern systems developed over the last 10-15 years, Satellite Laser Ranging (SLR) - (e.g., Smith, et al, 1978), LLR (e.g., Mulholland, 1978) and the Global Positioning System (GPS) - (e.g., Fell, 1981) all having the potential to meet the present earth orientation accuracy requirements, mark the urgency of defining a new reference system.

The geodetic community reacted quickly to these developments in theory and instrumentation and since that time the problem of defining reference systems for a deformable earth has been the impetus for much activity. Two IAU colloquia (Kołaczek and Weiffenbach, 1975; Gaposchkin and Kołaczek, 1981) have been entirely devoted to this topic and summarize developments in this area. A general consensus has emerged on how to remedy the present situation which will serve as a guideline for this investigation.

It is generally agreed that two reference systems are needed. The first, using the nomenclature of (Mueller, 1981), is a Conventional Terrestrial System (CTS) in which positions and deformations on the earth could be described. The second is a Conventional Inertial System (CIS) to which the rotations and translations of the CTS could be referred. The first one is the focus of this investigation.

The problem on the deformable earth is to establish an adequate representation of the earth's crust both spatially and temporally. We will follow the kinematic approach as outlined, for example, in (Gaposchkin, 1981; Kovalevsky and Mueller, 1981) which is the most conceptually simple and unambiguous. As in the CIO-BIH system, the CTS should be attached to observatories on the earth's surface. However, in the new system the stations cannot be assumed to be motionless with respect to each other. Furthermore, they should not be tied to the direction of the local plumblines but rather to directions tied to the crust. These stations define a polyhedron whose edges or baseline lengths are accurately and directly estimable from VLBI and laser ranging observations. An adopted set of coordinates for these stations at an arbitrary initial epoch define what will be termed the fundamental polyhedron. Implicit in the coordinates of its vertices are conventional spatial Cartesian axes that define the reference frame. These are accessible at any epoch through global transformations (rotations between the true frame to which the nutations refer, and translations with respect to the initial origin). The problem is to relate the rotated, translated and deformed polyhedron at a later epoch to the fundamental polyhedron, or equivalently to both the CIS and CTS. Therefore, the functions of the CTS stations are twofold. First, an extension of the present BIH system is to monitor those motions common to all points on earth (the polyhedron - CIS connection). The second function is to monitor what is left, i.e., the deformations of the polyhedron (the polyhedron - CTS connection). Therefore, by definition, the deformations do not contain any common rotations or translations (that are statistically significant).

It is conceptually possible to directly monitor the orientation parameters between the CTS and CIS frames simply by 3 rotation angles between two purely kinematical reference frames (the CIS frame being defined by a unit sphere polyhedron of extra-galactic radio sources - Gaposchkin, 1981). These rotations include the combined effects of precession, nutation, polar motion and earth rotation. However, since precession and nutation can be computed quite accurately from adopted earth models, it is preferable to model these effects. The earth orientation parameters (polar motion and earth rotation) which at this stage cannot be modeled adequately are estimated through observations. Of course, any errors in precession and nutation as well as common rotations due to plate motions will be absorbed in these parameters. The latter could occur from an inadequate distribution of observing stations (for example, all stations on one plate). Considering that the requirements for earth orientation parameters call for 1-2 day resolution at the subdecimeter level, near continuous observations are required. However, since these parameters are global, it is possible that only a subset of the CTS stations will have to participate on a regular basis.

The orientation of the CTS reference frame axes are quite arbitrary. However, it is widely agreed that efforts should be made to maintain continuity with the BIH system. This could be accomplished most simply through VLBI observations as will be described in Chapter 2. Similarly, the origin could be defined arbitrarily but it is preferable that it be at the center of mass of the earth, as determined by satellite or lunar laser observations. Changes in the center of mass could be determined in the same way, or by absolute gravity measurements (Heiskanen and Moritz, 1967; Mather, 1973; Moritz, 1979; Zieliński, 1981).

For the monitoring of deformations, it is anticipated that displacements due to the tidal potential and loading effects can be computed to within centimeters and therefore will be modeled (Mueller, 1981). Furthermore, although the stations should be located on stable regions of their respective plates, any possible site stability problems should be monitored by on-site instruments such as gravimeters and by local geodetic networks, particularly using GPS interferometry (Counselman and Shapiro, 1978). The deformation monitoring functions of the CTS stations, then, will primarily be directed to interplate motions. As for modeling plate tectonic motion, this is a controversial question since it is uncertain whether the average long-term plate motions inferred from geophysical and geological data relate to current rates of motion. The adoption of a plate model has been advocated most strongly in (Bender, 1974; Bender and Goad, 1979; Bender, 1981) and will be addressed quite extensively in this investigation.

The fundamental polyhedron as defined by the adopted CTS station coordinates at an initial epoch has a certain size and shape. At a later epoch, the deformed configuration is completely determined from the changes in the baseline lengths which are thus the key to monitoring deformations. Since the CTS at any epoch is defined by the coordinates of all the stations, and considering that deformations due to plate motions are predicted at the decimeter per year level, only periodic re-observations of the baseline lengths need to be taken but from all stations (including the ones that monitor earth orientation on a regular basis).

The realization of the CTS should have low sensitivity to changes in the distribution of the observing stations and in the frequency of observations from individual stations, considering that the number of stations and observations is likely to change from time to time. It should avoid as much as possible any dependence on geophysical hypotheses although it is recognized that the CTS will require some geophysical information, for example (earth tide or plate motion parameters as mentioned above). The definition of the CTS, however, should not depend on this information.

The definition of the reference system should be compatible with simple operational descriptions of how the system should be utilized. There should be established procedures or algorithms for acquisition, reduction, and application of observational data. This includes the adoption of earth models and fundamental constants to be adhered to in all computations.

1.2 Purpose and Objectives

It now appears that a new CTS (and CIS) will soon be established possibly (and hopefully) as an outcome of the upcoming 1983 MERIT (to Monitor Earth Rotation and Intercompare the Techniques of observation and analysis) main campaign (Wilkins, 1981). In anticipation of this event, this investigation addresses certain aspects of the problem of how to define and maintain a new CTS on the deformable earth.

The approach to be followed in this investigation, using as guidelines the consensus described in the previous section, is outlined in Fig. 1. The fundamental polyhedron and the reference frame are defined at an initial epoch t_0 by the adopted (fundamental) coordinates X_{t_0} of a particular set of stations. The coordinates are to be estimated from a dedicated observation campaign of VLBI, SLR, LLR and possibly GPS stations as described in Chapter 2.

The level I stations include a dedicated subset of the CTS observatories which monitor earth orientation parameters (EOP) on a

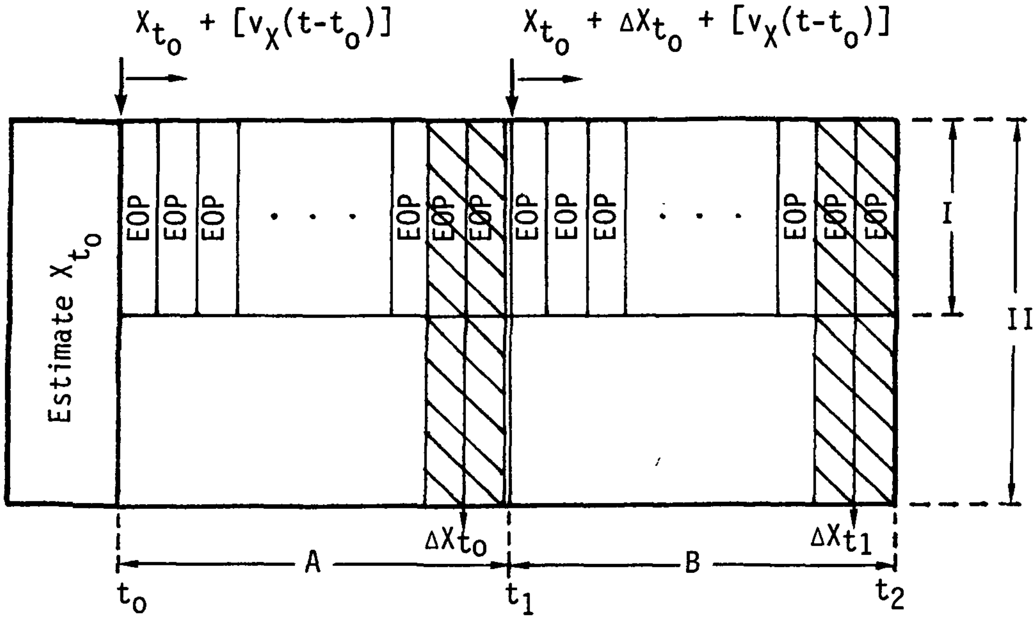


Fig. 1. Schematic CTS Operations

continuous basis. The polar motion and earth rotation parameters are averages over say 1-2 day periods as indicated by the short intervals in Fig. 1. They are rotations between the true frame to which the nutations refer and the reference frame axes defined by X_{t_0}. In order to insure that these parameters continue to refer to the same set of axes, the deformations of the polyhedron need to be estimated periodically but from measurements at all the CTS stations (level II) as indicated by the shaded portions of Fig. 1. For example, at the end of observation interval A, the deformations ΔX_{t_0} are estimated as will be described in Chapter 3 and are then added to the fundamental coordinates X_{t_0} to define the CTS for the next observational period B . In this way, the reference system is maintained, i.e., the earth orientation parameters refer to the same reference frame defined by X_{t_0}. As indicated by the addition of the terms in brackets to X_{t_0}, we allow the possibility of updating the station coordinates on the basis of an adopted plate motion model. This could also be viewed as a correction to be added retroactively to refine the earth orientation parameters estimated over interval A. It should be mentioned that the number and distribution of the CTS stations of level I can vary without affecting the reference system, assuming that the deformations are estimated from a well distributed network of level II CTS stations.

As mentioned in the previous section, any changes in the earth's center of mass, relative to that defined by X_{t_0}, can be determined from satellite dynamics or absolute gravity measurements. In addition, a scale parameter which would represent an expansion or contraction of the earth could be determined from re-measured baseline lengths or from gravity observations (Heiskanen and Moritz, 1967). Both the translational and scale parameters unlike the rotational ones, would have to be monitored from a well distributed global network, preferably from all CTS stations.

It should be clear that the reference frame defined by X_{t_0}, once chosen, does not change. It can be thought of as fixed to the initial (at t_0) positions of the CTS stations (the fundamental polyhedron). The frame, therefore, consists only of a set of spatial Cartesian axes with a particular orientation and origin. It is the reference system that is changing and moving with the deformed polyhedron. The fundamental polyhedron-CIS connection is given by the EOP. The fundamental polyhedron-CTS (deformed polyhedron) connection is given by the estimated deformations. Therefore, the CTS and its frame coincide in general only at the initial epoch. However, the CTS is not only a set of changing station coordinates. It must contain a well-defined description of anything that would influence these coordinates. This includes, of course, X_{t_0} (the reference frame), the CTS stations, adopted earth models (precession, nutation, tides, plate tectonics, etc.) and related fundamental constants, parameter estimation models and established procedures for all CTS operations as mentioned in section 1.1. The purpose of the CTS, then, is to make the reference frame accessible to the user who can then determine positions and detect motions of the earth.

It is useful to present here an excerpt from the concluding comments from the 56th IAU Colloquium (Gaposchkin and Kolaczek, 1981) as summarized by Kovalevsky and Mueller. These points concern the actions required before final decisions are made with regard to the new CTS. They include

1. Selection of observatories whose catalogue will define the CTS.
2. Initiation of measurements at the observatories.
3. Recommendation on the observational and computational maintenance of the CTS (e.g. permanent versus temporary and repeated station occupations, constants to be used).
4. Decision on how far and which way the earth deformation should be modeled initially.
5. Plans and recommendation for the establishment of new international services to provide users with the appropriate information regarding the use of the CTS frame.

The objectives of this investigation address certain aspects of points 1, 3 and 4 listed above. The primary focus will be on the choice of algorithms for the estimation of earth deformations and how this relates to the estimation of earth orientation parameters and the maintenance of the CTS (point 3). The choice of algorithm will depend on the question of whether or not plate motion models will be adopted. An attempt is made to answer the question of whether adding a model will improve the deformation monitoring capabilities of the system (point 4). This will in turn influence somewhat, as will be investigated, the selection of observatories whose coordinates will define the CTS (point 1).

1.3 Organization and Scope

Chapter 2 discusses the definition and maintenance of the CTS. A method is outlined for estimating the coordinates of the fundamental polyhedron from a network containing different measurement systems. Next, the separation of global and deformation parameters is addressed, that is the polyhedron-CIS and polyhedron-CTS connections. An example with a VLBI network is given for which a particularly suitable parameterization for CTS operations is proposed. Different approaches are discussed for maintaining the reference frame. A set of constraints are derived that maintain a discrete Tisserand's mean axes of crust. The physical implications of adopting an absolute plate motion model are described.

In Chapter 3, four estimation algorithms are presented for the analyses of polyhedron deformations. Two are chosen from the class of biased estimators considering the inherent singularity in maintaining a reference frame solely from geodetic observations. Two conditionally unbiased estimates are then presented. The optimal properties of these four estimators are outlined as well as their physical meaning. All are able to deal with the incorporation of geophysical data, particularly provided by an adopted plate tectonic model. In fact, three of the estimators require some a priori deformation information in addition to the geodetic observations. Finally, least squares collocation is applied to deal with the addition or temporary loss of several CTS stations.

In Chapter 4, numerical tests are performed to gain a better appreciation of the optimal properties of each estimator. The main thrust of these tests is to determine the best deformation estimation algorithm, of the four presented in Chapter 3, for combining geodetic and geophysical data. In anticipation of the planned MERIT main campaign scheduled for 1983-84, we extend the optimal polyhedron design analysis of (Mueller, et al, 1982). There it was assumed that no geophysical model for deformations is available. Here, we study the effects of adopting an absolute motion plate model.

The appendices outline some recent results in the approximate theory of optimal design which are applied to the polyhedron design problem. In addition, several results are presented dealing with weighted pseudoinverses.

2 DEFINITION AND MAINTENANCE OF THE CTS

2.1 Introduction

In this chapter, a method is outlined to separate global and deformation parameters in the maintenance of the CTS. Recall that it is necessary to monitor the deformations and update the initial coordinates of the CTS stations so that the transformation parameters (global rotations and translations) will always refer to the same reference frame. In order to do this, either some constraints are needed to insure that the deformations indeed do not contain any global motions, or some external information on the expected deformations is required. Both of these approaches are discussed. In the former, the reference frame axes are seen to be a discrete version of a Tisserand's mean axes of crust. In the latter approach, the reference frame axes are fixed in that part of the earth to which the a priori deformations refer, for example the mantle when using absolute plate motion models. We begin with a method to estimate the fundamental coordinates X_0 (we drop the subscript t of Chapter 1) that define the reference frame.

2.2 Definition of the CTS Frame

As discussed in Chapter 1, the main need for a new CTS comes from the anticipated accuracy of the new geodetic measurement systems and subsequently their ability to monitor deformations of the magnitude predicted by plate tectonic theory. The systems can be divided into baseline methods and coordinate methods. In the former, primarily interferometric methods such as VLBI and GPS interferometry, the origin of a terrestrial coordinate system is not sensed by the observables. The smallest unit for this type of measurement is a baseline (hence, baseline method) although it is a triangle for earth orientation analysis (Bock, 1980). The estimable parameters are baseline lengths and variations in earth orientation (relative to a well-defined initial orientation, i.e. a CTS frame).

The coordinate methods include SLR, LLR and GPS in the Doppler and pseudo-ranging modes (e.g., Fell, 1980). The basic unit for these systems is one station. An origin is sensed and thus these systems provide Cartesian coordinates. SLR observations are sensitive to polar motion variations but basically insensitive to variations in earth rotation (Van Gelder, 1978). On the other hand, LLR observations are sensitive to all three earth orientation parameters but particularly to variations in earth rotation (Calame, 1982).

In all of the new systems, though, coordinates (or in the case of interferometry, coordinate differences) are inseparable from earth orientation parameters. This means that to estimate coordinates (or coordinate differences) an external source of earth orientation is required and vice versa. By using BIH earth orientation values the CTS frame can be made continuous with the present BIH frame at the initial epoch as will be described below.

Considering the different measurement systems available, it will be necessary to merge several networks, each one defining essentially its own reference frames both CTS and CIS, into a common set. Suppose the relations between two CIS's is

$$x^{II} = R_1(\alpha_1)R_2(\alpha_2)R_3(\alpha_3)x^I \tag{2.2-1}$$

Similarly, the relation between two CTS's is

$$X^{II} = R_1(\beta_1)R_2(\beta_2)R_3(\beta_3)X^I \tag{2.2-2}$$

The transformation from CIS to CTS is (Mueller, 1969)

$$\begin{aligned} X^I &= S^I NPx^I \\ X^{II} &= S^{II} NPx^{II} \end{aligned} \tag{2.2-3}$$

where common nutation (N) and precession (P) matrices are assumed to be used in both techniques. The earth orientation matrix is given by

$$S = R_2(-\xi)R_1(-\eta)R_3(\theta) \tag{2.2-4}$$

in which ξ,η are the coordinates of the pole and θ is the Greenwich Apparent Sidereal Time. After some reduction and neglecting second-order terms, we have from (Mueller, et al, 1982)

$$-\Delta\eta = -(\eta^I - \eta^{II}) = -\beta_1 + \alpha_1\cos\theta + \alpha_2\sin\theta \tag{2.2-5}$$

$$-\Delta\xi = -(\xi^I - \xi^{II}) = -\beta_2 - \alpha_1\sin\theta + \alpha_2\cos\theta \tag{2.2-6}$$

$$W\Delta UT1 = W(UT1^I - UT1^{II}) = -\beta_3 + \alpha_3 \tag{2.2-7}$$

where W is the ratio of universal (UT1) to sidereal time. By station collocation, i.e., maintaining different instrument types at common sites, one determines the CTS difference (β angles). Then through the earth rotation parameter differences one finds the CIS differences (α angles). This indirect approach has been suggested by (Kovalevsky, 1980). The determination of the CIS differences is treated in (Mueller, et al, 1982).

In the following, a method to estimate the CTS differences and simultaneously to define the CTS frame based on the estimation of a unique well-defined set of coordinates for the fundamental polyhedron is outlined. Suppose that one baseline method, VLBI, and two coordinate methods, SLR and LLR, are to participate in a campaign to estimate the fundamental global spatial Cartesian coordinates, X_0. VLBI observations are insensitive to the initial orientation of the CTS frame, making coordinate differences and earth orientation parameters (polar motion and UT1) inseparable. In practice, this dependency is broken by initializing earth orientation over, say, the first day of observations of a particular campaign. As shown, e.g., in (Bock, 1980), the estimation of baseline components is then biased by this initialization but in this case the bias can be used to our advantage. Continuity with the present terrestrial system can be achieved by the input of BIH polar motion and earth rotation values for the initial step. In this way, at the fundamental epoch t_0 the new CTS frame can be aligned with the BIH frame through the estimated VLBI coordinate differences (within the errors in the BIH values). The first axis (x) of the BIH frame is in the direction of the Greenwich mean astronomic meridian and the third axis (z) is towards the average (mean) north terrestrial pole. The second axis (y) completes a right-handed global spatial Cartesian coordinate system. From the SLR or LLR estimated coordinates, the origin of the CTS frame could be made geocentric.

One is then led to the following three sets of transformation equations from which X_0 can be estimated

$$X_{L_i} = (1 + c_1)(X_i)_L + \begin{bmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{bmatrix} (X_i)_L + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \quad (2.2\text{-}8)$$

$$X_{S_i} = (1 + c_2)(X_i)_S + \begin{bmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{bmatrix} (X_i)_S \quad (2.2\text{-}9)$$

$$\Delta X_{V_{ij}} = (X_j)_V - (X_i)_V \quad (2.2\text{-}10)$$

The first set (2.2-8), one for each LLR station, has as observations X_{L_i} the geocentric coordinates of site i. The parameters are the CTS LLR site coordinates $(X_i)_L$, three rotation angles β_1, β_2, β_3 (connecting LLR to VLBI), a scale factor c_1 (LLR to VLBI) and three translation parameters δ_1, δ_2, δ_3 (LLR to SLR origin). For the second set (2.2-9), the parameters are the CTS SLR site coordinates $(X_i)_S$, three rotational angles γ_1, γ_2, γ_3 (SLR to VLBI) and a scale factor c_2 (SLR to VLBI). For the third set (2.2-10), the parameters are the VLBI site coordinates $(X_i)_V$, and the observations are any independent subset of coordinates differences ΔX_V from the VLBI estimated parameters. Different combinations of equations (2.2-8) to (2.2-10) could be formulated although those given here reflect today's situation (origin defined by SLR, orientation and scale by VLBI). Additional sets of equations could be added for other measurement systems (e.g., GPS). Considering (2.2-8) to (2.2-10) as observation equations and computing a weight matrix from the covariance matrices of the laser ranging and VLBI adjustments, one could then perform a least squares adjustment to estimate X_0, a consistent set of CTS coordinates (at the collocated sites $(X_i)_L = (X_i)_S = (X_i)_V$ is constrained) that would define the new reference frame. It would be geocentric and aligned with the BIH frame at t_0.

2.3 Separation of Global and Deformation Parameters

As mentioned in the previous section, all the available modern measurement systems suffer from the position-orientation inseparability problem which is inherent in all strictly earth based observations. However, it is required for the definition of the CTS to estimate periodically station coordinate changes (deformations) which are free from global rotations and translations. In order to separate global motions and deformations one approach is the addition of a set of constraints to overcome the singularity problem. These constraints are derived in the next section.

Without any external information of expected deformations, one is led to a free adjustment with orientation parameters of which several approaches are possible (Fritsch and Schaffrin, 1981). Here a two step analog to the "classical approach" is presented. First, the earth orientation parameters are eliminated, more precisely those motions common to all stations. Then, the deformations are estimated from periodically repeated measurements of the baseline lengths by minimizing a weighted norm in the parameter space of coordinate changes. Simultaneous adjustments (one step approaches) have been proposed for single measurement types particularly VLBI (Cannon, 1979; Manabe, 1982). An example, with a VLBI network will help elucidate these ideas.

Consider the basic VLBI mathematical model as given in (Bock, 1980). For baseline i-j observing source 1 at epoch k the path difference (time delay times the speed of light) which the incoming signal must travel after its reception at station i till its arrival at station j is given by

$$
d_{ijk\ell} = -\begin{bmatrix} \Delta x_{ij} & \Delta y_{ij} & \Delta z_{ij} \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_k \\ 0 & 1 & -\eta_k \\ -\xi_k & \eta_k & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 \\ -\sin\theta_k & \cos\theta_k & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\cdot \begin{bmatrix} \cos\delta_\ell \cos\alpha_\ell \\ \cos\delta_\ell \sin\alpha_\ell \\ \sin\delta_\ell \end{bmatrix} + c(\Delta C_{0_i} + \Delta C_{1_{ij}}(t_k - t_0)) \qquad (2.3\text{-}1)
$$

where (α,δ) are the source coordinates, $\Delta C_0, \Delta C_1$ are clock parameters and c the speed of light. As mentioned above, the coordinate differences Δx, Δy, Δz and the earth orientation parameters of polar motion (ξ,η) and earth rotation (θ) are inseparable. This can be seen by an examination of the linear relationships between the coefficients of the design matrix A

$$A_\xi = \Delta x A_{\Delta z} - \Delta z A_{\Delta x}$$

$$A_\eta = -\Delta y A_{\Delta z} + \Delta z A_{\Delta y} \tag{2.3-2}$$

$$A_\theta = -\Delta y A_{\Delta x} + \Delta x A_{\Delta y}$$

where the A's denote the partial derivative of the observable with respect to the subscripted parameter which can be found in (Bock, 1980). There, a different parameterization is proposed in which the total observation period is split up into earth orientation steps. For the m'th step, 3 rotation angles $\xi_m - \xi_1$, $\eta_m - \eta_1$, $\theta_m - \theta_1$ are estimated, relative to one fixed step, where ξ_1, η_1, θ_1 are input into the adjustment as determined from external sources (e.g.,BIH values). That is, the earth orientation parameters are averages over certain time intervals (e.g., 1 day averages). One set of coordinate differences are also estimated for the entire observational period. This parameterization is useful in some situations, one was mentioned in the previous section. However, it is particularly unsuited for CTS operations since the function of the CTS is to separate earth rotation variations and station displacements in a well-defined manner. In this parameterization, it is difficult to maintain orientation continuity and furthermore the size and distribution of observations of the first step are arbitrary. Moreover, this fixed step biases the estimation of coordinate differences by any errors $d\xi_1$, $d\eta_1$, $d\theta_1$ in the external earth orientation information according to

$$\begin{bmatrix} d\tau \\ d\varepsilon \\ d\sigma \end{bmatrix} = \begin{bmatrix} d\Delta x \\ d\Delta y \\ d\Delta z \end{bmatrix} + \begin{bmatrix} 0 & d\theta_1 & -d\xi_1 \\ -d\theta_1 & 0 & d\eta_1 \\ d\xi_1 & -d\eta_1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{2.3-3}$$

where τ, ϵ, σ are the "biased" baseline components (Bock, 1980)

The one step free adjustment approach for VLBI is as follows. Consider the parameter vector

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.3-4}$$

where X_1 represent the coordinate changes, X_2 the earth orientation parameters (3 per step) and X_3 other "non-geodetic" parameters such as the right ascension differences $(\alpha_1-\alpha_0)$, declinations (δ_1) and clock parameters (C_0,C_1) appearing in (2.3-1). The normal equations can be written as

$$\begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}^T & N_{22} & N_{23} \\ N_{13}^T & N_{23} & N_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{2.3-5}$$

where

$$N_{ij} = A_i^T P A_j \tag{2.3-6}$$

$$U_i = A_i^T P L_i \tag{2.3-7}$$

P is the weight matrix of the observations L. These are rank deficient by six due to the origin (VLBI is insensitive to an origin) and orientation (coordinate-orientation inseparability problem) defects.

One proceeds in two steps. First, the orientation and model parameters are eliminated such that

$$\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} N_{22} & N_{23} \\ N_{23}^T & N_{33} \end{bmatrix}^{-1} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} - \begin{bmatrix} N_{12} \\ N_{13} \end{bmatrix} X_1 \tag{2.3-8}$$

Note that only the Cayley inverse is required in this step. Next, and most generally, a weighted norm is minimized

$$\| X \|_M = (X^T M X)^{\frac{1}{2}} \tag{2.3-9}$$

where M is a weight matrix constructed from an adopted deformation model as described in Chapters 3 and 4. Denoting

$$N = N_{11} - [N_{12} \quad N_{13}] \begin{bmatrix} N_{22} & N_{23} \\ N_{23}^T & N_{33} \end{bmatrix}^{-1} \begin{bmatrix} N_{12}^T \\ N_{13}^T \end{bmatrix} \tag{2.3-10}$$

$$U = U_1 - [N_{12} \quad N_{13}] \begin{bmatrix} N_{22} & N_{23} \\ N_{23}^T & N_{33} \end{bmatrix}^{-1} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} \tag{2.3-11}$$

the coordinates of the CTS stations are estimated by (see section 3.3.1.2)

$$\hat{X}_1 = M^{-1} N (N M^{-1} N)^+ U \tag{2.3-12}$$

where $+$ denotes the pseudoinverse. Here the matrix M is assumed to be positive definite, the more general case of a positive semidefinite M is treated in Chapter 3. If there is no a priori information model, i.e., $M = I$

$$\hat{X}_1 = N^+ U \tag{2.3-13}$$

the ordinary pseudoinverse solution. This choice of M-norm for X_1 is equivalent (see the next section) to defining a discrete Tisserand's mean axes of crust. Instead of using the pseudoinverse to compute X_1, a set of constraints derived and explained in the next section could be augmented to the set of normal equations. In the second step, X_2 and X_3 are estimated by (2.3-8).

As mentioned above, several other norm choices are possible. Some of these including the "classical" approach presented here are applied in (Dermanis, 1977, 1981; Manabe, 1982) although in the less general case of M = I. Other approaches are possible, too. One might consider X_1 as a random variable vector and X_2 and X_3 deterministic. This would lead to a least squares collocation approach (section 3.6). Other combinations are possible. In order to circumvent choosing from all these different possibilities, the number of which indicates the problems encountered from the coordinate-orientation inseparability problem, an analogous but unambiguous approach is outlined below. It takes into consideration that the X_1 parameters need to be monitored periodically from all CTS stations while the X_2 parameters must be monitored continuously from possibly a subset of the stations. Furthermore, the impractical simultaneous adjustment of several observation types (e.g., VLBI, SLR, LLR) is avoided.

Consider the following two step approach continuing the VLBI network example. Once the fundamental coordinates X_0 are adopted, the following parameterization is particularly suitable for the first step of monitoring earth orientation. Instead of parameterizing the coordinate differences that appear in the mathematical model of equation

(2.3-1), we parameterize baseline lengths for the set X_1. The set X_2 includes 3 orientation parameters per step (e.g., one day earth orientation and two day polar motion averages), but in this case there is no need to fix one step. These earth orientation estimates are with respect to the defined reference frame axes implicit in the CTS coordinates that are input into the adjustment. Thus, continuity in orientation is maintained consistently even with a gap in the observations and no bias is introduced as in the previous parameterization.

The partial derivatives for the baseline lengths (the X_1 set) can be derived from the coordinate difference partials

$$\begin{aligned} A_{\Delta x_{ij}} &= -\cos\delta_\ell \cos(\theta_k - \alpha_\ell) \\ A_{\Delta y_{ij}} &= \cos\delta_\ell \sin(\theta_k - \alpha_\ell) \\ A_{\Delta z_{ij}} &= -\sin\delta_\ell \end{aligned} \tag{2.3-14}$$

Then,

$$A_{\ell_{ij}} = A_{\Delta x_{ij}} \frac{\partial \Delta x_{ij}}{\partial \ell_{ij}} + A_{\Delta y_{ij}} \frac{\partial \Delta y_{ij}}{\partial \ell_{ij}} + A_{\Delta z_{ij}} \frac{\partial \Delta z_{ij}}{\partial \ell_{ij}} \tag{2.3-15}$$

The baseline vector can be written as

$$\bar{\ell}_{ij} = \begin{bmatrix} \Delta x_{ij} \\ \Delta y_{ij} \\ \Delta z_{ij} \end{bmatrix} = \ell_{ij} \begin{bmatrix} \cos\alpha_{ij} \cos\beta_{ij} \\ \cos\alpha_{ij} \sin\beta_{ij} \\ \sin\alpha_{ij} \end{bmatrix} \tag{2.3-16}$$

where α and β are baseline direction angles, from which

$$\frac{\partial \Delta x_{ij}}{\partial \ell_{ij}} = \cos\alpha_{ij}\cos\beta_{ij} = \frac{\Delta x_{ij}}{\ell_{ij}} \tag{2.3-17}$$

$$\frac{\partial \Delta y_{ij}}{\partial \ell_{ij}} = \cos\alpha_{ij}\cos\beta_{ij} = \frac{\Delta y_{ij}}{\ell_{ij}} \tag{2.3-18}$$

$$\frac{\partial \Delta z_{ij}}{\partial \ell_{ij}} = \sin\alpha_{ij} = \frac{\Delta z_{ij}}{\ell_{ij}} \tag{2.3-19}$$

so that

$$A_{\ell_{ij}} = [A_{\Delta x_{ij}} \Delta x_{ij} + A_{\Delta y_{ij}} \Delta y_{ij} + A_{\Delta z_{ij}} \Delta z_{i}]/\ell_{ij} \tag{2.3-20}$$

All the other elements of the design matrix A are the same as in the coordinate difference adjustment (Bock, 1980).

This parameterization allows a clean separation of rotation and deformations. The baseline lengths (X_1) can detect any possible short term deformations but recall that only a subset of the CTS stations is observing. Although some have argued that VLBI observations are sufficient for monitoring earth orientation (e.g., Gaposchkin, 1981) other measurement types will most likely be involved, too. In that case, the CTS coordinates input into, say, SLR observation adjustments, define the reference frame to which the earth orientation parameters refer. Recall that a consistent set of polyhedron coordinates at the initial epoch would be determined from several measurement systems as outlined in section 2.2. These coordinates are updated periodically using the deformations estimated from the second step of this approach (see below), in order that the earth orientation parameters of the first step refer always to the same reference frame. Thus, the coordinates of the stations are never included as parameters for the earth orientation adjustments of any participating measurement system. In this baseline length-orientation angle parameterization, the orientation angles include any common rotations of the crust due to plate motions relative to the mantle (although this is not expected as discussed below). The baseline lengths may include a global expansion or contraction. These types of motion will be absorbed in the CTS by definition (Kovalevsky and Mueller, 1981).

For the periodic monitoring of deformations, the parameters of interest are the baseline lengths (X_1) of the deformed polyhedron defined by all the CTS stations with possibly different measurement systems. The baseline lengths contain only deformation information since they are independent of coordinate system (assuming that the coordinate system is not changing over the interval of time in which the baselines are estimated, for example, over a long arc laser ranging solution). They indicate the change in size and shape of the polyhedron relative to the fundamental polyhedron. Furthermore, the baseline lengths are the only strictly estimable parameters from all the different measurement systems and also the most accurate ones. The procedure, then, is to pool the baseline length estimates from the separate adjustments of the respective measurement systems, along with their covariance matrices. In a second adjustment, which is the subject of Chapter 3, the deformations of the CTS stations can be estimated by comparing the deformed baseline lengths to their initial epoch values.

It is in the second adjustment that a plate motion model could be introduced to improve the estimation of the deformations, and to test its consistency with the geodetic observations. If no plate motion model is adopted, then in order to maintain the CTS, a set of constraints is required in the estimation of deformations to insure that they include no global motions. These are derived in the next section and are generalized to include geophysical models of deformations. In the presence of such models, other approaches to maintain the CTS will also be described.

2.4 Approaches to Maintaining the CTS

2.4.1 Reference Frame Constraints - Geometrical Approach

The reference frame axes at an initial epoch are conventionally defined by the adopted coordinates of the CTS stations (the fundamental polyhedron). At a later epoch, the deformed polyhedron will similarly define a different set of axes. In order to maintain a consistent reference frame one approach is that these two sets of axes are constrained to differ by only global transformations as least in some weighted least squares sense. In this section, the constraints for this approach are derived from geometric considerations.

At the fundamental epoch the polyhedron coordinates are given by the initial coordinates X_0. At a later epoch, the new coordinates X_1 from which global rotations and translations have been either modeled (precession and nutation) or estimated (polar motion and earth rotation) and removed, differ from X_0. Furthermore, any translations due to changes in the center of mass have been corected for. Any remaining differences between these two sets of coordinates are due to deformations. The constraints necessary to insure that the reference frame axes implicit in X_0 and X_1 are equivalent can be derived as follows (expanding on an example in (Leick and Taylor, 1980)). Consider that the two sets of coordinates for station i

$$X_{0_i} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad ; \quad X_{1_i} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{2.4-1}$$

are still related through 3 infinitesimal rotations α_1, α_2, α_3 and 3 translations δ_1, δ_2, δ_3 by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}_i = R_3(\alpha_3)R_2(\alpha_2)R_1(\alpha_1) \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}_i + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \tag{2.4-2}$$

The scale problem will be addressed at a later stage. Here the assumption is that any scale differences between the various systems have been determined and reconciled as in section 2.2. The above equations can be rewritten as 3 observations equations per station

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 & -z_0 & y_0 & 1 & 0 & 0 \\ z_0 & 0 & -x_0 & 0 & 1 & 0 \\ -y_0 & x_0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} + V \tag{2.4-3}$$

where V is the noise vector. In matrix form,

$$L = AX + V \tag{2.4-4}$$

The "observation vector", L includes the changes in the coordinates with respect to the initial epoch, i.e., the deformations. The "parameter vector", X includes the common rotation and translation components (what is left after removing precession, nutation, center of mass changes, etc.). Introducing a weight matrix P for the observations we arrive at the standard least squares estimate (Uotila, 1967)

$$\hat{X} = (A^T PA)^{-1} A^T PL \tag{2.4-5}$$

In order for the reference frame implied in X_0 and X_1 to be maintained, it is required that $\hat{X} = 0$. This occurs in two non-trivial cases. First, when $L = 0$, which would occur only for a rigid earth (within the observational noise). The second, more interesting case would be when

$$A^T PL = 0 \tag{2.4-6}$$

Now, let us reconsider the problem. In reality, the deformations are the parameters to be estimated so that (2.4-6) becomes our reference system maintenance constraint

$$CM\hat{X} = 0 \tag{2.4-7}$$

Consider M as a weight matrix derived from an adopted model for station deformations as described in section 4.2. It defines a weighted norm in the parameter space

$$\| X \|_M = (X^T MX)^{1/2} \tag{2.4-8}$$

The former design matrix A is now the constraint matrix C which has the following form

$$C = \begin{bmatrix} S_1 & S_2 & \dots & S_p \\ I & I & \dots & I \end{bmatrix} \tag{2.4-9}$$

where

$$S_i = \begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \tag{2-4-10}$$

I is the 3×3 identity matrix and p is the number of polyhedron stations. To understand the significance of (2.4-7), let us assume for a moment that M is diagonal and the weight for the i'th station is m_i. The constraints can then be divided into two sets

$$\sum_{i=1}^{p} m_i \begin{bmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} x_i - x_0 \\ y_i - y_0 \\ z_i - z_0 \end{bmatrix} = 0 \tag{2.4-11}$$

$$\sum_{i=1}^{p} m_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i - x_0 \\ y_i - y_0 \\ z_i - z_0 \end{bmatrix} = 0 \tag{2.4-12}$$

These constrain, respectively, the orientation and origin defined by X_0 and X_1 to coincide, in the weighted least squares sense.

Consider again the original problem. We know that the standard least squares estimate has the property that

$$\hat{V}^T P \hat{V} = \text{minimum} \tag{2.4-13}$$

Since $\hat{X} = 0$, it follows from

$$\hat{V}^T P \hat{V} = L^T P L - \hat{X}^T U \tag{2.4-14}$$

that

$$L^T P L = \text{minimum} \tag{2.4-15}$$

In our reversed problem this implies that

$$\hat{X}^T M \hat{X} = \text{minimum} \tag{2.4-16}$$

i.e., a minimum norm solution.

Thus, in the general case, we can say at this point that the reference frame axes are fixed in the earth's crust, through a discrete number of CTS stations, in a minimum M-norm sense. In the next chapter, we present estimation techniques that incorporate these constraints.

The constraints (2.4-7) are given above in terms of global spatial coordinates. Let us express them in a local geodetic system (u,v,w) where the u-axis is positive north along the geodetic meridian, the v-axis is positive east and w-axis in the direction of the ellipsoidal normal. From (Rapp, 1976), the relation between the two systems is given by

$$\begin{bmatrix} u \\ -v \\ w \end{bmatrix}_i = R_2(\phi_i - 90°)R_3(\lambda_i - 180°) \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}_i \tag{2.4-17}$$

where ϕ_i, λ_i are the geodetic coordinates of station i. Assuming again that M is a diagonal matrix and approximating the ellipsoid by a sphere of mean earth radius, we arrive at the two sets of constraints

$$\sum_{i=1}^{p} m_i \begin{bmatrix} \sin\lambda & -\sin\phi\cos\lambda & 0 \\ \cos\lambda & -\sin\phi\sin\lambda & 0 \\ 0 & \cos\phi & 0 \end{bmatrix}_i \begin{bmatrix} u \\ v \\ (w) \end{bmatrix}_i = 0 \tag{2.4-18}$$

$$\sum_{i=1}^{p} m_i \begin{bmatrix} -\sin\phi\cos\lambda & -\sin\lambda & \cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & \cos\phi\sin\lambda \\ \cos\phi & 0 & \sin\phi \end{bmatrix}_i \begin{bmatrix} u \\ v \\ w \end{bmatrix}_i = 0 \tag{2.4-19}$$

that correspond to (2.4-11) and (2.4-12), respectively. The constraints (2.4-18) correspond to those given in (Bender and Goad, 1979; Bender, 1981). The w coordinate (which indicates geometric height) is in parentheses since it appears in the matrix multiplication. This means that an infinitesimal rotation only causes horizontal coordinate changes, i.e., the height is insensitive to such rotations. However, the opposite is not correct. Rotation is sensitive to height changes, the proof of which is due to an unpublished manuscript of S. Y. Zhu and is reproduced here. This will indicate that the constraints (2.4-18) and (2.4-19) should be taken as a complete set.

An infinitesimal rotation $\bar{\alpha}$ and a translation $\bar{\delta}$ will change the coordinates according to (2.4-3). The corresponding height change for a particular station is

$$\Delta w = \frac{\Delta x \cdot x + \Delta y \cdot y + \Delta z \cdot z}{[x^2 + y^2 + z^2]^{1/2}} \tag{2.4-20}$$

If only an infinitesimal rotation exists, that is $\bar{\delta} = 0$, then we will find $\Delta w = 0$ as stated above.

On the other hand, it can be shown that a height change Δw can influence rotation significantly. Consider (2.4-3) again as observation equations where the parameters are the infinitesimal rotations $\overline{\alpha}$ and translations $\overline{\delta}$, and the observations are changes in the coordinates. The normal equations are

$$NX = U \tag{2.4-21}$$

where

$$N = \begin{bmatrix} \Sigma(z^2+y^2) & -\Sigma xy & -\Sigma xz & 0 & \Sigma z & -\Sigma y \\ & \Sigma(x^2+z^2) & -\Sigma yz & -\Sigma z & 0 & \Sigma x \\ & & \Sigma(x^2+y^2) & \Sigma y & -\Sigma x & 0 \\ & & & n & 0 & 0 \\ & \text{Symmetric} & & & n & 0 \\ & & & & & n \end{bmatrix} \tag{2.4-22}$$

$$U = \begin{bmatrix} \Sigma(\Delta z \cdot y - \Delta y \cdot z) \\ \Sigma(\Delta x \cdot z - \Delta z \cdot x) \\ \Sigma(\Delta y \cdot x - \Delta x \cdot y) \\ \Sigma\Delta x \\ \Sigma\Delta y \\ \Sigma\Delta z \end{bmatrix} \tag{2.4-23}$$

Compare the expression for U to (2.4-7) and (2.4-9). In general, as seen in (2.4-22), the off-diagonal elements of N are not equal to zero. We write N^{-1} as

$$N^{-1} = [a_{ij}]$$

Transforming the above equation into the local geodetic system and setting $u = v = 0$ (see (2.4-19)), that is considering only a height change,

$$U = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Sigma\Delta w \cos\phi\cos\lambda \\ \Sigma\Delta w \cos\phi\sin\lambda \\ \Sigma\Delta w \sin\phi \end{bmatrix} \tag{2.4-24}$$

and

$$\alpha_i = a_{i4}U_4 + a_{i5}U_5 + a_{i6}U_6 \tag{2.4-25}$$

Since a_{i4}, a_{i5}, a_{i6} of N^{-1} generally are not equal to zero, neither is α_i. Simulation shows that α_i can be of the same order ot magnitude as Δw. This indicates that in general rotation is sensitive to height change.

In addition, the weight matrix M cannot be assumed to be diagonal but a full symmetric matrix as we shall see later on. Therefore, the two sets of constraints (2.4-11) and (2.4-12), in global spatial coordinates and (2.4-18) and (2.4-19) in local coordinates are a complete set as indicated in (2.4-7). This is the case regardless of which 3-D measurement is being considered, and even if we are only interested in rotations.

It is useful at this point to outline the approach taken by (Cannon, 1979; Cannon and Rochester, 1981) in establishing a terrestrial reference frame by means of VLBI observatories since it leads to the same set of constraints. It is based on analogies to the study of differential deformations in continuum mechanics.

Define the displacement (deformation) vector for baseline i

$$D_i = X_{1_i} - X_{0_i} \tag{2.4-26}$$

This quantity is in general not equal to zero due to deformations of the polyhedron and to measurement errors. An analog to the differential tensor of the infinitesimal displacement field is given by the unitless matrix for the k'th baseline

$$C_i = \frac{D_i X_{0_i}^T}{\left\| X_{0_i} \right\|^2} \tag{2.4-27}$$

This matrix can be written as the sum of two matrices

$$C_i = \frac{1}{2} e_i + \frac{1}{2} \Omega_i \tag{2.4-28}$$

where

$$e_i = C_i + C_i^T = \frac{1}{\left\| X_{0_i} \right\|^2} (D_i X_{0_i}^T + X_{0_i} D_i^T) \tag{2.4-29}$$

is analogous to the strain tensor and describes deformations of the i'th baseline and

$$\Omega_i = C_i - C_i^T = \frac{1}{\left\| X_{0_i} \right\|^2} (D_i X_{0_i}^T - X_{0_i} D_i^T) \tag{2.4-30}$$

is analogous to the rotation tensor of the displacement field. The displacement vector is then the sum of the "strain" matrix and the "rotation" matrix times the initial polyhedron coordinate vector of station i

$$D_i = \frac{1}{2} e_i X_{0_i} + \frac{1}{2} \Omega_i X_{0_i} = C_i X_{0_i} \tag{2.4-31}$$

A weighted mean global strain matrix is defined over the p polyhedron stations by

$$E = \frac{1}{\sum_{i=1}^{p} m_i} \sum_{i=1}^{p} m_i e_i \tag{2.4-32}$$

as well as a weighted mean global rotation matrix

$$\Lambda = \frac{1}{\sum_{i=1}^{p} m_i} \sum_{i=1}^{p} m_i \Omega_i \tag{2.4-33}$$

where m_i is a weighting factor. We can then define

$$e_i = E + \varepsilon_i \tag{2.4-34}$$

$$\Omega_i = \Lambda + \omega_i \tag{2.4-35}$$

where ε_i and ω_i are residual deformations and rotations such that

$$\frac{1}{\sum_{i=1}^{p} m_i} \sum_{i=1}^{p} m_i \varepsilon_i = 0 \tag{2.4-36}$$

$$\frac{1}{\sum_{i=1}^{p} m_i} \sum_{i=1}^{p} m_i \omega_i = 0 \qquad (2.4\text{-}37)$$

If the polyhedron stations were well distributed over the earth's surface the matrix E would be representative of the global deformations of the earth. As can be easily shown, if trace (E) would differ significantly from zero (basically a weighted average of the distance changes) this would indicate global earth expansion or contraction, i.e., a scale change. The off diagonal elements would indicate a global skewing ("shear") of the polyhedron. These effects could be absorbed by the CTS coordinates by computing new spanning base vectors as described by Cannon. The residuals ε_i, ω_i therefore should be due to non-global phenomena, i.e., deformations.

Any global rotation of the stations would be indistinguishable from errors in the SNP transformation (i.e., in the estimation of polar motion and earth rotation or modeling of precession and nutation). Therefore, the mean global rotation matrix should be identical to zero, i.e.,

$$\Lambda = 0 \qquad (2.4\text{-}38)$$

and thus (2.4-35) becomes

$$\Omega_i = \omega_i \qquad (2.4\text{-}39)$$

i.e., residual rotations should be almost entirely due to non-global phenomena.

Let us now examine the residual equations (2.4-36) and (2.4-37). They look quite similar to equations (2.4-11) and (2.4-12). If e_i and Ω_i are tensor analogs, it should follow that so are ε_i and ω_i. This can be seen from

$$\omega_i = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} = \frac{1}{2}\,(\gamma_i - \gamma_i^T) \qquad (2.4\text{-}40)$$

$$\varepsilon_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\,(\gamma_i + \gamma_i^T) \qquad (2.4\text{-}41)$$

where

$$\gamma_i = \begin{bmatrix} 1 & -z_i & y_i \\ z_i & 1 & -x_i \\ -y_i & x_i & 1 \end{bmatrix} \qquad (2.4\text{-}42)$$

is the sum of a symmetric (ε_i) and antisymmetric (ω_i) matrix. The constraints then become

$$\sum_{i=1}^{P} m_i\,(\gamma_i - \gamma_i^T)(X_1 - X_0)_i = 0 \qquad (2.4\text{-}43)$$

$$\sum_{i=1}^{P} m_i\,(\gamma_i + \gamma_i^T)(X_1 - X_0)_i = 0 \qquad (2.4\text{-}44)$$

These equations are equivalent to (2.4-11) and (2.4-12) as well as to

(2.4-36) and (2.4-37). In the study of differential deformations in continuum mechanics, the differential motion of the particle displacement field can be split into two independent components, the strain tensor and the rotation tensor. It is true that, in our case, we are considering small changes over baselines of possibly several thousand kilometer extent. However, the tensor analogy ends since as described above the 6 constraints form a complete set and are not independent. Furthermore, the M matrix is generally not diagonal. Therefore, our constraints should be written as

$$\begin{bmatrix} \gamma_1 - \gamma_1^T & \gamma_2 - \gamma_2^T & \cdots & \gamma_P - \gamma_P^T \\ \gamma_1 + \gamma_1^T & \gamma_2 + \gamma_2^T & \cdots & \gamma_P + \gamma_P^T \end{bmatrix} M(X_1 - X_0) = 0 \qquad (2.4\text{-}45)$$

or equivalently

$$CMX = 0 \qquad (2.4\text{-}7)$$

Thus, we see that these constraints have appeared in the literature in different but usually less general forms (see also (Moritz, 1979; Richter, 1981)) particularly in the analysis of horizontal deformations (e.g., Brunner, et al, 1981).

2.4.2 Reference Frame Constraints - Physical Approach

In the previous section, we presented a set of constraints that insure that the reference frame defined through the fundamental polyhedron is maintained, in an M-norm sense. In the following, we show that this corresponds to maintaining a discrete analog of Tisserand's mean axes of crust.

The time derivative of a position vector fixed in the earth is given by (e.g., Goldstein, 1981)

$$\left[\frac{dX}{dt}\right]_I = \left[\frac{dX}{dt}\right]_T + \omega \times X \tag{2.4-46}$$

The subscripts I and T indicate that the time derivatives are with respect to an inertial and terrestrial frame, respectively. The vector ω is the instantaneous angular velocity of the earth or its rotation vector. Equation (2.4-46) can be written as

$$V_I = V_T + \omega \times X \tag{2.4-47}$$

For the rigid body, $V_T = 0$ since there is no rotation of points relative to the terrestrial frame. In this case, V_I results solely from the rotation of the earth in space. For the deformable earth V_T denotes those motions relative to the chosen terrestrial frame, i.e., deformations. Furthermore, the rotation vectors are in general different for each point, although for points on the same tectonic plate, they may be nearly the same. In any case, deviations in the rotation vector from one point to another are small and thus it is useful to define an instantaneous mean rotation vector, $\bar{\omega}$ such that

$$\iiint_E V_T \cdot V_T \, \rho dE = \text{minimum} \tag{2.4-48}$$

where ρ denotes density. This condition defines the motion of the reference frame termed the Tisserand mean axes of body (Munk and MacDonald, 1960). According to (Smith, 1981), this integral should be evaluated only over the earth's crust (solid outer surface) since this is where our observations are taken and station coordinates assigned. In this case, one has a Tisserand's mean axes of crust. Since observations are only available at a finite number of observing stations (a polyhedron) condition (2.4-48) is unrealizable in practice. In this case, only a discrete analog of this condition is attainable

$$\sum_{i=1}^{p} m_i (V_{T_i} \cdot V_{T_i}) = \text{minimum} \tag{2.4-49}$$

where m_i are mass elements. Hopefully, the distribution and number of these stations will make this approximation meaningful, in the sense of being representative of the motion of the earth's surface.

It is useful to present a set of constraints equivalent to (2.4-49). Without loss of generality we continue with summation instead of integration for the reason mentioned above. That is, we shall consider the polyhedron as a system of discrete mass particles with internal motions and rotating in inertial space. Its angular momentum vector H is related to the torques L by Euler's equation

$$L = \left[\frac{dH}{dt}\right]_I = \left[\frac{dH}{dt}\right]_T + \omega \times H \tag{2.4-50}$$

The total angular momentum is given by

$$H = \sum_{i=1}^{p} m_i \; (X_i \times V_i) \tag{2.4-41}$$

which from (2.4-47)

$$H = \sum_{i=1}^{p} m_i \, [X_i \times (\omega \times X_i + V_T)] \tag{2.4-52}$$

$$= \sum_{i=1}^{p} m_i \, [X_i \times (\omega \times X_i)] + \sum_{i=1}^{p} m_i \, (X_i \times V_T) \tag{2.4-53}$$

$$= I \cdot \omega + h \tag{2.4-54}$$

Thus, the angular momentum splits into a rigid body motion

$$H_R = I \cdot \omega \tag{2.4-55}$$

where I is the inertia tensor, and into a relative angular momentum, h due to the deformation of the system of particles. This is analogous to splitting the gravity field into a normal and disturbing potential, or a satellite orbit into a Keplerian orbit and its perturbations. Carrying the satellite analogy further, the fundamental polyhedron corresponds to the Keplerian orbit. If at the fundamental epoch, the earth became rigid, then the axes defined by the polyhedron would continue to rotate with angular momentum H_R.

It can be shown that the constraints (2.4-48) are equivalent to the condition

$$h = 0 \tag{2.4-56}$$

Working along the lines of (Jeffreys, 1970) denote for the discrete case

$$T = \sum_{i=1}^{p} m_i (V_{T_i} \cdot V_{T_i}) \tag{2.4-57}$$

$$= \sum_{i=1}^{p} m_i \; (V_I - \omega \times X_i)\cdot(V_I - \omega \times X_i) \tag{2.4-58}$$

$$\begin{aligned} = \sum_{i=1}^{p} m_i \; [&(V_1 - \omega_2 z + \omega_3 y)_i^2 \\ &+ (V_2 - \omega_3 x + \omega_1 z)_i^2 \\ &+ (V_3 - \omega_1 y + \omega_2 x)_i^2] \end{aligned} \tag{2.4-59}$$

Minimizing T with respect to the components of ω, the instantaneous rotation vector, yields

$$\frac{\partial T}{\partial \omega_1} = \sum_{i=1}^{p} m_i [(y^2 + z^2)\omega_1 + (yv_3 - zv_2) - x(\omega_3 z + \omega_2 y)]_i = 0 \tag{2.4-60}$$

$$\frac{\partial T}{\partial \omega_2} = \sum_{i=1}^{p} m_i [(x^2 + z^2)\omega_2 + (xv_3 - zv_1) - y(\omega_1 x + \omega_3 z)]_i = 0 \tag{2.4-61}$$

$$\frac{\partial T}{\partial \omega_3} = \sum_{i=1}^{p} m_i [(x^2 + y^2)\omega_3 + (yv_1 - xv_2) - z(\omega_1 x + \omega_2 y)]_i = 0 \tag{2.4-62}$$

In matrix form,

$$\sum_{i=1}^{p} m_i \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & y^2 + x^2 \end{bmatrix}_i \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \sum_{i=1}^{p} m_i \begin{bmatrix} yv_3 - zv_2 \\ xv_3 - zv_1 \\ yv_1 - xv_2 \end{bmatrix}_i \tag{2.4-63}$$

or,

$$I \cdot \omega = H_R \tag{2.4-64}$$

implying that h = 0 from (2.4-54). This means that

$$h = \sum_{i=1}^{p} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}_i \begin{bmatrix} V_{T_1} \\ V_{T_2} \\ V_{T_3} \end{bmatrix}_i = 0 \tag{2.4-65}$$

$$= \sum_{i=1}^{p} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}_i \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}_i = 0 \tag{2.4-66}$$

where dX_i are the differential displacements of the polyhedron stations with respect to the fundamental epoch. Considering that only periodic re-observations of the polyhedron deformations will be available over finite time intervals, the differential displacements can be approximated by a finite displacement vector ΔX_i. Finally, if we interpret the mass elements m_i as weights, the discrete approximation of the conditions (2.4-48) and (2.4-56) is equivalent to the set of orientation constraints (2.4-11) of the previous section. As pointed out by (Munk and MacDonald, 1960) only the motion of the Tisserand axes are defined by the above constraints, the choice of origin and orientation being arbitrary. Following the requirements outlined in Chapter 1, we can choose the orientation of the polyhedron to be consistent with the BIH system and the origin to be at the center of mass, both at the fundamental epoch. The translation constraints (2.4-12) maintain the origin definition.

Thus, we have seen that the CTS reference frame as maintained by the constraints $CMX = 0$ is a discrete Tisserand's mean axes of crust or geographic axes as defined by (Munk and MacDonald, 1960).

2.4.3 Alternate Approaches

One drawback to Tisserand axes is precisely the problem of relative motions as pointed out by (Moritz, 1979, 1980a). In the presence of secular motions the Tisserand axes will rotate with respect to the observatories. In other words, the constraints $CX = 0$ (all stations are weighted equally, $M = I$) will introduce inconsistencies in the maintenance of the reference frame since in general the constraints have no relation to physical processes. This is the situation if one is restricted to geodetic observations on the earth's crust.

It would be an improvement if there was some model available for the expected deformations from which the model weight matrix M could be constructed. This can be done through absolute motion models from which plate motion velocities with respect to the mantle could be computed (section 4.2). Minster and Jordan (1978) refer to a mean mesospheric frame which "is fixed with respect to the average position of the deep mantle, assumed to be rigid or at least to have typical internal motions with slower motion than the motion of the plates." They construct a model based on the Wilson-Morgan fixed hot spot hypothesis (Wilson, 1963, 1965; Morgan, 1971, 1972) which is used in the simulations of Chapter 4. Another model is constructed by (Solomon and Sleep, 1974) by constraining the crust to have no net rotation with respect to the mantle. This is a requirement for any absolute motion model that is being considered for the CTS. It fits in with the generally accepted hypothesis that the crust and mantle very nearly rotate together (Smith, 1981). In any case, (Bender, 1981) indicates that the absolute plate motion velocities appear to differ by about 1 cm/year among the available absolute motion models even though they are derived from different plate motion assumptions. It should be noted though that there is a controversy regarding the hot spot hypothesis and absolute reference frames (e.g., Le Pichon, 1973).

Suppose a model weight matrix is constructed from an adopted absolute motion plate model (section 4.2). It can be constructed for any number of stations on the crust so one can define M_G as the global model matrix of infinite dimension. In this case

$$CM_G X = 0 \tag{2.4-67}$$

for a model that does not contain any net rotations of the crust with respect to the mantle. Recall that $X = X_1 - X_0$ are the coordinate changes (deformations). For a particular polyhedron of stations a finite dimensional M is constructed, a subset of M_G. Therefore, one could use the constraints $CM\hat{X} = 0$ developed in the previous section as an alternative to $C\hat{X} = 0$ when a model is available.

Alternatively, one could use the constraints $C\hat{X} = Y$ where Y is computed from the model. If $Y \neq 0$, this does not mean that there are common rotations (or translations) between the crust and mantle. Rather, for this particular distribution of stations, the absolute plate motion model implies, for example, that the coordinate changes due to plate motions of the crust over the mantle do not sum to zero. Another approach would be to use the M matrix directly in the deformation analysis without imposing any constraints. All these approaches have a corresponding deformation estimation algorithm which will be discussed in Chapter 3. Here we discuss in general terms the physical significance of using an M matrix.

Recall the assumption that the mantle and crust rotate in a mean sense together, or equivalently that the M_G matrix includes no net rotations between the crust and mantle. The CTS reference frame axes, defined at the initial epoch by the fundamental coordinates X_0, can then be considered fixed with respect to the crust and mantle. At a later epoch, the CTS observatories will have different coordinates X_1 (where all global rotations and translations have been removed) with respect to this frame. These new coordinates in the crust and mantle fixed frame define the CTS. In order to maintain the reference frame consistently, it is necessary to estimate the deformations $X = X_1 - X_0$.

Summarizing, periodically all CTS stations observe in a short campaign for the purpose of monitoring deformations. From the baseline lengths and the model matrix M (if one is adopted), the deformation vector X is estimated in a second adjustment using one of the four algorithms of the next chapter. Any inconsistency between the re-estimated baseline lengths and the plate model will show up here.

Of course, the baseline lengths being crust fixed observations cannot be used to construct an absolute motion model. On the other hand, if there were no inconsistencies there would be no reason to challenge the assumption that the reference frame axes are crust and mantle fixed. After this short deformation analysis campaign, the subset of earth orientation monitoring stations continues its regular operations. But now, input into the earth orientation estimation algorithms (using the parameterization of section 2.3) are the updated CTS coordinates in the crust and mantle fixed system. Thus, polar motion and earth rotation still refer to the same reference frame axes as defined by X_0, i.e., the reference system is maintained. If one did not correct for the deformations, the reference frame axes would rotate with respect to the CTS and degrade the earth orientation estimates. Applying the constraints $C\hat{X} = 0$ without an M matrix means that the reference frame is only fixed in the crust; and if the distribution of stations is inadequate ($CX \neq 0$ in reality), then the earth orientation parameters would be degraded but to a lesser extent than not correcting for deformations at all.

Of course, one could use an absolute motion model and not talk about the mantle at all. But implicit in M would be the mantle fixed frame and the assumption of no common rotations with the crust. In fact, it is possible to adopt only a relative plate motion model (e.g., Minster and Jordan, 1974, 1978) and then no assumption about an absolute reference frame in the mantle would be involved at all. Everything would then refer to the crust, an approach that probably has many advocates (e.g., Smith, 1981). However, a relative motion model provides information only, of course, on the relative motions between stations not on changes in station coordinates (deformations). In any case, it seems preferable to estimate deformations using some plate model along with the re-estimated baseline lengths than to artificially impose the constraints $C\hat{X} = 0$. This conjecture will be tested in the simulations of Chapter 4. It is important to estimate the deformations using the geophysical data in a weak way in line with, as stated in section 1.1, the requirement that the CTS should not be dependent on geophysical hypotheses. In other words, the deformation estimates should be as insensitive as possible to errors in the geophysical model. Furthermore, any inconsistencies between the geodetic and geophysical data should be detectable in the estimation process. The purpose of the remainder of this investigation is to find such an estimation algorithm.

3 DEFORMATION ANALYSIS ALGORITHMS

3.1 Introduction

One of the requirements for the establishment of a reference system is the adoption of well-defined computational and estimation algorithms. We investigate in this chapter four possible algorithms for the analysis of polyhedron deformations.

The analysis of deformation is accomplished in a two step procedure. Periodically, observations are taken from all polyhedron stations. Each measurement system analyzes its own data from which the estimated baseline lengths, along with their covariance matrices, are pooled into one common set. These baseline lengths are then compared to their corresponding values at the initial epoch which determines how the polyhedron has deformed. However, the absolute location of the deformed polyhedron, i.e. its new coordinates (or rather the change in coordinates relative to the initial epoch), is undetermined from just the length of its edges, but this is what we seek. Without any other information the estimation of the deformation vector in this form is singular due to the familiar origin and orientation defects and therefore, the best linear unbiased estimate (BLUE) does not exist. This leads us to investigate other classes of estimators. Two are chosen from the class of biased estimators and two from the class of conditionally unbiased estimators. All are general enough to incorporate adopted geophysical deformation models but do so in fundamentally different ways. Furthermore, considering that a particular set of station coordinates defines the reference system, it is necessary to deal with the possibility of the loss or addition of a particular number of CTS stations. This leads to an application of least squares collocation.

3.2 Mathematical Model

Given are the adopted fundamental coordinates X_0 and their corresponding set of fundamental baseline lengths L_0 at an initial epoch. By comparing the estimated baseline lengths L_t at a later epoch t to L_0, the deformation of the polyhedron can be estimated.

The mathematical model for the deformation analysis is simply the chord length of baseline i - j at epoch t

$$L_{t_{ij}} = [(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2]^{1/2} \tag{3.2-1}$$

It is fundamental that the linearization of this model be performed about the initial coordinates, X_0, that define a datum for monitoring the time variations of the polyhedron. This is especially the case for our problem since we will be dealing with estimates that are sensitive to the initial parameter approximations. Linearization of (3.2-1) about X_0 yields for baseline i - j

$$L_{t_{ij}} = L_{0_{ij}} + \left.\frac{\partial L_{t_{ij}}}{\partial x_i}\right|_{X_0} (x_i - x_{i_0}) + \ldots + \left.\frac{\partial L_{t_{ij}}}{\partial z_j}\right|_{X_0} (z_j - z_{j_0}) \tag{3.2-2}$$

Adding a true noise vector V yields the basic observation equation

$$L = AX + V \qquad (3.2\text{-}3)$$

The observation vector is

$$L = L_t - L_0 \qquad (3.2\text{-}4)$$

the changes in the baseline lengths. We denote $\sigma_0^2 P^{-1}$ as the covariance matrix of the observed baseline lengths L_t, where σ_0^2 is the variance of unit weight. The assumption is that any uncertainty in L_0 is not considered (see section 4.4). The parameter vector X includes the deformations of the polyhedron, i.e., the change in coordinates between the initial epoch and a later one (e.g., ΔX_{t_0}, ΔX_{t_1}, ... of Fig. 1). The design matrix A contains the partial derivatives appearing in (3.2-2) evaluated at X_0. It has dimensions $n \times u$ where $n = p(p-1)/2$ is the number of polyhedron baselines (assuming all are observed), $u = 3p$ is the number of polyhedron coordinates, and p is the number of vertices. The column rank of A is deficient by 6, due to the orientation and origin defects so that

$$R(A) = 3p - 6 \qquad (3.2\text{-}5)$$

where R denotes the rank of the matrix.

Without any a priori information on the parameter vector X one is limited to the Generalized Gauss-Markoff (GGM) estimation model (L, AX, Q_V) where (e.g., Rao and Mitra, 1971)

$$E\{V\} = 0 \quad ; \quad D[V] = Q_V = \sigma_0^2 P^{-1} \qquad (3.2\text{-}6)$$

from which

$$E\{L\} = AX \quad ; \quad D[L] = \sigma_0^2 P^{-1} \tag{3.2-7}$$

The operator E stands for expectation and D for dispersion.

The rank deficiency of the A matrix implies that the best linear unbiased estimate (BLUE) for X does not exist. This leads us initially to investigate the class of biased estimators. In the next section, we examine two estimators of this class.

3.3 Biased Estimation

3.3.1 Best Linear Minimum Bias Estimation

3.3.1.1 A Deterministic Approach

It is well known that the method of least squares can be developed in a purely deterministic manner, most easily using the concept of inner product spaces. The starting point is an inconsistent set of linear (in our case linearized) equations

$$Y = AX \tag{3.3-1}$$

The matrix A represents a linear transformation of the vector X in E^u, a u-dimensional (parameter) vector space, to a vector Y in E^n, an n-dimensional (observation) vector space. E^n becomes an inner product space with the definition of an inner product, in the most general case a weighted inner product

$$\langle y_1, y_2 \rangle_P = y_1^T P y_2 \ ; \ y_1, y_2 \in E^n \tag{3.3-2}$$

It must fulfill the following properties (Davis, 1975)

(1) $\langle x,x \rangle_P \geq 0$, $\langle x,x \rangle_P = 0$ if and only is $x = 0$ (Positivity)

(2) $\langle x_1,x_2 \rangle_P = \langle x_2,x_1 \rangle_P$ (Symmetry)

(3) $\langle \alpha x_1,x_2 \rangle_P = \alpha \langle x_1,x_2 \rangle_P$; α real (Homogeneity)

(4) $\langle x_1 + x_2,x_3 \rangle_P = \langle x_1,x_3 \rangle_P + \langle x_2,x_3 \rangle_P$ (Linearity)

For the weighted inner product to be properly defined, the weight matrix P must be positive definite. An inner product is essential for least squares solutions since it introduces the concept of projection.

An inner product space is also a normed vector space, the weighted (ellipsoidal) norm being defined through the inner product as

$$\| y \|_P = (y^T P y)^{1/2} \; ; \; y \in E^n \tag{3.3-3}$$

providing the concept of length. It must fulfill the properties (Davis, 1975)

(1) $\| y \|_P \geq 0$ (Positivity)

(2) $\| y \|_P = 0$ if and only if $y = 0$ (Definiteness)

(3) $\| \alpha y \|_P = |\alpha| \, \| y \|_P$ for every scalar α (Homogeneity)

(4) $\| x+y \|_P \leq \| x \|_P + \| y \|_P$ (Triangle Inequality)

Again, P must be positive definite.

Normed spaces are also metric spaces, the weighted metric defined as

$$d(y_1,y_2)_P = \| y_1 - y_2 \|_P = [(y_1-y_2)^T P (y_1-y_2)]^{1/2} \tag{3.2-4}$$

providing a generalized concept of distance.

Likewise, in E^u we define

$$<x_1, x_2>_M = x_1^T M x_2 \ ; \ x_1, x_2 \in E^u \tag{3.3-5}$$

$$\| x \|_M = (x^T M x)^{1/2} \ ; \ x \in E^u \tag{3.3-6}$$

$$d(x_1, x_2)_M = \| x_1 - x_2 \|_M = [(x_1 - x_2)^T M (x_1 - x_2)]^{1/2} \tag{3.3-7}$$

In the most general case, and the one that we will encounter in the analysis of deformations, the A matrix of equation (3.2-3) and the weight matrix M in (3.3-5) - (3.3-7) are rank deficient. For non-positive definite M and P, their corresponding weighted inner product and normed spaces are improperly defined as will be discussed later. In this investigation, though, P is always assumed to be positive definite.

Consider a solution to the set of equations (3.3-1)

$$X = GY \tag{3.3-8}$$

In our general case, we would like to preserve the property of least squares and this can be accomplished by minimizing the weighted norm

$$\| Y - AX \|_P = [(Y - AX)^T P (Y - AX)]^{1/2} \tag{3.3-9}$$

This leads to the familiar normal equations

$$NX - U = 0 \tag{3.3-10}$$

where

$$N = A^T PA \ ; \ U = A^T PY \tag{3.3-11}$$

Since A is in our case rank deficient, so is N. From (3.2-5)

$$R(N) = 3p - 6 \tag{3.3-12}$$

and therefore (3.3-10) cannot be solved for X using Cayley inversion of N. In order to arrive at a solution of (3.3-10), generalized matrix algebra is required, particularly that of the pseudoinverse so that the solution is unique, the primary prerequisite property. The ordinary pseudoinverse solution $X = N^+U$ where $+$ denotes the pseudoinverse is well known and has been used, for example, in the analysis of local horizontal deformations (e.g., Brunner et al., 1979). However, the most general case of a weighted M-norm, particularly for a singular M matrix, has not been treated let alone applied to any geodetic problem. Let us, then, develop the case of an M-norm in E^u, considering first that M is positive definite, along the lines of (Rao and Mitra, 1971).

In order to introduce the weighted norm $\|X\|_M$ we minimize the Lagrangian function

$$\phi = X^T MX - 2K^T(NX-U) \tag{3.3-13}$$

so that the solution vector will have the additional property of minimizing $X^T MX$ subject to the P least squares property implicit in the normal equations. Minimizing ϕ with respect to X and the Lagrangian multiplier K yields two matrix equations

$$MX - NK = 0 \tag{3.3-14}$$

$$NX - U = 0 \tag{3.3-15}$$

From (3.3-14)

$$X = M^{-1}NK \tag{3.3-16}$$

and substituting into (3.3-15)

$$NM^{-1}NK - U = 0 \tag{3.3-17}$$

Since

$$R(NM^{-1}N) = R(N) = 3p - 6 \tag{3.3-18}$$

then

$$K = (NM^{-1}N)^{+} U \tag{3.3-19}$$

Substituting into (3.3-16) yields the following unique solution

$$X = M^{-1}N(NM^{-1}N)^{+}U \tag{3.3-20}$$

where for (3.3-8)

$$G = M^{-1}N(NM^{-1}N)^{+}A^{T}P \tag{3.3-21}$$

Note that the $+$ could be replaced by any generalized (g) inverse (Appendix A.1). Using the symbolism of (Rao and Mitra, 1971)

$$G = A^{+}_{PM} \tag{3.3-22}$$

is the minimum M-norm P least squares g-inverse of A . *It has also* been referred to as the weighted pseudoinverse (Boullion and Odell, 1971). In the case of $M = I$,

$$G = A^{+}_{PI} = N^{+}A^{T}P \tag{3.3-23}$$

and

$$X = N^{+}U \tag{3.3-24}$$

the ordinary pseudoinverse solution.

In order to prove that G is indeed an A^{+}_{PM} the following four conditions must hold (Appendix A.1)

$$AGA = A \tag{3.3-25}$$

$$GAG = G \tag{3.3-26}$$

$$(GA)^{T}M = MGA \tag{3.3-27}$$

$$(AG)^{T}P = PAG \tag{3.3-28}$$

Conditions (3.3-25) and (3.3-28) are equivalent to (see Appendix A.1)

$$A^{T}PAG = A^{T}P \tag{3.3-29}$$

and (3.3-26) and (3.3-27) to

$$G^{T}MGA = G^{T}M \tag{3.3-30}$$

The proof that (3.3-29) and (3.3-30) are fulfilled for (3.3-21) can be found in Appendix A.2.

It can be shown (Appendix A.3) that

$$N^{+}_{IM} = M^{-1}N(NM^{-1}N)^{+} \tag{3.3-31}$$

the corresponding G matrix for the consistent set of normal equations so that

$$X = N^{+}_{IM}U \tag{3.3-32}$$

from which we get the relationship

$$A^{+}_{PM} = N^{+}_{IM}A^{T}P \tag{3.3-33}$$

Summarizing, the solution X = GY for the set of inconsistent linear equations (3.3-1) is unique and has the property of minimum M-norm P least squares. Substituting X = GY into (3.3-1) yields a new vector

$$Y' = AGY \tag{3.3-34}$$

where $Y' = AX$ is a consistent set of equations. The matrix AG can be easily shown to be idempotent and therefore a projection operator. It projects any vector Y in E^u into Y' a vector in the image (the column space) of A . In fact, the consistency of $Y' = AX$ is used in defining the concept of generalized inverse (Rao and Mitra, 1971) which indicates why it is important that A^{+}_{PM} be a g-inverse. This will concern us further in section 3.3.1.4 where the more general case of positive semidefinite M is treated.

3.3.1.2 Estimation Model

The results of the previous section can be applied to the linear estimation problem essentially through an appropriate choice of inner products by considering the weight matrices P and M as inverses of moment matrices. Consider again the set of observation equations

$$L = AX + V \tag{3.2-3}$$

We assume an expanded GGM model $(L, AX, Q_V, Q_{\bar{X}})$ where

$$E\{V\} = 0 \tag{3.3-35}$$

$$Q_V = E\{VV^T\} = D[V] = \sigma_0^2 P^{-1} \tag{3.3-36}$$

and

$$\begin{aligned} Q_{\bar{X}} &= E\{\bar{X}\bar{X}^T\} = (\tau_0^2 M^{-1} \quad \text{if } Q_{\bar{X}} \text{ positive definite}) \\ &= \Sigma_{\bar{X}} + \mu_{\bar{X}}\mu_{\bar{X}}^T \end{aligned} \tag{3.3-37}$$

such that

$$\mu_{\bar{X}} = E\{\bar{X}\} = X \tag{3.3-38}$$

$$\Sigma_{\bar{X}} = E\{(\bar{X}-X)(\bar{X}-X)^T\} \tag{3.3-39}$$

It follows, as in the GGM model (3.2-6,7), that

$$E\{L\} = AX \quad ; \quad D[L] = \sigma_0^2 P^{-1} \tag{3.3-40}$$

In this setup, $\bar{X}$ is an independent estimate of the parameter vector and is stochastic in nature. On the other hand X is deterministic. In this section, we assume that $Q_{\bar{X}}$ is positive definite. Note the two variances of unit weight σ_0^2 and τ_0^2, the latter related to the parameter space. Note that $\bar{X}$ does not appear in (3.3-40), but enters only through the E^u inner product weight matrix M.

The solution vector X (3.3-20) then becomes our first deformation estimate

$$\hat{X}_1 = Q_{\bar{X}} N (N Q_{\bar{X}} N)^{+} U \tag{3.3-41}$$

$$= M^{-1} N (N M^{-1} N)^{+} U \tag{3.3-42}$$

$$= G_1 L = A^{+}_{PM} L \tag{3.3-43}$$

G_1 being defined as in (3.3-21). This is derived from minimization of (3.3-9)

$$\| Y - AX \|^2_P = \| V \|^2_P = V^T P V \tag{3.3-44}$$

followed by, as before,

$$\| X \|_M = \text{minimum} \tag{3.3-45}$$

Thus, the statistical meaning of $\hat{X}_1$ follows from the model defined by (3.3-35) - (3.3-40). The P and M matrices are defined now through the moment matrices Q_V and $Q_{\bar{X}}$. In our case, Q_V is the covariance matrix for the baseline lengths (since $E\{V\} = 0$). In the following then, we denote a covariance matrix by Σ and a moment matrix by Q , e.g., Σ_L instead of Q_V. The moment matrix $Q_{\bar{X}}$ can be constructed from an adopted model that predicts the deformations of the polyhedron stations (an example is given in section 4.2). This can be done only approximately for this estimation model since in (3.3-37) $\mu_{\bar{X}} = X$ is unknown and we must approximate $\mu_{\bar{X}}$ by $\bar{X}$.

Now that $\hat{X}_1$ has statistical meaning, using (3.3-40)

$$E\{\hat{X}_1\} = G_1 E\{L\} = G_1 AX = M^{-1} N (N M^{-1} N)^{+} NX \tag{3.3-46}$$

But the rank of A is not full so that

$$G_1 A \neq {}_uI_u \tag{3.3-47}$$

since in general for the product of two matrices A and B

$$R(AB) \leq R(A) \quad \text{and} \quad \leq R(B) \tag{3.3-48}$$

Thus,

$$E(\hat{X}_1) \neq X \tag{3.3-49}$$

and $\hat{X}_1$ is a biased estimate. However, it can be shown that $\hat{X}_1$ results from minimizing the bias norm

$$\| I - GA \|_{M^{-1}} \tag{3.3-50a}$$

or alternatively

$$tr[(I - GA)M^{-1}(I - GA)^T] \tag{3.3-50b}$$

where tr denotes the trace operator (Chipman, 1964). For this to hold, it is necessary and sufficient that

$$\begin{aligned} AGA &= A \\ (GA)^T M^{-1} &= M^{-1} GA \end{aligned} \tag{3.3-51}$$

An alternative formulation for the minimum bias estimator is as follows (Rao and Mitra, 1971). Considering the model (3.3-35) - (3.3-40) find a linear estimate $G^T L$ for X such that

$$tr(\Sigma_{G^T L}) = \sigma_0^2 \, tr\,(G^T \Sigma_L G) = \sigma_0^2 \, \|G^T\|^2_{\Sigma_L} \tag{3.3-52}$$

is a minimum (this is the minimum norm condition) in the class of estimators which minimize the bias (the least squares condition)

$$\| A^T G^T - I \|_{Q_{\bar{X}}} = [tr((A^T G^T - I)^T Q_{\bar{X}} (A^T G^T - I))]^{1/2} \tag{3.3-53}$$

where $Q_{\bar{X}}$ is positive definite. Then G^T is a minimum Q_V (Σ_L) -norm, $Q_{\bar{X}}$ -least squares solution of $A^T G^T = I$ (which is inconsistent) such that

$$\begin{aligned} G^T &= (A^T)^+_{Q_{\bar{X}} Q_V} \\ &= [A^+_{Q_V^{-1} Q_{\bar{X}}^{-1}}]^T \end{aligned} \tag{3.3-54}$$

using a result from (Rao and Mitra, 1971) whereby

$$G = A^+_{PM}$$

and

$$\hat{X}_1 = G_1 L$$

From the above, it can also be seen that $\hat{X}_1$ is also a minimum variance estimate in the class of minimum bias estimators (see also (Chipman, 1964)).

The covariance matrix for $\hat{X}_1$ is given from (3.3-43) by

$$\begin{aligned} \Sigma_{\hat{X}_1} &= G_1 \Sigma_L G_1^T \\ &= \sigma_0^2 M^{-1} N (NM^{-1} N)^+ N (NM^{-1} N)^+ NM^{-1} \\ &= \sigma_0^2 N^+_{IM} N (N^+_{IM})^T \end{aligned} \tag{3.3-55}$$

where N^+_{IM} is given by (3.3-31). An unbiased estimate for the a posteriori variance of unit weight is (Uotila, 1967)

$$\hat{\sigma}_0^2 = \frac{\hat{V}^T P \hat{V}}{n - R(A)} \tag{3.3-56}$$

where

$$\hat{V}^T P \hat{V} = L^T P L - \hat{X}_1^T U \tag{3.3-57}$$

In our case, for p stations

$$n = \frac{p(p-1)}{2} \tag{3.3-58}$$

and

$$R(A) = 3p - 6 \tag{3.3-59}$$

from which it follows that

$$n - R(A) = \frac{(p-4)(p-3)}{2} = \frac{1}{2}\,\frac{(p-3)!}{(p-5)!} \tag{3.3-60}$$

It is evident that in order to achieve redundancy at least five polyhedron stations are required.

An estimate for τ_0^2 of (3.3-37) is approximately given by

$$\hat{\tau}_0^2 = \frac{\hat{X}^T M \hat{X}}{u} \tag{3.3-61}$$

where, in this case, u is the number of parameters (and the rank of M).

Summarizing, $\hat{X}_1$ has the deterministic properties of minimum M-norm in the class of P least squares estimators and the stochastic properties of minimum variance in the class of minimum bias estimators. Therefore, $\hat{X}_1$ is known as the Best (minimum variance) Linear Minimum Bias Estimate or BLIMBE (Rao and Mitra, 1971). In the next section, we see that $\hat{X}_1$ can be computed by augmenting the singular normal equations (3.3-10) with the set of constraints CMX = 0 derived in Section 2.4.1.

3.3.1.3 The Weighted Inner Constraint Estimate

It is well known that the ordinary pseudoinverse estimate

$$\begin{aligned} \hat{X} &= N^{+}U \\ &= A^{+}_{PI}L \end{aligned} \tag{3.3-62}$$

is equivalent to augmenting the system of singular normal equations $NX = U$ by a matrix C such that the conditions

$$NC^T = 0 \quad (\text{or } AC^T = 0) \tag{3.3-63}$$

$$C\hat{X} = 0 \tag{3.3-64a}$$

are fulfilled (Meissl, 1969; Blaha, 1971). The set (3.3-64a) is known as an inner constraint. For our particular case, we need six such constraints, the number of rank deficiencies of N. We generalize the results to deal with the definition of an M-norm in the parameter space. We assume that M is positive definite. The derivation of this weighted inner constraint estimate is a generalization of the development in (Blaha, 1971). Instead of (3.3-64a) we substitute the condition

$$CM\hat{X} = 0 \tag{3.3-64b}$$

We start by minimizing the Lagrangian function

$$\phi = V^TPV - 2K_1^T(AX + V - L) - 2K_2^T(CMX) \tag{3.3-65}$$

with respect to the unknowns V, X, K_1, K_2. After some matrix manipulations we arrive at

$$\begin{bmatrix} N & MC^T \\ CM & 0 \end{bmatrix} \begin{bmatrix} X \\ -K_2 \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix} \tag{3.3-66}$$

The resulting augmented normal matrix is non-singular and we could solve for X directly. In order to eliminate K_2, and derive an explicit expression for X we can write

$$\begin{bmatrix} N & MC^T \\ CM & 0 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.3-67}$$

Explicitly, this represents four matrix equations

$$ND_1 + MC^TD_3 = I \tag{3.3-68}$$

$$ND_2 + MC^TD_4 = 0 \tag{3.3-69}$$

$$CMD_1 = 0 \tag{3.3-70}$$

$$CMD_2 = I \tag{3.3-71}$$

Pre-multiplying (3.3-69) by C

$$CND_2 + CMC^TD_4 = 0 \tag{3.3-72}$$

But $CN = 0$ from (3.3-63) and since CMC^T is full rank (which follows from the assumption of M positive definite), we have

$$D_4 = 0 \tag{3.3-73}$$

Pre-multiplying (3.3-68) by C

$$CND_1 + CMC^TD_3 = C \tag{3.3-74}$$

from which

$$D_3 = (CMC^T)^{-1} C \tag{3.3-75}$$

Inserting $D_2 = D_3^T$ into (3.3-71)

$$(CMC^T)(CMC^T)^{-1} = I$$

which implies that

$$D_2 = D_3^T = C(CMC^T)^{-1} \tag{3.3-76}$$

Substituting into (3.3-68) yields

$$ND_1 + MC^T(CMC^T)^{-1}C = I \tag{3.3-77}$$

This is equivalent to

$$[N+kMC^TCM][D_1+k^{-1}C^T(CMC^T)^{-1}(CMC^T)^{-1}C] = I \tag{3.3-78}$$

This follows from (3.3-63) and (3.3-70). The scale factor k is arbitrary as can be seen by performing the multiplications on the left hand side of (3.3-78). It is useful for improving the condition of $N+MC^TCM$ for inversion. It can be shown following the same reasoning as in (Blaha, 1971) that the matrix $N+kMC^TCM$ is non-singular. Therefore

$$D_1 = (N+kMC^TCM)^{-1} - k^{-1}C^T(CMC^T)^{-1}(CMC^T)^{-1}C \tag{3.3-79}$$

From (3.3-66) we have

$$\hat{X} = D_1 U \qquad (3.3\text{-}80)$$

$$= [(N+kMC^TCM)^{-1} - k^{-1}C^T(CMC^T)^{-1}(CMC^T)^{-1}C]U \qquad (3.3\text{-}81)$$

$$= (N+kMC^TCM)^{-1}U \qquad (3.3\text{-}82)$$

since $CA^T = 0$, and

$$\Sigma_{\hat{X}} = \sigma_0^2[(N+kMC^TCM)^{-1} - k^{-1}C^T(CMC^T)^{-1}(CMC^T)^{-1}C] \qquad (3.3\text{-}83)$$

In order to demonstrate that this estimate is equivalent to the BLIMBE estimate $X_1 = A^+_{PM}L$ it is necessary to prove that $G = D_1A^TP$ is A^+_{PM}. For this to hold, it must satisfy the conditions (3.3-29) and (3.3-30). First, we show that

$$A^TPAG = A^TP \qquad (3.3\text{-}29)$$

Post-multiply (3.3-77) by A^TP

$$A^TPAD_1A^TP + MC^T(CMC^T)^{-1}CA^TP = A^TP \qquad (3.3\text{-}84)$$

which follows from $CA^T = 0$. Second, we prove that

$$G^TMGA = G^TM \qquad (3.3\text{-}30)$$

Transpose (3.3-77) and pre-multiply by $(D_1A^TP)^TM$ $(=G^TM)$

$$\begin{aligned}(D_1A^TP)^TMD_1A^TPA &+ (D_1A^TP)^TMC^T(CMC^T)^{-1}CM \\ &= G^TMGA + PAD_1MC^T(CMC^T)^{-1}CM \qquad (3.3\text{-}85) \\ &= G^TM\end{aligned}$$

which follows from (3.3-70). Therefore, the weighted inner constraint estimate (3.3-82) is equivalent to the BLIMBE (3.3-43). However, the former is computationally more efficient considering that only a Cayley inverse is required.

A note should be added concerning the units for the computation of $\hat{X}$ by (3.3-82). In order for (3.3-64b) to be consistent, C and M must be unitless so that the 0-vector will be in units of length. Therefore, we divide the length unit elements of C by a mean earth radius R, i.e. (see 3.3-90 below),

$$S_i = \begin{bmatrix} 0 & \frac{-z_i}{R} & \frac{-y_i}{R} \\ \frac{z_i}{R} & 0 & \frac{-x_i}{R} \\ \frac{-y_i}{R} & \frac{x_i}{R} & 0 \end{bmatrix} \tag{3.3-86}$$

In addition, M is divided by its Euclidean norm

$$\| M \| = (\Sigma m_{ij})^{1/2} \tag{3.3-87}$$

so that

$$\bar{M} = \frac{M}{\| M \|} \tag{3.3-88}$$

Therefore, in (3.3-82) we should write strictly

$$\hat{X} = (N + \bar{M}C^T C\bar{M})^{-1} U \tag{3.3-89}$$

but this has no effect on the estimate which is independent of a scale factor k as shown above.

The constraints C for this particular problem can be derived from the condition $AC^T = 0$. It is found that

$$C = \begin{bmatrix} S_1 & S_2 & \cdots & S_P \\ I & I & \cdots & I \end{bmatrix} \tag{3.3-90}$$

fulfills this condition which is just (2.4-9), the reference frame maintenance constraints. Therefore, the BLIMBE (or equivalently the weighted inner constraint) estimate provides an algorithm to maintain a discrete version of the Tisserand mean axes of crust which incorporates also the stochastic properties of non-perfect measurements and adopted geophysical models (that come through the weighted parameter norm). In this case, we can say that the reference frame axes are maintained in a minimum M-norm P least squares and BLIMBE sense by a specified number of CTS stations.

3.3.1.4 A Generalization for M-Seminorms

In the previous sections, we have derived X under the assumption that P and M are positive definite, i.e.,

$$\| x \|_M = (x^T M x)^{1/2} > 0 \qquad \text{for all } x \in E^u \tag{3.3-91}$$

$$\| y \|_P = (y^T P y)^{1/2} > 0 \qquad \text{for all } y \in E^n \tag{3.3-92}$$

except for the trivial cases $x = 0$ and $y = 0$ for which equality holds. As we shall see, it is quite possible in our applications that the moment matrix $Q_{\bar{X}}$ is only positive semidefinite (it must be at least positive semi-definite since it is a moment matrix). This more general case raises several problems. First, M can no longer be defined as the Cayley inverse of $Q_{\bar{X}}$ (in general, non-positive definiteness does not necessarily imply singularly, however, since we are dealing with moment matrices, non-positive definiteness does imply singularity). Second, assuming an appropriate M could be found, the properties of the weighted inner product and norm listed in section 3.3.1.1 may no longer hold for E^u.

If M is a positive definite matrix, it can be expressed as

$$M = U \Lambda U^T \tag{3.3-93}$$

by singular value decomposition where Λ is a diagonal matrix of dimension u whose non-zero elements are the real eigenvalues λ_i of M, and the columns of the n-dimensional orthogonal matrix U ($U^T U = I$, $UU^T = I$) are the corresponding normalized eigenvectors U_i. In this case, all the eigenvalues are positive (nonzero). The eigenvectors form an orthogonal basis for E^u. Therefore, the weighted norm $\| x \|_M$ and inner product $\langle x_1, x_2 \rangle_M$ can be defined properly over the entire space E^u. However, if M is only positive semidefinite, then it can be decomposed as (e.g., Lanczos, 1961)

$$\underset{u\,u}{M} = \underset{u\,p}{\tilde{U}}\ \underset{p\,p}{\tilde{\Lambda}}\ \underset{p\,u}{\tilde{U}^T} \tag{3.3-94}$$

where $\tilde{\Lambda}_p$ is a p-dimensional diagonal matrix whose diagonal elements are the p non-zero eigenvalues of M (there are u-p zero eigenvalues) and the p columns of the semiorthogonal matrix $\tilde{U}$ ($\tilde{U}^T\tilde{U} = I$, $\tilde{U}\tilde{U}^T \neq I$) which contain their corresponding p eigenvectors, i.e., the reduced eigenspace

$$[(\lambda_i,\tilde{U}_i) \mid \lambda_i > 0 \ , \quad i = 1,\ldots,p] \tag{3.3-95}$$

The $\tilde{U}_i$ are also called the principal vectors of M (Ben Israel and Greville, 1974). In this case, the principal vectors form on orthogonal basis only for a subspace of E^u, call it $E^{\tilde{u}}$.

It follows that the weighted (ellipsoidal) norm and inner product can be defined properly over the same subspace $E^{\tilde{u}}$. That is, for the inner product

$$\langle x_1,x_2\rangle_M = x_1^T M x_2 \ ; \ x_1,x_2 \in E^{\tilde{u}} \tag{3.3-96}$$

and for the norm

$$\| x \|_M = (x^T M x)^{1/2} \ ; \ x \in E^{\tilde{u}} \tag{3.3-97}$$

However, the parameter vector $\hat{X}$ may not necessarily be an element of $E^{\tilde{u}}$. Therefore, one is led to define an "improper" weighted (hyperbolic) norm for the entire space E^u although it can "properly" be applied to physical problems (Pease, 1965). Property 1 of a proper inner product (see section 3.3.1.1) must be modified to

(1') $\langle x,x\rangle_M \geq 0$ (Positive Semidefiniteness)

since the inner product could be equal to zero for $x \neq 0$. This means that a vector x could conceivably be orthogonal to itself but in the semidefinite weighted sense. Likewise, the property (2) of definiteness no longer holds for a weighted norm. Rao and Mitra (1971) refer to such norms as seminorms. Furthermore, as shown in (Pease, 1975), the triangle inequality property (4) of section 3.3.1.1 may also fail.

Consider Q_X as only positive semidefinite. Then

$$Q_X = \tilde{U}_{\bar{X}}\tilde{\Lambda}_{\bar{X}}\tilde{U}^T_{\bar{X}} \tag{3.3-98}$$

as described above. In this case, a reasonable choice for M is

$$M = Q^+_X = \tilde{U}_X\tilde{\Lambda}^{-1}_X\tilde{U}^T_X \tag{3.3-99}$$

a well known result for the computation of a pseudoinverse. It can be shown that M is also positive semidefinite (Lewis and Newman, 1968). Consider $Q_{\bar{X}}$ as an operator

$$Q_X \; : \; E^{\tilde{u}_1} \longrightarrow E^{\tilde{u}_2} \tag{3.3-100}$$

where $E^{\tilde{u}_1}$ and $E^{\tilde{u}_2}$ are p-dimensional inner product spaces. For $u \notin E^{\tilde{u}_1}$

$$Q_X u = 0 \tag{3.3-101}$$

Similarly,

$$M : E^{\tilde{u}_2} \longrightarrow E^{\tilde{u}_1} \tag{3.3-102}$$

This holds since $M = Q^+_X$ is a unique operator which makes the above choice of M suitable for our purposes (in fact, a reflexive g-inverse A^r (Appendix A.1) also has this property). In general though, for any matrix A

$$(A^g)^g \neq A$$

which makes just a g-inverse unsuitable.

Now that we have defined a seminorm $\| X \|_M$ and found a suitable choice for M, let us address the problem of generalizing the deformation estimate (3.3-43). This discussion will be primarily based on (Rao and Mitra, 1971; Mitra and Rao, 1974).

Let us first consider the simplest case of an ordinary least squares estimate (i.e. N is full rank) with a weighted parameter norm defined in the parameter space where M is positive definite. We start from the normal equations

$$NX - U = 0 \qquad (3.3\text{-}15)$$

and minimize

$$\phi = X^T MX - 2K^T (NX-U) \qquad (3.3\text{-}103)$$

with respect to X and K . This yields the relationship

$$MX - NK = 0 \qquad (3.3\text{-}104)$$

in addition to (3.3-15). It follows that

$$X = M^{-1} NK \qquad (3.3\text{-}105)$$

Into (3.3-10) and solving for K

$$K = (NM^{-1}N)^{-1}U \qquad (3.3\text{-}106)$$

Substituting into (3.3-105) gives

$$\hat{X} = M^{-1}N(NM^{-1}N)^{-1}U \qquad (3.3\text{-}107)$$
$$= N^{-1}U$$

Of course, this could have been directly obtained from (3.3-5). However, it does prove that the ordinary least squares estimate is invariant with respect to any defined positive definite norm in E^u. This is not surprising since both N and M are full rank and are not restricted to any subspace of E^u. This is really what we mean by an unbiased estimate since $E(\hat{X}) = X$ can conceivably hold for any $\hat{X} \in E^u$. In fact, it is enough that N be full rank even if M is not positive definite. That is, the best linear unbiased estimate (if it exists) is unaffected by any weighted norm in the parameter space. This follows from the fact that $I - GA = 0$ so that the bias norm (3.3-50) is zero independent of any weight matrix.

A more general case will now be described where the M matrix is positive semidefinite and the design matrix A is rank deficient. As pointed out in (Rao and Mitra, 1971) care must now be taken since both M and N are not of full rank. Let us return to the two matrix equations

$$MX - NK = 0 \qquad (3.3\text{-}14)$$

$$NX - U = 0 \qquad (3.3\text{-}15)$$

Combining (3.3-14) and (3.3-15)

$$(N + M)X - NK = U \qquad (3.3\text{-}109)$$

from which

$$X = (N+M)^{-1}(NK+U) \qquad (3.3\text{-}110)$$

assuming that N+M is full rank the necessity of which for our application will be described below. Into (3.3-15) yields

$$N(N+M)^{-1}(NK+U) = U \qquad (3.3\text{-}111)$$

that can be solved for K as

$$K = [N(N+M)^{-1}N]^{+}[I-N(N+M)^{-1}]U \qquad (3.3\text{-}112)$$

Substituting back into (3.3-110)

$$\begin{aligned}\hat{X} &= (N+M)^{-1}N[N(N+M)^{-1}N]^{+}U \\ &- (N+M)^{-1}N[N(N+M)^{-1}N]^{+}N(N+M)^{-1}U \\ &+ (N+M)^{-1}U\end{aligned} \qquad (3.3\text{-}113)$$

The last two terms can be shown to cancel using the relations (Rao and Mitra, 1971)

$$A(A^{T}PA)^{g}A\ PA = A \qquad (3.3\text{-}114)$$

$$(A^{T}PA)(A^{T}PA)^{g}A^{T} = A^{T} \qquad (3.3\text{-}115)$$

for any matrix P such that $R(A^{T}PA) = R(A)$. This holds automatically for positive definite P for which the above results could be modified as

$$PA(A^TPA)^gA^TPA = PA \tag{3.3-116}$$

$$(A^TPA)(A^TPA)^gA^TP = A^TP \tag{3.3-117}$$

Using (3.3-117) the second term in (3.3-113) can be written as

$$(N+M)^{-1}N[N(N+M)^{-1}N]^+N(N+M)^{-1}NN^gA^TPL$$

which reduces from (3.3-114) to

$$(N+M)^{-1}NN^gA^TPL$$

Applying (3.3-117) again, the second term is seen to be equal to the third term in (3.3-113) so that

$$\begin{aligned}\hat{X} &= (N+M)^{-1}N[N(N+M)^{-1}N]^+U \\ &= GL\end{aligned} \tag{3.3-118}$$

where

$$G = (N+M)^{-1}N[N(N+M)^{-1}N]^+A^TP \tag{3.3-119a}$$

Note that the same estimation model is assumed as in section 3.3.1.2. The relation (3.3-117) can be used to show that $\hat{X}$ is invariant with respect to any g-inverse for the term in brackets. It gives

$$G = (N+M)^{-1}N[N(N+M)^{-1}N]^gNN^gA^TP \tag{3.3-119b}$$

Another result from (Rao and Mitra, 1971) is that $A(A^TPA)^gA^T$ is invariant for any choice of $(A^TPA)^g$ which applied to $N[N(N+M)^{-1}N]^gN$ proves the above assertion.

According to (Mitra and Rao, 1974) for the possibly inconsistent set of linear equations $L = AX$

$$G = (N+M)^{g}N[N(N+M)^{g}N]^{g}A^{T}P$$

called an A_{PM} inverse of A, is one choice for a minimum M-seminorm P-semileast squares inverse of A. It follows that $X = GL$ has minimum M-seminorm among the semileast squares solutions of $L = AX$. According to the same theorem, A_{PM} is unique if and only if $N+M$ is positive definite explaining our use of Cayley inversion in the above derivation for $\hat{X}$. However, uniqueness is not sufficient for our purposes since A_{PM} may not even be a g-inverse ($AA_{PM}A \neq A$). We require this property in order to make (3.3-34)

$$L' = AGL$$

consistent. Also, we would like A_{PM} to be reflexive so that $R(A_{PM}) = R(A)$ (Rao and Mitra, 1971). If these properties hold, then A_{PM} is called an A^{+}_{PM}; but recall that unlike the situation in section 3.3.1.1, P and M may be positive semidefinite. If both are positive definite, A^{+}_{PM} is equivalent to the G matrix of BLIMBE.

For A^{+}_{PM} to exist it is necessary and sufficient that

$$C(M) \cap C(A^{T}) \subset A^{T}P \qquad (3.3\text{-}120a)$$

where C denotes column space, according to theorem 3.6 of (Mitra and Rao, 1974). In our application P is always positive definite so that this condition could be modified to

$$C(M) \cap C(A^T) \subset A^T \tag{3.3-120b}$$

which always holds. Therefore, in order for G to be A^+_{PM}, according to theorem 3.2 in the same paper, the following four conditions are necessary and sufficient

$$PAGA = PA \tag{3.3-121}$$

$$MGAG = MG \tag{3.3-122}$$

$$(GA)^T M = MGA \tag{3.3-123}$$

$$(AG)^T P = PAG \tag{3.3-124}$$

That G fulfills these four conditions is proven in Appendix A.4. For P positive definite (3.3-121) is equivalent to (3.3-25). Conditions (3.3-123) and (3.3-124) are equivalent to (3.3-27) and (3.3-28), respectively. Therefore, the difference between the M positive definite case of section 3.3.1.2 and M positive semidefinite is just condition (3.3-122). Unfortunately, without the fulfillment of (3.3-26) which can occur only if M is positive definite, the solution $\hat{X} = GL$ for G in (3.3-119) although M-seminorm P least squares is no longer BLIMBE as we shall see in the simulations of Chapter 4. Thus, we will call $\hat{X}^*_1 = G^*_L L$ a MINOLESS (minimum norm (seminorm in this case) least squares solution) using the terminology of (Schaffrin, 1975). For positive definite M , $\hat{X}^*_1 \equiv X_1$ as shown in 3.3.1.2. Nevertheless, the solution (3.3-119) may still be applicable to deformation analysis as will be tested in Chapter 4.

The covariance matrix of $\hat{X}_1^*$ is, using (3.3-118),

$$\begin{aligned}\hat{\Sigma}_{\hat{X}_1^*} &= G\Sigma_L G^T \\ &= \hat{\sigma}_0^2(N+M)^{-1}N[N(N+M)^{-1}N]^+N[N(N+M)^{-1}N]^+N(N+M)^{-1}\end{aligned} \tag{3.3-126}$$

where $\hat{\sigma}_0^2$ is given by (3.3-56). Here, $\hat{\tau}_0^2$ is given approximately by

$$\hat{\tau}_0^2 = \frac{\hat{X}^T M \hat{X}}{R(M)} \tag{3.3-127}$$

where $Q_{\overline{X}} = \hat{\tau}_0^2 M^+$ (compare to (3.3-37) and (3.3-61)).

The deformation estimate $\hat{X}_1^*$ of this section is no longer equivalent to the weighted inner constraint solution of section 3.2.1.3 (recall that there M was assumed positive definite). Therefore the most one can say is that the reference system is maintained in a minimum M-seminorm P-least squares (MINOLESS not BLIMBE) sense by a specified set of CTS stations.

3.3.2 Best Linear Estimation

Another possible biased estimator is called the Best Linear Estimate or the BLE although several versions are available in the literature (e.g. Rao, 1973). Recall the multivariate definition of the mean square error

$$\begin{aligned}MSE(\hat{X}) &= E\{(\hat{X}-X)(\hat{X}-X)^T\} \\ &= \Sigma_{\hat{X}} + [X - E\{\hat{X}\}][X - E\{\hat{X}\}]^T\end{aligned} \tag{3.3-128}$$

as a sum of covariance and bias squared. As shown in section 3.3.1, the BLIMBE minimizes the bias term. Conditional on this property is the minimum variance one. The BLE, on the other hand, minimizes the mean square error. Unlike the BLIMBE, it can only be approached in a probabilistic manner. Its existence is dependent on some a priori knowledge of the parameter vector as we shall see below.

As before we will derive a homogeneous estimate of the form

$$X = GL$$

We assume now that both X and L are random variables with moment matrices

$$Q_X = E\{XX^T\} \tag{3.3-129}$$

$$Q_L = E\{LL^T\} \tag{3.3-130}$$

$$Q_{XL} = E\{XL^T\} \tag{3.3-131}$$

where Q_L is non-singular (note that this is not assumed for Q_X). By the Gauss-Markoff theorem, the linear minimum mean square error estimate of X is (Liebelt, 1967)

$$\hat{X} = Q_{XL}Q_L^{-1}L \tag{3.3-132}$$

with mean square error matrix

$$MSE(\hat{X}) = Q_X - Q_{XL}Q_L^{-1}Q_{XL}^T \tag{3.3-133}$$

Note that some knowledge of the moment matrices is required.

For the observation equations

$$L = AX + V \tag{3.2-3}$$

we assume an estimation model of the form $(L, A\bar{X}, Q_V, Q_X)$ where X and V have a priori probability distributions. For V

$$E\{V\} = 0 \tag{3.3-134}$$

$$Q_V = E\{VV^T\} = D[V] = \sigma_0^2 P^{-1} \tag{3.3-135}$$

and for X

$$E\{X\} = \bar{X} \tag{3.3-136}$$

$$D[X] = E\{(X-\bar{X})(X-\bar{X})^T\} = \Sigma_X \tag{3.3-137}$$

(recalling that $\bar{X}$ is an independent estimate of X) so that

$$Q_X = E\{XX^T\} = \Sigma_X + \bar{X}\bar{X}^T = \sigma_0^2 M^{-1} \tag{3.3-138}$$

where for positive semidefinite Q_X, $Q_X = \sigma_0^2 M^+$. Here X is stochastic and $\bar{X}$ deterministic, the oppositve of the BLIMBE model. The conditional distributions of L are then given by (Chipman, 1964)

$$E\{L|V\} = A\bar{X} + V \; ; \; D[L|V] = A\Sigma_X A^T \tag{3.3-139}$$

$$E\{L|X\} = AX \quad ; \; D[L|X] = \sigma_0^2 P^{-1} \tag{3.3-140}$$

from which

$$E\{L\} = E\{E\{L|V\}\} = E\{A\bar{X} + V\} = A\bar{X} \tag{3.3-141}$$

$$\begin{aligned} D[L] &= E\{D[L|V]\} + D[E\{L|V\}] \\ &= E\{A\Sigma_X A^T\} + D[A\bar{X} + V] \\ &= A\Sigma_X A^T + \sigma_0^2 P^{-1} = \Sigma_L \end{aligned} \tag{3.3-142}$$

Therefore,

$$Q_L = E\{LL^T\} = E\{(AX + V)(AX + V)^T\}$$

$$= A\ Q_X\ A^T + \sigma_0^2\ P^{-1} \tag{3.3-143}$$

$$\begin{pmatrix} = A\ \Sigma_X\ A^T + A\ \bar{X}\ \bar{X}^T + \sigma_0^2\ P^{-1} \\ = \Sigma_L + E\{L\}\ E\{L^T\} \end{pmatrix}$$

Furthermore,

$$Q_{XL} = E\{XL^T\} = E\{X(AX + V)^T\}$$

$$= E\{XX^T\}\ A^T + E\{XV^T\}$$

$$= Q_X\ A^T \tag{3.3-144}$$

where we have assumed for both (3.3-143) and (3.3-144) that

$$Q_{XV} = E\{XV^T\} = 0 \tag{3.3-145}$$

In our context, this means that baseline measurement errors and deformations are uncorrelated. As we shall see, it is useful to assume as indicated in (3.3-138) that $Q_{\hat{X}}$ is also known to within a scale factor σ_0^2, which can be taken as unity. The deviation of an unbiased estimate for σ_0^2, derived later, from unity will indicate the degree of compatibility between the baseline measurements and a chosen geophysical deformation model. With this in mind and from (3.3-132) it follows that

$$\hat{X}_S = Q_X\ A^T\ (A\ Q_X\ A^T + P^{-1})^{-1}\ L \tag{3.3-146}$$

where S denotes that X is assumed stochastic. The mean square error matrix can be computed from (3.3-133) as

$$MSE(\hat{X}_S) = \sigma_0^2\ [Q_X - Q_X\ A^T(AQ_XA^T + P^{-1})^{-1}\ A\ Q_X] \tag{3.3-147}$$

Note that strictly Q_X should be replaced by M^+ in (3.3-146) and (3.3-147) although we will continue with this notation since using M^+ may indicate misleadingly that pseudoinversion is required.

We see that this estimate is general enough to deal with rank deficient A and Q_X matrices encountered in the analysis of deformations. In the case that Q_X is nonsingular, using the identities (Liebelt, 1967)

$$Q_X A^T (A Q_X A^T + P^{-1})^{-1} = (A^T P A + Q_X^{-1})^{-1} A^T P \qquad (3.3\text{-}148)$$

and

$$(A^T P A + Q_X^{-1})^{-1} = Q_X - Q_X A^T (A Q_X A^T + P^{-1})^{-1} A^T Q_X \qquad (3.3\text{-}149)$$

both for possibly rank deficient A , we get in simpler form

$$\hat{X}_S = (N+M)^{-1} U \qquad (3.3\text{-}150)$$

and

$$MSE(\hat{X}_S) = \sigma_0^2 (N+M)^{-1} \qquad (3.3\text{-}151)$$

where $M = Q_X^{-1}$.

We see that the approach here to overcoming the singularity problem due to the rank deficiency of A (and N) is to use a priori information for the deformation vector X . In BLIMBE, the approach was to use pseudoinverse algebra or equivalently to augment N with a set of inner constraints.

Now, it is quite useful to derive another estimate which is a limiting case of the above estimate X_S, following essentially (Bibby and Toutenberg, 1977). We minimize the quadratic loss function

$$R(\hat{X}) = E\{(\hat{X}-X)^T B(\hat{X}-X)\} \qquad (3.3\text{-}152)$$

The expression in the brackets represents a generalized distance (compare to (3.3-7) for the metric defined by B (assuming that B is non-singular), or a weighted norm

$$\| e \|_{B^{-1}} \quad ; \quad e = \hat{X} - X \tag{3.3-153}$$

where B indicates the relative importance of the different elements of X . We start with the GGM model (L, AX, $\sigma_0^2 P^{-1}$) of section 3.2

$$L = AX + V \quad ; \quad V \sim (0, \sigma_0^2 P^{-1}) \tag{3.3-154}$$

The estimate X will be a function of L

$$\hat{X} = GL = GAX + GV \tag{3.3-155}$$

from which

$$\hat{X} - X = (GA - I)X + GV \tag{3.3-156}$$

and

$$\begin{aligned} R(\hat{X}) &= E\{[(GA-I)X + GV]^T B[(GA - I)X + GV]\} \\ &= X^T (A^T G^T - I) B(GA-I)X + E\{V^T GBG^T V\} \end{aligned} \tag{3.3-157}$$

where it can be shown that

$$E\{V^T GBG^T V\} = \sigma_0^2 \text{ trace } (BGP^{-1}G^T) \tag{3.3-158}$$

Note that $\sigma_0^2 GP^{-1}G^T$ is the covariance matrix of $\hat{X}$. Minimizing $R(\hat{X})$ with respect to G yields

$$G = XX^T A^T (AXX^T A^T + \sigma_0^2 P^{-1})^{-1} \tag{3.3-159}$$

so that

$$\hat{X}_D = XX^T A^T (AXX^T A^T + \sigma_0^2 P^{-1})^{-1} L \tag{3.3-160}$$

where D denotes that X is assumed deterministic. Note that this result is independent of B. The estimate doesn't seem to be of any use since it depends upon the true value of X which, of course, we do not know.

Let us examine (3.3-160) more closely. By the identity (3.3-149)

$$[AXX^T A^T + \sigma_0^2 P^{-1}]^{-1} = \frac{1}{\sigma_0^2}[P - PAX(X^T A^T PAX + \sigma_0^2)^{-1} X^T A^T P] \tag{3.3-161}$$

Note that although this identity holds also for a rank deficient A matrix, in the literature full rank is usually assumed in the derivations of BLE's. Substituting into (3.3-160)

$$\begin{aligned} \hat{X}_D &= \frac{1}{\sigma_0^2} XX^T A^T [P - PAX(X^T NX + \sigma_0^2)^{-1} X^T A^T P] L \\ &= \frac{1}{\sigma_0^2} X \left[1 - \frac{X^T NX}{X^T NX + \sigma_0^2}\right] X^T U \\ &= X \frac{X^T U}{X^T NX + \sigma_0^2} \end{aligned} \tag{3.3-162}$$

Now,

$$
\begin{aligned}
E\{\hat{X}_D\} &= \frac{X}{X^T N X + \sigma_0^2} X^T E(U) \\
&= X \frac{X^T N X}{X^T N X + \sigma_0^2} \qquad (3.3\text{-}163) \\
&= X \left[1 + \frac{\sigma_0^2}{X^T N X} \right]^{-1}
\end{aligned}
$$

which shows that on the average, $\hat{X}_D$ (3.3-160) is an underestimate of X.

Let us then compare the estimates $\hat{X}_S$ (3.3-146) and $\hat{X}_D$ (3.3-160). First, $\hat{X}_S$ is derived under the assumption that X is stochastic while for $\hat{X}_D$ it is assumed deterministic. Thus, for $\hat{X}_D$, XX^T is required while for $\hat{X}_S$ only its expected value, i.e. the moment matrix $Q_X = E\{XX^T\}$. Furthermore, $\hat{X}_D$ is on the average an underestimate of X.

How can one alleviate the unfortunate situation of $\hat{X}_D$ for which the true value X is required. Several approaches are possible. If there is a priori information on X, say $\bar{X}$, one could replace X by $\bar{X}$ in (3.3-160). This is essentially the approach advocated by (Rao, 1973). In practice, a covariance matrix may be available for $\bar{X}$. Therefore, a preferable approach is to replace XX^T in (3.3-160) by its expectation $Q_X = E\{XX^T\}$ given in (3.3-138). This means using $\hat{X}_S$ instead of $\hat{X}_D$. Therefore, we shall refer to $\hat{X}_D$ as our Best Linear Estimate (BLE).

Consider the limiting case when the expected deformations $\bar{X}$ are known perfectly. Then $\Sigma_{\bar{X}} = 0$, $X = \bar{X}$ and $\hat{X}_S$ becomes $\hat{X}_D$ which yields, as shown above, an underestimate of X. On the other extreme if $\Sigma_{\bar{X}} \to \infty$, it is as if one has no a priori information on X so that neither $\hat{X}_S$ nor $\hat{X}_D$ are useful. In this case, only BLIMBE is available with $M = k^2 I$ (it is invariant with respect to k^2 as is easily seen by examining (3.3-42)). We can then write the I-norm BLIMBE or the ordinary pseudoinverse solution as

$$\hat{X} = N^{+}U = \lim_{k^2 \to 0} [(N + k^2 I)^{-1} U] \tag{3.3-164}$$

for singular N. Then $k^2 = 0$ in the case of no a priori information. Or more generally it is a limit for BLE when $M \to 0$. The expression in brackets in (3.3-164) is a special case of BLE, called the ridge estimator (see (Pavlis, 1979) for a good review).

Considering these two limiting cases it should follow that

$$\| \hat{X}_{BLE} \| < \| \hat{X}_{BLIMBE} \| \tag{3.3-165}$$

as we shall show below. Recall that the minimum norm property of BLIMBE is conditional on P-least squares.

Consider the minimization of the Lagrangian function

$$\phi = V^T P V + X^T M X - 2K^T (AX + V - L) \tag{3.3-166}$$

with respect to X , V , K which gives (we assume here that M is positive definite)

$$PV - K = 0 \tag{3.3-167}$$

$$MX - A^T K = 0 \tag{3.3-168}$$

$$AX + V - L = 0 \tag{3.3-169}$$

Solving (3.3-167) for K and substituting into (3.3-168) gives

$$MX - A^T PV = 0 \tag{3.3-170}$$

Substituting V from (3.3-169) into (3.3-170) yields

$$\hat{X} = (N + M)^{-1} U$$

as in (3.3-150). Therefore, the BLE has the property of

$$\hat{V}^T P \hat{V} + \hat{X}^T M \hat{X} = \text{minimum} \tag{3.3-171}$$

Denote

$$\hat{X}_1 \equiv \text{BLIMBE}$$

$$\hat{X}_2 \equiv \text{BLE}$$

As shown by (Hoerl and Kennard, 1970)

$$\begin{aligned} &(L - A\hat{X}_2)^T P (L - A\hat{X}_2) \\ &\quad = (L - A\hat{X}_1)^T P (L - A\hat{X}_1) + H \\ &H = (\hat{X}_2 - \hat{X}_1)^T N (\hat{X}_2 - \hat{X}_1) \end{aligned} \tag{3.3-172}$$

Since H is a quadratic form and in general $\hat{X}_1 \neq \hat{X}_2$ (see (3.3-175) below) then

$$(\hat{V}^T P \hat{V})_2 - (\hat{V}^T P \hat{V})_1 > 0 \tag{3.3-173}$$

which already follows from the P-least squares property of $\hat{X}_1$. From (3.3-166)

$$(\hat{V}^T P \hat{V})_2 + (\hat{X}^T M \hat{X})_2 < (\hat{V}^T P \hat{V})_1 + (\hat{X}^T M \hat{X})_1 \tag{3.3-174a}$$

or

$$0 < (\hat{V}^T P \hat{V})_2 - (\hat{V}^T P \hat{V})_1 < (\hat{X}^T M \hat{X})_1 - (\hat{X}^T M \hat{X})_2 \qquad (3.3\text{-}174b)$$

which proves (3.3-165), i.e., between BLE and BLIMBE, BLE has the minimum norm property. Therefore, the BLE for the deformation vector is on the average smaller than the BLIMBE, it is closer to zero. On the other hand, the BLIMBE has minimum bias. Note that this comparison has been performed for only M positive definite recalling the discussion of section 3.3.1.4 for M positive semidefinite.

Since the vector U is the same for $\hat{X}_1$ and $\hat{X}_2$ it easily follows that

$$\hat{X}_2 = (N+M)^{-1} N \hat{X}_1 \qquad (3.3\text{-}175)$$

indicating that in general $\hat{X}_1 \neq \hat{X}_2$, in the case when N is singular, except as explained before if $M=0$ in which case

$$\hat{X}_2 = N^+ N \hat{X}_1 = N^+ U = \hat{X}_1$$

As a summary, it can be shown (Chipman, 1964; Rao, 1976) that the BLE can be obtained from minimizing the mean square error matrix (3.3-128) also called the risk function

$$\begin{aligned} R(G) &= (I - GA)Q_X(I - GA)^T + G\Sigma_L G^T \\ &= (\text{bias})^2 + \text{covariance} \end{aligned} \qquad (3.3\text{-}176)$$

This yields

$$\hat{X}_2 = G_2 L$$

$$G_2 = Q_X A^T (A Q_X A^T + P^{-1})^{-1} \tag{3.3-146}$$

$$\hat{X}_2 = (N+M)^{-1} U \quad \text{(if } Q_X \text{ non-singular)} \tag{3.3-150}$$

as before. The minimum risk then becomes

$$R(G_2) = (I - G_2 A) Q_X \tag{3.3-177}$$

which is just the mean square error matrix (3.3-133).

It is useful to compute an expression for $\hat{V}^T P \hat{V}$. In general,

$$\hat{V}^T P \hat{V} = (L - A\hat{X})^T P (L - A\hat{X}) \tag{3.3-178}$$

$$= L^T P L - 2\hat{X}^T U + \hat{X}^T N \hat{X} \tag{3.3-179}$$

$$= L^T P L - \hat{X}^T U + (\hat{X}^T N - U^T)\hat{X} \tag{3.3-180}$$

When Q_X is positive definite and from (3.3-150)

$$(N+M)\hat{X}_2 - U = 0 \tag{3.3-181}$$

or

$$\hat{X}_2^T N + \hat{X}_2^T M - U^T = 0 \tag{3.3-182}$$

and

$$(\hat{X}_2^T N - U^T)\hat{X}_2 = -\hat{X}_2^T M \hat{X}_2 \tag{3.3-183}$$

Substituting into (3.3.180) yields

$$\hat{V}_2^T P \hat{V}_2 = L^T P L - \hat{X}_2^T U - \hat{X}_2^T M \hat{X}_2 \tag{3.3-184}$$

From (3.3-166) it follows that the BLE minimizes (Q_X positive definite)

$$\hat{V}_2^T P \hat{V}_2 + \hat{X}_2^T M \hat{X}_2 = L^T PL - \hat{X}_2^T U \tag{3.3-185}$$

Now,

$$E\{\hat{V}_2^T P \hat{V}_2 + \hat{X}_2 M \hat{X}_2\} = E\{L^T PL\} - E\{\hat{X}_2^T U\}$$

Since

$$\begin{aligned} E\{L^T PL\} &= E\{tr[L^T PL]\} \\ &= tr[P\ E\{LL^T\}] \\ &= tr[P(\sigma_0^2\ P^{-1} + AQ_X A^T)] \\ &= n\ \sigma_0^2 + tr(P\ A\ Q_X\ A^T) \end{aligned} \tag{3.3-186}$$

and

$$\begin{aligned} E\{\hat{X}_2^T\ U\} &= E\{L^T(AQ_X A^T + \sigma_0^2\ P^{-1})^{-1}\ AQ_X A^T PL\} \\ &= tr[(AQ_X A^T + \sigma_0^2\ P^{-1})^{-1}\ AQ_X A^T P\ E\{LL^T\}] \\ &= tr[(AQ_X A^T + \sigma_0^2\ P^{-1})^{-1}\ AQ_X A^T P\ (AQ_X A^T + \sigma_0^2\ P^{-1})] \\ &= tr[AQ_X A^T P] \\ &= tr[PAQ_X A^T] \end{aligned} \tag{3.3-187}$$

we find that

$$E\{\hat{V}_2^T P \hat{V}_2 + \hat{X}_2^T M \hat{X}_2\} = n\ \sigma_0^2 \tag{3.3-188}$$

This leads us to an unbiased estimate for the BLE variance of unit weight

$$\hat{\sigma}_0^2 = \frac{\hat{V}_2^T P \hat{V}_2 + \hat{X}_2^T M \hat{X}_2}{n} \tag{3.3-189}$$

which can be shown to hold also for Q_X positive semidefinite. As mentioned above and as will be seen in the simulations of Chapter 4, this a posteriori value will indicate the compatibility of the baseline observations and the geophysical model introduced through the moment matrix Q_X.

The physical meaning of the BLE corresponds to one of the alternate approaches discussed in Section 2.4.3. In this case information is available on the expected deformations of the polyhedron stations, for example, from an adopted absolute motion plate model (see Section 4.2). This would essentially eliminate the singularity of the CTS problem since the expected deformations refer to an absolute frame of reference fixed in the mantle (or crust and mantle). In order to improve on these expected deformations and to test the deformation model, geodetic observations are taken (repeated baseline lengths). By estimating the deformations of the polyhedron stations periodically, the reference frame is maintained since the CTS stations can now be assigned updated positions in that frame.

Any adopted model should also include model parameter standard errors. This information and the expected deformations themselves are included in the moment matrix Q_X for the BLE. Another approach is to simply correct the station coordinates directly and then use the stochastic portion of Q_X in estimating any residual motions that remain. The next estimator will follow this approach.

3.4 Unbiased Estimation

3.4.1 Bayesian Estimation

In the previous section, two possible biased estimators were presented. In the case of no a priori information on the deformation vector X, these reduce to the ordinary pseudoinverse (or equivalently inner constraint) estimate, $X = N^{+}U$. In the case of the availability of a Q_X matrix, we investigate whether an unbiased estimate of X exists, and under what assumptions.

Consider another extended GGM model (L, AX, Q_V, $\bar{X}$, $\Sigma_{\bar{X}}$) similar to the BLIMBE estimation model except that the moment matrix $Q_{\bar{X}}$ of (3.3-37) is split into $\bar{X}$ and $\Sigma_{\bar{X}}$. For random variables $\bar{X}$ and V

$$E\{\bar{X}\} = X \quad ; \quad D[\bar{X}] = \Sigma_{\bar{X}} = E\{(\bar{X} - X)(\bar{X} - X)^T\} \tag{3.4-1}$$

and

$$E\{V\} = 0 \quad ; \quad D[V] = Q_V = E\{VV^T\} = \Sigma_L \tag{3.4-2}$$

Given the probability distributions of V and $\bar{X}$ we obtain the conditional distributions (Chipman, 1964)

$$E\{L|\bar{X}\} = A\bar{X} \quad ; \quad D[L|\bar{X}] = \Sigma_{\bar{X}} \tag{3.4-3}$$

$$E\{L|V\} = AX + V \quad ; \quad D[L|V] = A\Sigma_{\bar{X}}A^T \tag{3.4-4}$$

It follows that the unconditional distribution of L is

$$E\{L\} = E\{E\{L|V\} = E\{E\{L|\bar{X}\}\} = AX \tag{3.4-5}$$

$$\begin{aligned} D[L] &= E\{D[L|V]\} + D[E\{L|V\}] \\ &= A\Sigma_{\bar{X}}A^T + \Sigma_L \end{aligned} \tag{3.4-6}$$

Compare these results to (3.3-141) and (3.3-142).

In order to introduce the random expected deformation vector directly and not through the moment matrix as for BLIMBE and BLE we define a new random vector

$$V_X = \bar{X} - X \tag{3.4-7}$$

There now can be written two sets of observation equations

$$\begin{bmatrix} L \\ L_X \end{bmatrix} = \begin{bmatrix} A \\ I \end{bmatrix} X + \begin{bmatrix} V \\ V_X \end{bmatrix} \tag{3.4-8}$$

where

$$L_X = \bar{X} \tag{3.4-9}$$

and

$$E\begin{Bmatrix} V \\ V_X \end{Bmatrix} = 0 \quad ; \; D\begin{bmatrix} V \\ V_X \end{bmatrix} = \begin{bmatrix} \Sigma_L & 0 \\ 0 & \Sigma_{\bar{X}} \end{bmatrix} \tag{3.4-10}$$

A linear estimator of X is given as a combination of L and L_X by

$$\hat{X} = GL + G_X L_X = [G \quad G_X] \begin{bmatrix} L \\ L_X \end{bmatrix} \tag{3.4-11}$$

Let us construct an estimate that is unbiased and of minimum variance following (Chipman, 1964). From (3.4-1) and (3.4-5)

$$\begin{aligned} E(\hat{X}) &= GAX + G_X X \\ &= (GA + G_X)X \end{aligned} \tag{3.4-12}$$

By equating to X, for unbiasedness

$$G_X = I - GA \tag{3.4-13}$$

The minimum variance condition then requires

$$\mathrm{tr}\left\{[G \quad I - GA] \begin{bmatrix} \Sigma_L & 0 \\ 0 & \Sigma_{\bar{X}} \end{bmatrix} \begin{bmatrix} G^T \\ (I - GA)^T \end{bmatrix}\right\} = \text{minimum} \tag{3.4-14a}$$

or

$$\mathrm{tr}\{G\Sigma_L G^T + (I - GA)\Sigma_{\bar{X}}(I - GA)^T\} = \text{minimum} \tag{3.4-14b}$$

i.e., the trace of the covariance matrix is minimized. Comparing this expression to the risk function (3.3-176) for BLE it follows that

$$G = \Sigma_{\bar{X}} A^T (A\Sigma_{\bar{X}} A^T + \Sigma_L)^{-1} \tag{3.4-15}$$

This differs from G_2 of (3.3-146) in that $\Sigma_{\bar{X}}$ replaces Q_X. It follows from (3.4-9, 11, 13, 15) that the estimate for X, call it $\hat{X}_3$, is given by

$$\begin{aligned} \hat{X}_3 &= \Sigma_{\bar{X}} A^T (A\Sigma_{\bar{X}} A^T + \Sigma_L)^{-1} L \\ &\quad + [I - \Sigma_{\bar{X}} A^T (A\Sigma_{\bar{X}} A^T + \Sigma_L)^{-1} A]\bar{X} \end{aligned} \tag{3.4-16}$$

Thus, $\hat{X}_3$ has the properties of unbiasedness, under the assumption $E\{\bar{X}\} = X$, and minimum variance (but is not BLUE being heterogeneous). In other words, it has minimum mean square error in the class of unbiased estimators. Recall that the BLE has the property of minimum mean square error but in the class of biased (homogeneous) estimators.

For positive definite $\Sigma_{\bar{X}}$

$$M = \Sigma_{\bar{X}}^{-1} \tag{3.4-17}$$

$$\hat{X}_3 = (N+M)^{-1}U + [I - (N+M)^{-1}N]\bar{X} \tag{3.4-18}$$

$$= \hat{X}_2^* + [I - (N+M)^{-1}N]\bar{X} \tag{3.4-19}$$

where $\hat{X}_2^*$ is similar to the BLE (3.3-150) but with $\Sigma_{\bar{X}}$ instead of Q_X. The second term in (3.4-19) is the bias of $\hat{X}_2^*$.

Estimate $\hat{X}_3$ (in 3.4-18) can be written in another form as

$$\hat{X}_3 = \bar{X} + (N+M)^{-1}A^TP(L - A\bar{X}) \tag{3.4-20}$$

which involves a correction term to the expected deformation. In other words, the second term in (3.4-20) can be considered an estimate of the residual deformation

$$X_t - (X_0 + \bar{X}) \tag{3.4-21}$$

where X_t are the CTS coordinates at an epoch t. Examination of (3.4-20) indicates that linearization of the mathematical model (3.2-1) is about the fundamental coordinates X_0. In the no noise case

$$L = A\bar{X} \tag{3.4-22}$$

and

$$X_3 = \bar{X} \tag{3.4-23}$$

Finally, in yet another form, for $\Sigma_{\bar{X}}$ positive definite and using identity (3.3-149) on (3.4-16)

$$\begin{aligned}
\hat{X}_3 &= (N+M)^{-1}U + [I - \Sigma_{\bar{X}}A^T(A\Sigma_{\bar{X}}A^T + \Sigma_L)^{-1}A]\bar{X} \\
&= (N+M)^{-1}U + [M^{-1} - M^{-1}A^T(AM^{-1}A^T + \Sigma_L)^{-1}AM^{-1}]M\bar{X} \\
&= (N+M)^{-1}U + (N+M)^{-1}M\bar{X} \\
&= (N+M)^{-1}(U + M\bar{X})
\end{aligned} \tag{3.4-24}$$

the familiar weighted parameter estimate (Uotila, 1973) or the Bayesian estimate (Bossler, 1972).

The covariance matrix is given by (3.4-14) and (3.4-15) as

$$\begin{aligned}
\Sigma_{\hat{X}_3} &= G\Sigma_L G^T + (I - GA)\Sigma_{\bar{X}}(I - GA)^T \\
&= G\Sigma_L G^T + \Sigma_{\bar{X}} - GA\Sigma_{\bar{X}} - \Sigma_{\bar{X}}A^TG^T + GA\Sigma_{\bar{X}}A^TG^T \\
&= G(A\Sigma_{\bar{X}}A^T + \Sigma_L)G^T + \Sigma_{\bar{X}} - GA\Sigma_{\bar{X}} - \Sigma_{\bar{X}}A^TG^T \\
&= \Sigma_{\bar{X}}A^T(A\Sigma_{\bar{X}}A^T + \Sigma_L)^{-1}A\Sigma_{\bar{X}} + \Sigma_{\bar{X}} \\
&\quad - \Sigma_{\bar{X}}A^T(A\Sigma_{\bar{X}}A^T + \Sigma_L)^{-1}A\Sigma_{\bar{X}} \\
&\quad - \Sigma_{\bar{X}}A^T(A\Sigma_{\bar{X}}A^T + \Sigma_L)^{-1}A\Sigma_{\bar{X}} \\
&= \Sigma_{\bar{X}} - \Sigma_{\bar{X}}A^T(A\Sigma_{\bar{X}}A^T + \Sigma_L)^{-1}A\Sigma_{\bar{X}}
\end{aligned} \tag{3.4-25}$$

When $\Sigma_{\bar{X}}$ is positive definite, using identity (3.3-149),

$$\Sigma_{\hat{X}_3} = \sigma_0 (N+M)^{-1} \tag{3.4-26}$$

Since the Bayesian estimate results from a combination of two types of measurements (see (3.4-8), associated with each type could be a different variance of unit weight. Define

$$\Sigma_L = E\{VV^T\} = \sigma_0^2 R \ (=\sigma_0^2 P^{-1}, \text{ R positive definite}) \tag{3.4-27}$$

$$\Sigma_{\bar{X}} = E\{V_X V_X^T\} = \tau_0^2 S \ (=\sigma_0^2 M^{-1}, \text{ S positive definite}) \tag{3.4-28}$$

$$\rho^2 = \sigma_0^2/\tau_0^2 \tag{3.4-29}$$

Then, estimate (3.4-16) may be written more generally as

$$\begin{aligned}\hat{X}_3 &= \tau_0^2 SA^T(\tau_0^2 ASA^T + \sigma_0^2 R)^{-1} L \\ &+ [I - \tau_0^2 SA^T(\hat{\tau}_0^2 ASA^T + \sigma_0^2 R)^{-1} A]\bar{X}\end{aligned} \tag{3.4-30}$$

$$\begin{aligned}&= SA^T(ASA^T + \rho^2 R)^{-1} L \\ &+ [I - SA^T(ASA^T + \rho^2 R)^{-1} A]\bar{X}\end{aligned} \tag{3.4-31}$$

In the case of positive definite R and S (Chipman, 1964)

$$\begin{aligned}\hat{X}_3 &= [A^T PA + \rho^2 M)^{-1} A^T PL \\ &+ [I - (A^T PA + \rho^2 M)^{-1} A^T PA]\bar{X}\end{aligned} \tag{3.4-32}$$

Different assumptions could be made about σ_0^2 and τ_0^2 as outlined in (Bossler, 1972). For the purpose of this investigation we assume that

$$\tau_0^2 = \sigma_0^2 \quad (\rho^2 = 1) \tag{3.4-33}$$

(the case when $\tau_0^2 \neq \sigma_0^2$ will be investigated in a future study) so that for the a posteriori variance of unit weight (Bossler, 1972)

$$\hat{\sigma}_0^2 = \frac{[\hat{V}^T P \hat{V}]}{n - 2} \tag{3.4-34}$$

where, in general,

$$\begin{aligned} [\hat{V}^T P \hat{V}] &= \hat{V}^T P \hat{V} + \hat{V}_X^T M \hat{V}_X \\ &= (L - A\hat{X})^T P (L - A\hat{X}) + (L_X - X)^T M (L_X - X) \end{aligned} \tag{3.4-35}$$

and

$$[\hat{V}^T P \hat{V}] = L_1^T P L_1 + L_X^T M L_X - \hat{X}^T U - \hat{X}^T M L_X \tag{3.4-36}$$

when M is positive definite. In the expression for $\hat{\sigma}_0^2$, n is the number of geodetic (baseline length) observations. Note that for positive semidefinite $\Sigma_{\bar{X}}$, we use $M = \Sigma_{\bar{X}}^{\pm}$ for (3.4-35).

The physical interpretation of the Bayesian estimate ($\hat{X}_3$) is the same as that of the BLE ($\hat{X}_2$) as explained in the previous section. They differ in how the expected deformations $\bar{X}$ are incorporated into the adjustment. For $\hat{X}_3$, the expected deformations are added directly as corrections to the CTS coordinates (see (3.4-20)). Therefore, the estimated parameters are residual deformations. Since the deformation model is so directly introduced we refer to $\hat{X}_3$ then as a "strong" Bayesian estimate. For $\hat{X}_2$ (and $\hat{X}_1$ - BLIMBE), the expected deformations are introduced in a weaker manner via the moment matrix Q_X and the estimate is the total deformation. Therefore, we refer to $\hat{X}_2$ as a "weak" Bayesian estimate. In fact, BLE is sometimes referred to as the Bayes (instead of Best) Linear Estimate (Rao, 1976). Furthermore, $\hat{X}_3$ is an unbiased estimate though under the "strong" assumption $E(\bar{X}) = X$, while for the biased estimate $\hat{X}_2$ we make no such assumption. Finally, for $\hat{X}_3$, X is assumed deterministic and $\bar{X}$ stochastic as for $\hat{X}_1$. For $\hat{X}_2$ the opposite assumption is made. These differences will be studied in the simulations of Chapter 4.

3.4.2 Best Linear Conditionally Unbiased Estimate (BLICUE)

As seen above, we have found an unbiased estimate for X under the assumption that $E(\bar{X}) = X$. In this section, we present a conditionally unbiased estimate following essentially (Plackett, 1950; Chipman, 1964; Theil, 1971; Bossler, 1972).

The starting point is the model $(L, AX | CX = C\bar{X}, \Sigma_{\bar{X}}, Q_V)$ for the two sets of observation equations

$$\begin{bmatrix} L \\ L_X \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} X + \begin{bmatrix} V \\ V_X \end{bmatrix} \qquad (3.4\text{-}37)$$

where C is the constraint matrix (3.3-90). The second set of equations contain the weighted constraints

$$C\bar{X} = CX + V_X \ ; \qquad (L_X = C\bar{X}) \qquad (3.4\text{-}38)$$

so that

$$V_X = C(\bar{X} - X) \qquad (3.4\text{-}39)$$

where $\bar{X}$ is the expected deformation of the polyhedron stations. For this model

$$E\begin{Bmatrix} V \\ V_X \end{Bmatrix} = 0 \quad ; \quad D\begin{bmatrix} V \\ V_X \end{bmatrix} = \begin{bmatrix} \sigma_0^2 P^{-1} & 0 \\ 0 & C\Sigma_{\bar{X}}C^T \end{bmatrix} \tag{3.4-40}$$

where

$$D(V_X) = E\{V_X V_X^T\} = C\Sigma_{\bar{X}}C^T \tag{3.4-41}$$

so that

$$E\begin{Bmatrix} L \\ L_X \end{Bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} X \quad ; \quad D\begin{bmatrix} L \\ L_X \end{bmatrix} = \begin{bmatrix} \sigma_0^2 P^{-1} & 0 \\ 0 & C\Sigma_{\bar{X}}C^T \end{bmatrix}$$

In contrast to the Bayesian estimate $\hat{X}_3$ for which each station is treated individually in applying the expected deformations $\bar{X}$, here the a priori information is reduced to six constraints. For example, the sum of changes in the X coordinates of the CTS stations may not sum to zero. Of course, if $C = I$ then this model is equivalent to that of $\hat{X}_3$.

The least squares minimum variance solution of (3.4-37) is

$$\hat{X}_4 = [N + C^T P_X C]^{-1}[U + C^T P_X C\bar{X}] \tag{3.4-43}$$

where

$$P_X = (C\Sigma_{\bar{X}}C^T)^{-1} \tag{3.4-44}$$

Since $AC^T = 0$, it can be shown (Chipman, 1964) that using the notation of section 3.3.1

$$A_{PI}^{+} = [N + C^T P_X C]^{-1} A^T P \tag{3.4-45}$$

and

$$C^{+}_{P_X I} = [N + C^T P_X C]^{-1} C^T P_X \tag{3.4-46}$$

Recall that this means that (3.4-45) and (3.4-46) satisfy the four conditions of a minimum M-norm P-least squares g-inverse where in this case M = I. For (3.4-46), $C^{+}_{P_X I}$ is P_X-least squares. Therefore, X can be expressed as

$$\hat{X}_4 = A^{+}_{PI} L + C^{+}_{P_X I} C\bar{X} \tag{3.4-47}$$

or in a more revealing manner as

$$\hat{X}_4 = N^{+} U + C^{+}_{P_X I} C\bar{X} \tag{3.4-48}$$

applying a result similar to (3.3-33). Thus, we see that $\hat{X}_4$ is composed of the BLIMBE with the condition $C\hat{X} = 0$ (M = I) and a correction term that introduces the deviations of the reference frame conditions $C\hat{X}$ from zero due to the possible secular deformations of the polyhedron. If no deformations are expected ($\bar{X} = 0$) at an uncertainty level given by $\Sigma_{\bar{X}}$, $\hat{X}_4$ reduced to the standard pseudoinverse solution.

Now,

$$\begin{aligned} E\{\hat{X}_4\} &= [N + C^T P_X C]^{-1} \; [NX + C^T P_X CX] \\ &= X \end{aligned} \tag{3.4-49}$$

so that $\hat{X}_4$ is an unbiased estimate conditional on

$$E(C\bar{X}) = CX \tag{3.4-50}$$

Since it also has the minimum variance property (in this class of estimators) we shall refer to $\hat{X}_4$ as the Best Linear Conditionally Unbiased Estimate or BLICUE for short. Theil, (1971) refers to this as the mixed estimator and shows that it has a Bayesian inference interpretation which is apparent when comparing it to $\hat{X}_3$. Therefore, it can be interpreted as a combination of the BLIMBE and Bayesian approaches. Furthermore, it bridges the gap between weighted parameter and constrained estimation being a weighted constraint estimate.

For the computation of the BLICUE by (3.4-43) only Cayley algebra is required as long as $R(P_X) = R(C) = 6$ in which case $N + C^T P_X C$ has full rank. Otherwise, $\hat{X}_4$ could be computed from (3.4-48).

The covariance matrix for BLICUE is derived as follows. Consider

$$\hat{X}_4 = G_1 L + G_2 L_X \tag{3.4-51}$$

where

$$G_1 = [N + C^T(C\Sigma_{\bar{X}}C^T)^{-1}C]^{-1}A^T P \tag{3.4-52}$$

$$G_2 = [N + C^T(C\Sigma_{\bar{X}}C^T)^{-1}C]^{-1}C^T(C\Sigma_{\bar{X}}C^T)^{-1} \tag{3.4-53}$$

Then,

$$\Sigma_{\hat{X}_4} = [G_1 \quad G_2] \begin{bmatrix} \Sigma_L & 0 \\ 0 & \Sigma_{L_X} \end{bmatrix} \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} \tag{3.4-54}$$

where

$$\Sigma_L = \sigma_0^2 \, P^{-1} \tag{3.4-55}$$

$$\Sigma_{L_X} = (C\Sigma_{\bar{X}}C^T) \tag{3.4-56}$$

which reduces to

$$\Sigma_{\hat{X}_4} = \sigma_0^2[N + C^T(CP_XC^T)^{-1}C]^{-1} \tag{3.4-56}$$

Deriving an expression for V^TPV yields an interesting result. Denoting

$$\hat{V}^TP\hat{V} = \hat{V}_1^TP\hat{V}_1 + \hat{V}_X^TP_X\hat{V}_X \tag{3.4-57}$$

the second term is seen to be

$$\hat{V}_X^TP_X\hat{V}_X = [C(\bar{X} - \hat{X})]^T(C\Sigma_{\bar{X}}C^T)^{-1}[C(\bar{X} - \hat{X})] = 0 \tag{3.4-58}$$

since the weighted constraints require that $C\bar{X} = C\hat{X}$. Then,

$$\hat{V}^TP\hat{V} = \hat{V}_1^TP\hat{V}_1 = L^TPL - \hat{X}_4^T\,U \tag{3.4-59}$$

which is the same as for the BLIMBE, and therefore, so is $\hat{\sigma}_0^2$ (3.3-56).

By examining (3.4-48) one can interpret the BLICUE approach to monitoring deformations and maintaining the CTS, as dealing with the "systematic" part of the deformations in a P_X-least squares sense and the remaining "random" part in a P-least squares sense. As stated in Chapter 2, applying the constraints of the form $C\hat{X} = L_X$ ($L_X \neq 0$) does not imply that the deformations include global motions, rather secular motions that do not average out. That is, for the expected deformations $\bar{X}$, with respect to the reference frame implicit in the geophysical model, and for a particular station distribution, one expects that $CX = L_X$ and not $CX = 0$.

Table 1. Properties of Deformation Estimators

Property \ Estimator	BLIMBE[1,2]	BLE	Bayesian	BLICUE
Uniqueness	Yes	Yes	Yes	Yes
Homogeneity	Yes	Yes	No	No
P-least squares	$\hat{V}^T P \hat{V} = \min$	$\hat{V}^T P \hat{V} + \hat{X}^T Q_X^{-1} \hat{X} = \min$	$\hat{V}^T P \hat{V} + \hat{V}_X^T \Sigma_{\bar{X}}^{-1} \hat{V}_X = \min$	$\hat{V}^T P \hat{V} = \min$
Minimum M-norm	In the class of P-least squares	Yes[3]	No	No
Biasedness	Minimum bias	Biased	Unbiased assuming $E(\bar{X}) = X$	Unbiased conditional on $E(C\bar{X}) = CX$
Minimum Variance	In the class of minimum bias estimators	In the class of biased estimators	Yes	Conditional
Minimum Mean Square Error	No	In the class of biased estimators	Yes	No
Estimation Model	$(L, AX, Q_V, Q_{\bar{X}})$	$(L, A\bar{X}, Q_V, Q_X)$	$(L, AX, Q_V, \bar{X}, \Sigma_{\bar{X}})$	$(L, AX \mid CX = C\bar{X}, \Sigma_{\bar{X}}, Q_V)$

[1] For $Q_{\bar{X}}$ positive semidefinite the MINOLESS is not equivalent to the BLIMBE. It then has only the properties of minimum M-seminorm P-least squares.

[2] Reduces to a free adjustment for $M = I$ ($Q_{\bar{X}}$ not available).

[3] Among these four estimators.

3.5 Summarizing the Properties of the Four Estimators

Table 1 summarizes the properties of the four estimators presented above. In the next chapter, numerical comparisons will be made.

3.6 Addition and Temporary Deletion of CTS Stations

3.6.1 Introduction

The CTS frame is defined at an initial epoch by the adopted set of coordinates of a polyhedron of stations. In order to maintain the system, the CTS coordinates are updated periodically for the deformations of the polyhedron. It is very possible that from time to time, one or more of the stations will not be able to participate in a particular deformation analysis observing session. Furthermore, it must be anticipated that new stations will be added to the system periodically. Both of these occurrences must be accounted for, in order to maintain continuity and avoid ambiguity in the reference frame definition. We will adopt a least squares collocation approach to handle these situations.

3.6.2 Addition of New CTS stations

In order to deal with this situation, we will apply the general model for least squares collocation (Moritz, 1980b)

$$L = A_1 X + t + V \qquad (3.6\text{-}1)$$

The observation vector L is composed of different parts. The first contains X, a non-random parameter vector to be estimated and A_1 the usual design matrix. The second is the signal part t, of a random nature, and third, a noise vector V due to errors in the observations (possibly also model errors). In our application, X yields estimates for the new CTS station coordinates. The signal portion is

$$t = A_2 S \tag{3.6-2}$$

where the signals S are the deformations to be filtered, and A_2, their design matrix. The L vector is defined as before. Then, the expanded linearized equations for the mathematical model (3.2-1) are

$$\begin{aligned} L &= A_1 X + A_2 S + V \\ &= [A_1 \quad A_2] \begin{bmatrix} X \\ S \end{bmatrix} + V \end{aligned} \tag{3.6-3}$$

In order to isolate the stochastic portion of (3.6-3), define

$$\bar{L} = L - A_1 X \tag{3.6-4}$$

The covariance function (moment matrix) for $\bar{L}$ is then

$$\begin{aligned} Q_{\bar{L}} &= E\ \{\bar{L}\bar{L}^T\} \\ &= E\ \{(A_2 S + V)(A_2 S + V)^T\} \\ &= A_2 E\ \{SS^T\}\ A_2^T + E\{VV^T\} \end{aligned} \tag{3.6-5}$$

where we assume that signal and noise are uncorrelated. It follows that

$$Q_{\bar{L}} = A_2 Q_S A_2^T + Q_V \tag{3.6-6}$$

where

$$Q_S = E\ \{SS^T\}\ ; \qquad Q_V = E\ \{VV^T\} \tag{3.6-7}$$

are the signal and noise covariance functions (moment matrices) respectively.

Note that Q_S is constructed only for those stations whose deformations are being estimated while Q_V refers to all observations. The cross covariance function for the deformations and observations $\bar{L}$ is

$$
\begin{aligned}
Q_{S\bar{L}} &= E\{S\bar{L}^T\} \\
&= E\{S(A_2S + V)^T\} \\
&= E\{SS^T\}A_2^T \\
&= Q_SA_2^T
\end{aligned}
\tag{3.6-8}
$$

The collocation estimates for X and S are then given by (Moritz, 1980b)

$$\hat{X} = [A_1^T(A_2Q_SA_2^T + Q_V)^{-1}A_1]^+ \ A_1^T(A_2Q_SA_2^T + Q_V)^{-1}L \tag{3.6-9}$$

$$\hat{S} = Q_SA_2^T(A_2Q_SA_2^T + Q_V)^{-1}(L - A_1\hat{X}) \tag{3.6-10}$$

and

$$\Sigma_{\hat{X}} = [A_1^T(A_2Q_SA_2^T + Q_V)^{-1}A_1]^+ \tag{3.6-11}$$

$$
\begin{aligned}
\Sigma_{\hat{S}} = {} & Q_S - Q_SA_2^T(A_2Q_SA_2^T + Q_V)^{-1}A_2Q_S \\
& + Q_SA_2^T(A_2Q_SA_2^T + Q_V)^{-1}A_1\Sigma_{\hat{X}}A_1^T(A_2Q_SA_2^T + Q_V)^{-1}A_2Q_S
\end{aligned}
\tag{3.6-12}
$$

$$\Sigma_{\hat{X}\hat{S}} = -\Sigma_{\hat{X}}A_1^T(A_2Q_SA_2^T + Q_V)^{-1}A_2Q_S \tag{3.6-13}$$

The design matrix

$$ {}_{n}A_{u} = \left[{}_{n}{A_1}_{q} \qquad {}_{n}{A_2}_{u-q} \right] $$

has the general pattern

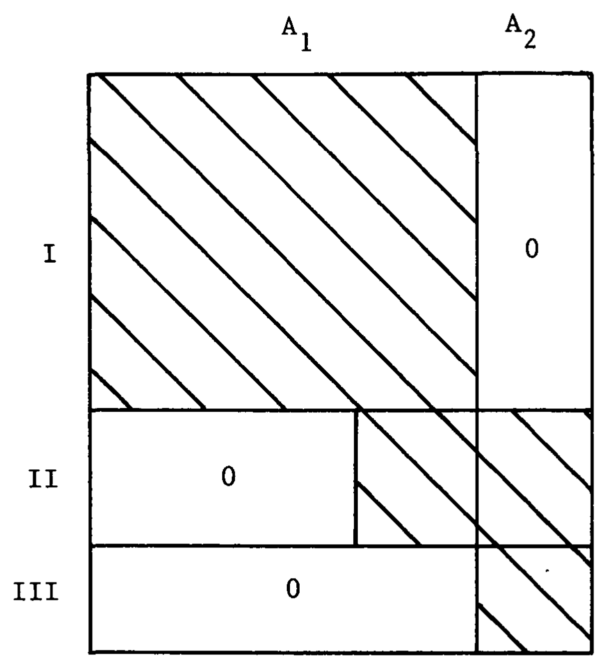


Fig. 2 A-Matrix Structure for Addition of CTS Stations

The number of rows in I is the number of baselines in the original polyhedron. The number of rows in II is the number of baselines with an original station at one end and a new one at the other. The number of rows in III is the number of altogether new baselines. Since both A_1 and A_2 are computed from the same baseline length model, they are both rank deficient. This explains the pseudoinverse in the expression for $\hat{X}$ (3.6-9).

Reiterating, the vector $\hat{X}$ contains the estimated coordinates of the new CTS stations. The linearization, in this case, is taken about some approximate station coordinates. The vector $\hat{S}$ contains the filtered signals (deformations) relative to the expected deformation and the linearization is taken about X_0, the fundamental coordinates. It should be mentioned that once new stations have been added to the system, in subsequent deformation analyses the linearization of the mathematical model (3.2-1) should be taken about X_t where t is the epoch at which the new stations were added. In any case, it may be useful to update the linearization point periodically from convergence considerations although the deformations are small compared to the baseline lengths.

An examination of the estimates for X and S is quite revealing. Consider for a moment that there are no new CTS stations. Then

$$\hat{S} = Q_S A^T (A Q_S A^T + Q_V)^{-1} L \tag{3.6-14}$$

which is just the BLE estimate (3.3-146). Thus, application of this estimator represents the philosophy of considering the deformations as signals to be filtered from the observational noise. Recall that in the estimation model for BLE, the deformation vector X was assumed to be random. Next, consider the $\hat{X}$ vector when there are no signals

$$\hat{X} = (A^T Q_V^{-1} A)^+ A^T Q_V^{-1} L \tag{3.6-15}$$

Adding an M-norm in the parameter space yields the BLIMBE (3.3-43).

Thus, the BLIMBE approach follows the philosophy of considering the deformations as parameters, i.e. deterministic quantities.

3.6.3 Temporary Deletion of Several CTS Stations

Some CTS stations may be unable to participate in a particular deformation analysis observing campaign. Provisions must be made for this in order to maintain the reference system. In this case, it is possible to apply the prediction capabilities of collocation, which is equivalent as shown by (Dermanis, 1976) to minimum mean square error prediction. Recall that the BLE yields minimum mean square error so that this method will be an extension of the BLE model.

Consider the BLE model with new notation reflecting its filtering interpretation

$$L = Bt + V \tag{3.6-17}$$

where now, t represents the signals (deformations) actually measured (at the participating CTS sites) and B their corresponding design matrix. Then

$$C_L = B\, Q_t B^T + Q_V \tag{3.6-18}$$

$$C_{SL} = Q_S B^T \tag{3.6-19}$$

from which

$$\hat{S} = Q_S B^T (B\, Q_t B^T + Q_V)^{-1} L \tag{3.6-20}$$

Here S , the signal vector has two components

$$S = \begin{bmatrix} t \\ u \end{bmatrix} \tag{3.6-21}$$

where t are the deformations to be filtered at the participating stations and u are the deformations to be predicted at the missing sites. The structure of the B matrix is as follows

$$B = {}_{m}\Big[\ {}_{m}{B_1}_{q} \quad : \quad {}_{m}0_{u-q}\ \Big]_{u}$$

where there are m baseline observations. The signal covariance matrix Q_t is constructed from the geophysical deformation model for the q observing stations. The full Q_S matrix is computed for all the CTS stations whether they have observed or not. Actually, both Q_t and Q_S are subsets of the global Q matrix that can be computed for any point on earth (compare to M_G of section 2.4.3).

The prediction is accomplished through the adopted geophysical model from which is derived the signal covariance function. In this context the covariance function has probabilistic justification. Dermanis (1976) shows that the choice of inner product (i.e. weighted norm) is equivalent to the choice of covariance function in collocation.

3.7 On the Estimability of the Baseline Length Change Estimates

The question raised in this section is whether or not the "adjusted" baseline length change vector

$$\hat{L} = A\hat{X} \tag{3.7-1}$$

is estimable, or in other words is it an unbiased estimate of L for each of the four estimators? In our context, is the estimated change in the size and shape of the polyhedron unbiased?

For BLIMBE,

$$\begin{aligned}\hat{L}_1 = A\hat{X}_1 &= AQ_{\bar{X}}N(NQ_{\bar{X}}N)^g A^T PL \\ &= AG_1 L = AA^+_{PM}L\end{aligned} \tag{3.7-2}$$

Then,

$$\begin{aligned}E\{\hat{L}_1\} &= AA^+_{PM}E\{L\} \\ &= AA^+_{PM}AX \\ &= AX = L\end{aligned} \tag{3.7-3}$$

Therefore, $\hat{L}_1$ is an unbiased estimate for L though $\hat{X}_1$ is biased.

For MINOLESS (differentiated from the BLIMBE by an asterisk)

$$\begin{aligned}\hat{L}^*_1 = A\hat{X}^*_1 &= A(N+Q^+_{\bar{X}})^{-1}N[N(N+Q^+_{\bar{X}})^{-1}N]^g A^T P(AX+V) \\ &= AN^g N(N+Q^+_{\bar{X}})^{-1}N[N(N+Q^+_{\bar{X}}N]^g[NX + A^T PV] \\ &= AN^g NX + AG^*_1 V = AX + AG^*_1 V\end{aligned} \tag{3.7-4}$$

using identities (3.3-114) and (3.3-115). It follows that

$$E\{\hat{L}^*_1\} = AX = L \tag{3.7-5}$$

Therefore, $\hat{L}^*_1$ is unbiased, too.

Recall that the MINOLESS and BLIMBE are equivalent when $Q_{\bar{X}}$ is positive definite. Note that AG_1 and AG^*_1 are idempotent and, therefore, both are projection operators. That is, they map the observation vector L into $\hat{L}$ an element of the column space of A.

For BLE,

$$\begin{aligned}\hat{L}_2 &= A\hat{X}_2 = AQ_X A^T(AQ_X A^T + \sigma_0^2 P^{-1})^{-1} L \\ &= AG_2 L\end{aligned} \tag{3.7-6}$$

and

$$E\{A\hat{X}_2\} = AG_2 A\bar{X} \neq AX \tag{3.7-7}$$

in general. Therefore $\hat{L}_2$ is biased as is $\hat{X}_2$. Note that since AG_2 is not idempotent, it is not a projection operator.

For the Bayesian estimate,

$$\hat{L}_3 = A\hat{X}_3 = A\bar{X} + A\Sigma_{\bar{X}} A^T(A\Sigma_{\bar{X}} A^T + \sigma_0^2 P^{-1})^{-1}(L - A\bar{X}) \tag{3.7-8}$$

and

$$E\{\hat{L}_3\} = E\{A\hat{X}_3\} = AX = L \tag{3.7-9}$$

if $E\{\bar{X}\} = X$, the assumption made for this estimate. Similarly, the BLICUE

$$E\{\hat{L}_4\} = L \tag{3.7-10}$$

under the assumption $E\{C\bar{X}\} = CX$. Therefore, both estimates yield conditionally unbiased estimates for $\hat{X}$ and $\hat{L}$. Note that both estimates are not homogeneous and, thus, one cannot express $\hat{X}$ and $\hat{L}$ as

$\hat{X} = GL$ or $\hat{L} = AGL$, as can be done for BLIMBE and BLE, except by adding another term to L (see (3.4-11) and (3.4-43)).

How does one interpret the fact that $\hat{L}_1$ is unbiased and $\hat{L}_2$ biased? For BLIMBE, it is apparent that the model moment matrix $Q_{\bar{X}}$ has no effect on the estimation of L. That is, $\hat{L}_1$ is invariant with respect to a weighted norm in the parameter space and would be the same as obtained in a free adjustment without any geophysical information at all. This also follows from examining the BLIMBE estimation model (3.3-35) - (3.3-40) in which there is no connection between L and $\bar{X}$. On the other hand, for BLE, $\hat{L}_2$ is influenced by the a priori information given by Q_X, or in other words by the expected value $\bar{L}$ of the new distances computed from $\bar{X}$ the expected deformations. Therefore, it comes down again to what is preferred a biased or unbiased estimate for some parameter, or does one prefer to ignore a priori information or not.

It is well known that a biased estimator can improve upon unbiased estimators if there is some a priori information about the unknown parameters (Rao, 1973). In our application we can show that $\hat{L}_2$ has minimum mean square error. That is,

$$MSE(\hat{L}_2) = E\{(\hat{L}_2 - L)(\hat{L}_2 - L)^T\} = \text{minimum} \tag{3.7-11}$$

From (3.7-6)

$$MSE(\hat{L}_2) = E\{[AG_2L - L][AG_2L - L]^T\} \tag{3.7-12}$$

$$= E\{[AG_2 - I]LL^T[AG_2 - I]^T\}$$

$$= E\{[AG_2 - I][AX + V][AX + V]^T[A\ G_2 - I]^T\}$$

(using $Q_{XV} = 0$)

$$= [AG_2 - I]AE\{XX^T\}A^T[AG_2 - I]^T + AG_2E\{VV^T\}G_2^TA^T$$

$$= A[G_2A - I]E\{XX^T\}[G_2A - I]^TA^T + AG_2E\{VV^T\}G_2^TA^T$$

$$= AE\{(\hat{X} - X)(\hat{X} - X)^T\}A^T \tag{3.7-13}$$

$$= A\ MSE(\hat{X})\ A^T \tag{3.7-14}$$

which follows from (3.3-128). Since the BLE estimator minimizes the mean square error (in the class of biased estimators), then it follows directly that (3.7-11) holds and improves upon $\hat{L}_1$. Using the same reasoning it follows that $\hat{L}_3$ for the strong Bayesian estimate has minimum mean square error (also minimum variance - being conditionally unbiased) in the class of conditionally unbiased (heterogeneous) estimators.

4 DEFORMATION ANALYSIS SIMULATIONS

4.1 Introduction

In Chapter 3, four possible estimators were presented as possible candidates for CTS deformation analysis. Each one has its particular optimal properties. In this chapter, we study the significance of these properties by running a series of simulations. The main question to be answered is whether or not an absolute motion plate model should be adopted, as opposed to performing an ordinary (unweighted) pseudoinverse solution (a free adjustment). In addition, we seek to determine the best way to combine geodetic baseline observations and geophysical models in the estimation of crustal deformations. Recall that in a free adjustment, the singular normal matrix is augmented with the constraint matrix C such that $C\hat{X} = 0$ (alternatively, pseudoinverse algebra is used to solve the set of normal equations. That is, these constraints arbitrarily impose no net translation or rotation for the estimated deformations $\hat{X}$ without any real physical justification. The introduction of a geophysical model can direct $\hat{X}$ to a physically more meaningful solution. A similar concept has been applied in (Prescott, 1981) for monitoring deformations along a strike slip fault.

We "adopt" the absolute motion model AM1-2 of (Minster and Jordan, 1978). Since using this model leads to a positive semidefinite moment matrix Q_X, we compare the four estimators first using a positive definite and then a positive semidefinite model matrix.

4.2 The Model Matrix

As mentioned in section 2.4.3, it is postulated by Wilson and Morgan that hot spots (or sources of volcanic magma) are fixed in the deep mantle and, thus, form a set of rigid points that serve as an absolute reference frame for plate motion. In AM1-2, rotation vectors for eleven major plates are given in this absolute frame. The components of these vectors are the poles of rotation of the plates (ϕ_P, λ_P) and their rates of rotation (ω_P). Standard errors are also provided for these estimates in Table 2, for which the data is obtained from (Minster and Jordan, 1978). As can be seen, the errors in the slow moving plates are quite large for the pole parameters. The intention is not to advocate a particular model but to adopt one for simulation purposes. We note that there is controversy regarding the hot spot hypothesis (e.g., Le Pichon, et al.,1973). Nevertheless, as mentioned before, Bender (1981) points out that absolute motion models derived from different geophysical assumptions about plate motions differ by about 1 cm/year.

A model moment matrix for the expected absolute velocities of the CTS stations can be constructed from an absolute motion model as follows. The velocity of station i on plate j in an absolute frame is given by (Minster and Jordan, 1974)

$$V_{ij} = \Omega_j \times X_i \qquad (4.2\text{-}1)$$

Table 2 AM1-2 Absolute Motion Plate Model
(Adapted from Table 7, Minster and Jordan, 1978)

Tectonic Plate	Absolute Rotation Vectors and Standard Errors					
	Rotation Poles				Rotation Rate	
	Latitude (ϕ) (Deg N)		Longitude (λ) (Deg E)		Rate (ω) (Deg/M.Y.*)	
	ϕ	σ_ϕ	λ	σ_λ	ω	σ_ω
1. African	18.76	33.93	338.24	42.20	0.139	0.055
2. Eurasian	0.70	124.35	336.81	146.67	0.038	0.057
3. North American	-58.31	16.21	319.33	39.62	0.247	0.080
4. Pacific	-61.66	5.11	97.19	7.71	0.967	0.085
5. South American	-82.28	19.27	75.67	85.88	0.285	0.084
6. Indian-Australian	19.23	6.96	35.64	6.57	0.716	0.076
7. Nazca	47.99	9.36	266.19	8.14	0.585	0.097
8. Arabian	27.29	12.40	356.06	18.22	0.388	0.067
9. Antarctic**	21.85	91.81	75.55	63.20	0.054	0.091
10. Carribean**	-42.80	39.20	66.75	40.98	0.129	0.104
11. Cocos**	21.89	3.08	244.29	2.81	1.422	0.119

* Million Years
** Not used in the simulations

where

$$\Omega_j = \omega_j \begin{bmatrix} \cos\phi_j \cos\lambda_j \\ \cos\phi_j \sin\lambda_j \\ \sin\phi_j \end{bmatrix} \tag{4.2-2}$$

is the j'th plate's rotation vector, and

$$X_i = R \begin{bmatrix} \cos\phi_i \cos\lambda_i \\ \cos\phi_i \sin\lambda_i \\ \sin\phi_i \end{bmatrix} \tag{4.2-3}$$

is the spatial Cartesian i'th station vector. A spherical earth approximation is sufficient for the description of plate kinematics where R is the radius. Then,

$$V_{ij} = \begin{bmatrix} V_{x_{ij}} \\ V_{y_{ij}} \\ V_{z_{ij}} \end{bmatrix} = R\omega_j \begin{bmatrix} \cos\phi_j \sin\phi_i \sin\lambda_j - \sin\phi_j \cos\phi_i \sin\lambda_i \\ \sin\phi_j \cos\phi_i \cos\lambda_i - \cos\phi_j \sin\phi_i \cos\lambda_j \\ \cos\phi_j \cos\phi_i \sin(\lambda_i - \lambda_j) \end{bmatrix} \tag{4.2-4}$$

By error propagation, the variance-covariance matrix for the station velocities is given by

$$\Sigma_V = G \Sigma_P G^T \tag{4.2-5}$$

where G is the partial derivative matrix of the form

$$G_{ij} = \frac{\partial(V_{x_{ij}}, V_{y_{ij}}, V_{z_{ij}})}{\partial(\phi_{p_j}, \lambda_{p_j}, \omega_{p_j})} \; ; \quad \begin{array}{l} i = 1,\ldots,k \\ j = 1,\ldots,\ell \end{array} \tag{4.2-6}$$

where k denotes the number of stations, ℓ the number of plates and Σ_p is the covariance matrix for the plate parameters given from Table 4.1. Here we assume a diagonal covariance matrix. It has been learned that a full covariance matrix is available (J. B. Minster, 1982, private communication) although too late to be used for these simulations. The partial derivatives are computed as

$$\frac{\partial V_{x_{ij}}}{\partial \phi_j} = R\omega_j[-\sin\phi_j \sin\phi_i \sin\lambda_j - \cos\phi_j \cos\phi_i \sin\lambda_i] \tag{4.2-7}$$

$$\frac{\partial V_{x_{ij}}}{\partial \lambda_j} = R\omega_j \cos\phi_j \sin\phi_i \cos\lambda_j \tag{4.2-8}$$

$$\frac{\partial V_{x_{ij}}}{\partial \omega_j} = R[\cos\phi_j \sin\phi_i \sin\lambda_j - \sin\phi_j \cos\phi_i \sin\lambda_i] \tag{4.2-9}$$

$$\frac{\partial V_{y_{ij}}}{\partial \phi_j} = R\omega_j[\cos\phi_j \cos\phi_i \cos\lambda_i + \sin\phi_j \sin\phi_i \cos\lambda_j] \tag{4.2-10}$$

$$\frac{\partial V_{y_{ij}}}{\partial \lambda_j} = R\omega_j \cos\phi_j \sin\phi_i \sin\lambda_j \tag{4.2-11}$$

$$\frac{\partial V_{y_{ij}}}{\partial \omega_j} = R[\sin\phi_j \cos\phi_i \cos\lambda_i - \cos\phi_j \sin\phi_i \cos\lambda_j] \tag{4.2-12}$$

$$\frac{\partial V_{z_{ij}}}{\partial \phi_j} = -R\omega_j \sin\phi_j \cos\phi_i \sin(\lambda_i - \lambda_j) \tag{4.2-13}$$

$$\frac{\partial V_{z_{ij}}}{\partial \lambda_j} = -R\omega_j \cos\phi_i \cos\phi_i \cos\ (\lambda_i - \lambda_j) \tag{4.2-14}$$

$$\frac{\partial V_{z_{ij}}}{\partial \omega_j} = R\cos\phi_j \cos\phi_i \sin\ (\lambda_i - \lambda_j) \tag{4.2-15}$$

It is apparent that

$$G_{ij} = 0$$

if station i is not located on plate j.

We assume the deformation vector X to be related linearly to the velocity vector by

$$X = V(t - t_0) \tag{4.2-16}$$

where $t - t_0$ is the time elapsed from the initial CTS epoch. The model moment matrix for the deformations is given by

$$\begin{aligned} Q_X &= E\{XX^T\} = (t - t_0)^2 E\{VV^T\} \\ &= \Sigma_X + \bar{X}\bar{X}^T \end{aligned} \tag{4.2-17}$$

using

$$Q_V = E\{VV^T\} = \Sigma_V + \bar{V}\bar{V}^T \tag{4.2-18}$$

where $\bar{V}$ denotes the expected velocity vector computed from (4.2-4), $\bar{X}$ is the expected deformation vector computed from (4.2-16), and Σ_V and Σ_X are the covariance matrices of V and X, respectively. Recall that in Chapter 3 the moment matrix Q_X is introduced differently for each estimator. Note that in (4.2-16) and (4.2-17) as the time interval gets shorter the deformation gets smaller and so does its covariances. This is somewhat misleading since the plate models are given as long-term average rates (over approximately 50 years to several million years (J. D. Minster, 1982 private communication)). However, Bender (1981) suggests that the present rates of motion should not be much different from the long-term average rates. This is then the assumption made in these simulations and the one that will eventually be tested by geodetic observations.

The matrix Q_X for a horizontal plate model, such as AM1-2, is always rank deficient. Implicit in the plate velocities (4.2-4) are horizontal motions between rigid plates. That is, no vertical motion is indicated and baseline lengths on the same plate should not change. The Q_X matrix is given though in terms of deformations in 3 components (x,y,z) per station. If we would express the deformations in a local system (see section 2.4.1), the height component would drop out. Therefore, there is one rank deficiency per station due to the vertical component. In addition, for stations on one rigid plate there is one rank deficiency per non-redundant baseline. For example, consider 4 stations on one plate. The model predicts 12 deformations, 3 per station. For 4 stations, a quadrilateral, there are five independent baselines out of six. Therefore, there are 9 rank deficiencies-- 4 vertical motions + 5 rigid independent baselines. This reasoning applies to the Σ_X portion of Q_X. The $\bar{X}\bar{X}^T$ portion always has a rank of one so that the sum of the two matrices, i.e., Q_X, can at most be increased by one over the rank of Σ_X.

The above example of 4 stations indicates also that the minimum number of stations per plate should not be less than four from the reliability point of view. For $n < 4$ there is no redundant baselines and therefore no check for systematic errors or site stability problems.

4.3 Numerical Tests

4.3.1 Simulation Procedure

In Chapter 3, four estimators were presented as possible candidates for deformation analysis. Each one has its optimal properties. In order to get a better feeling for the applicability of these estimates, this section describes some numerical tests. For example, it was shown that the BLIMBE is a minimum bias estimator. However, if this minimum bias is large, it makes little sense to use this estimate for deformation analysis. Through controlled experiments described below, it is possible to assess the magnitude of the bias, and similarly test the other properties of the different estimators.

Several numerical comparisons were made for the four estimators. The least squares property is examined through the computation of $\hat{V}^T P \hat{V}$ and the minimum norm property by $\hat{X}^T M \hat{X}$. For the two biased estimators, the trace of the bias matrix is computed, i.e.,

$$tr[(I - GA)Q_X(I - GA)^T] \tag{4.3-1}$$

where for BLIMBE (Q_X positive definite)

$$(GA)_1 = Q_X N(NQ_X N)^+ N \tag{4.3-2}$$

For the minimum M-seminorm P-least squares estimate (MINOLESS) with positive semidefinite Q_X

$$(GA)_1 = (N+Q_X^+)^{-1}N[N(N+Q_X^+)^{-1}N]^+N \tag{4.3-3}$$

For the BLE, Q_X positive definite

$$(GA)_2 = (N+Q_X^{-1})^{-1}N \tag{4.3-4}$$

and

$$(GA)_2 = Q_X A^T(AQ_X A^T + \Sigma_L)^{-1}A \tag{4.3-5}$$

for positive semidefinite Q_X. The minimum variance (mean square error) property is reflected by the trace of the different covariance (or mean square error) matrices. For the BLIMBE this quantity is added to the trace of the bias matrix to yield the mean square error. The ratio of maximum and minimum eigenvalues (referred to as the C-measure in section 4.3.4) assesses the condition of the covariance or mean square error matrices.

Another property, and a most essential one, is how close does the estimate come to the true value. This is tested in a simulation environment where the "true" value is known. For this purpose, we compute the root mean square of the deviations of the estimated value from the true value as

$$\mathrm{RSMPE} = \left[\frac{\sum_{i=1}^{3p} (\hat{X}_i - X_i)^2}{3p - 1}\right]^{1/2} \tag{4.3-6}$$

$$= \left[\frac{\sum_{i=1}^{3p} [(\hat{X}_i - X_i) - \overline{\hat{X}_i - X_i}]^2}{3p - 1} + (\bar{X} - \bar{\hat{X}})^2\right]^{1/2} \tag{4.3-7}$$

$$= [\mathrm{variance} + (\mathrm{bias})^2]^{1/2}$$

where the bar denotes an average and p is the number of stations. Both variance and bias terms are computed and added to yield what we will call the RSMPE (the root mean square of the sampling error).

The simulation procedure is as follows. For a particular network at an initial epoch t_0, we assign a set of "fundamental" coordinates X_0 and compute their corresponding baseline lengths. Next, by means of the AM1-2 absolute plate motion model, we compute the expected deformations $X = (X_t - X_0)$ at some later epoch and compute the corresponding expected baseline lengths. To these we add 3 cm Gaussian noise and subtract from them the initial baseline lengths to yield the "observed" baseline length changes. We assume that the baseline lengths are re-observed after two years ($t - t_0$ = 2 years).

The adjustment algorithms outlined in Chapter 3 are followed. In all cases, the linearization of the baseline length mathematical model (3.2.1) is taken about X_0. It is assumed that the re-observed baseline lengths are accurate to 3 cm and uncorrelated. This is a reasonable assumption considering that at the present, it is possible to estimate individual intercontinental VLBI baselines with nearly such precision (Herring, 1981). The model moment matrix Q_X is computed on the basis of the AM1-2 model as explained in the previous section.

The network depicted in Fig. 3 is the basis for the simulations. Three stations were chosen per each of six major tectonic plates (North American, South American, African, Eurasian, Indian-Australian, and Pacific) and one station for two smaller plates (Arabian and Nazca). The sites were chosen on the basis of several criteria. First, the station if possible should be operational or at least have been mentioned as a likely candidate. This criterion is met by at least eleven of the sites (stations 4-11, 13, 16, 17 of Table 3). The remaining sites were chosen in stable parts of each plate according to Lowman's global plate tectonic map reproduced in Fig. 4. Furthermore, each plate should be well represented which was determined by an examination of the expected plate motion vectors shown in Fig. 3 and listed in Table 3. This same network will be used in the experiments of section 4.3.4.

In order to compare the four estimators using first a positive definite Q_X matrix, a subset of 8 stations is chosen, one per each of the above mentioned tectonic plates (Table 3). In order to construct a non-singular model matrix a secular vertical motion model was selected arbitrarily (and therefore not reproduced here) and its corresponding model matrix added to that propagated from the AM1-2 model. It consists of vertical deformations in the range of ±3 cm per year with 1 cm accuracy. This eliminates the rank deficiencies due to the undefined vertical components. Since there is only one station per plate, there are no rigid baselines. The 8-station experiments are solely performed

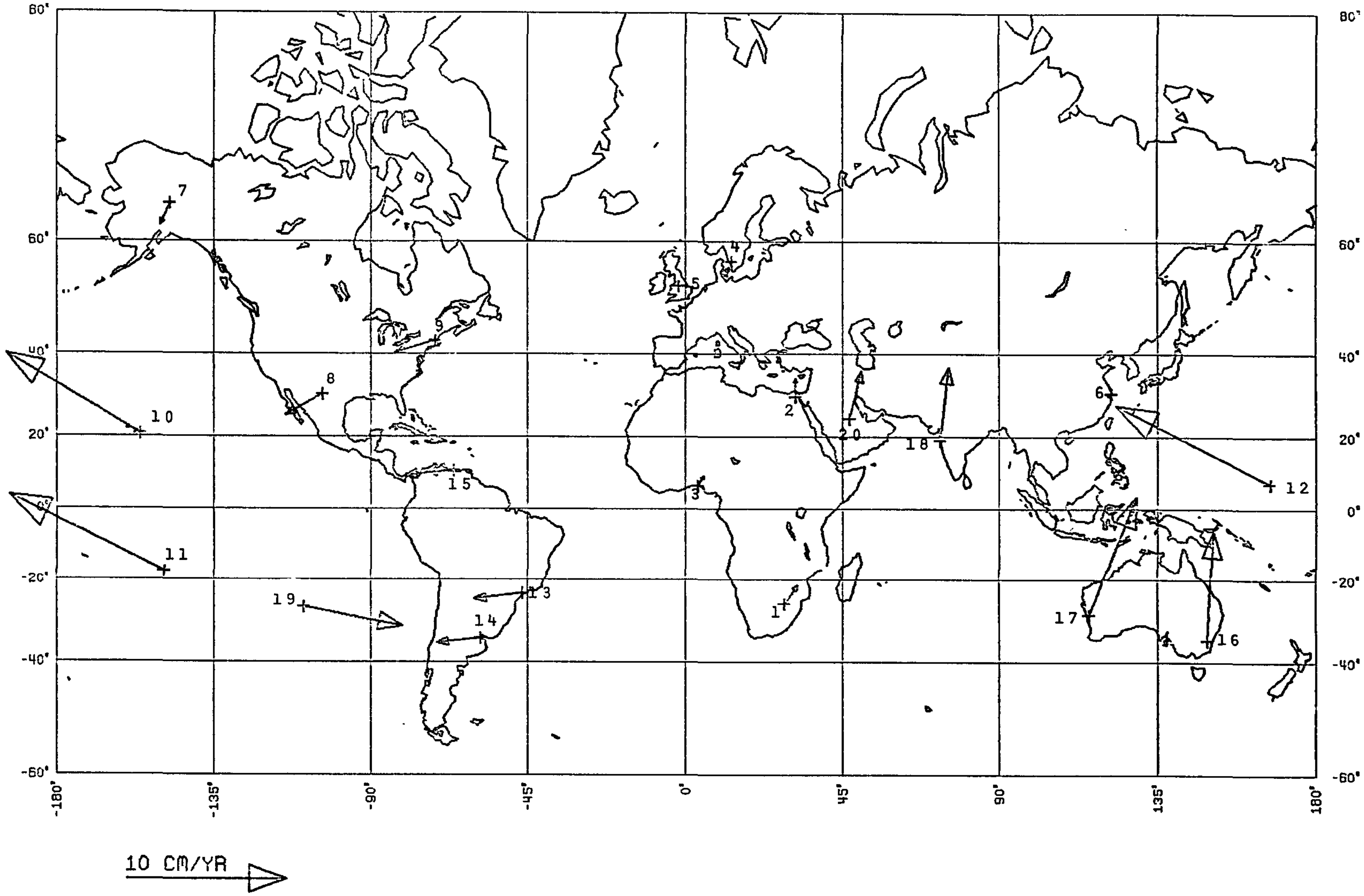


Fig. 3 20-Station 8-Plate Simulation Network and AM1-2 Computed Velocities (see Table 3)

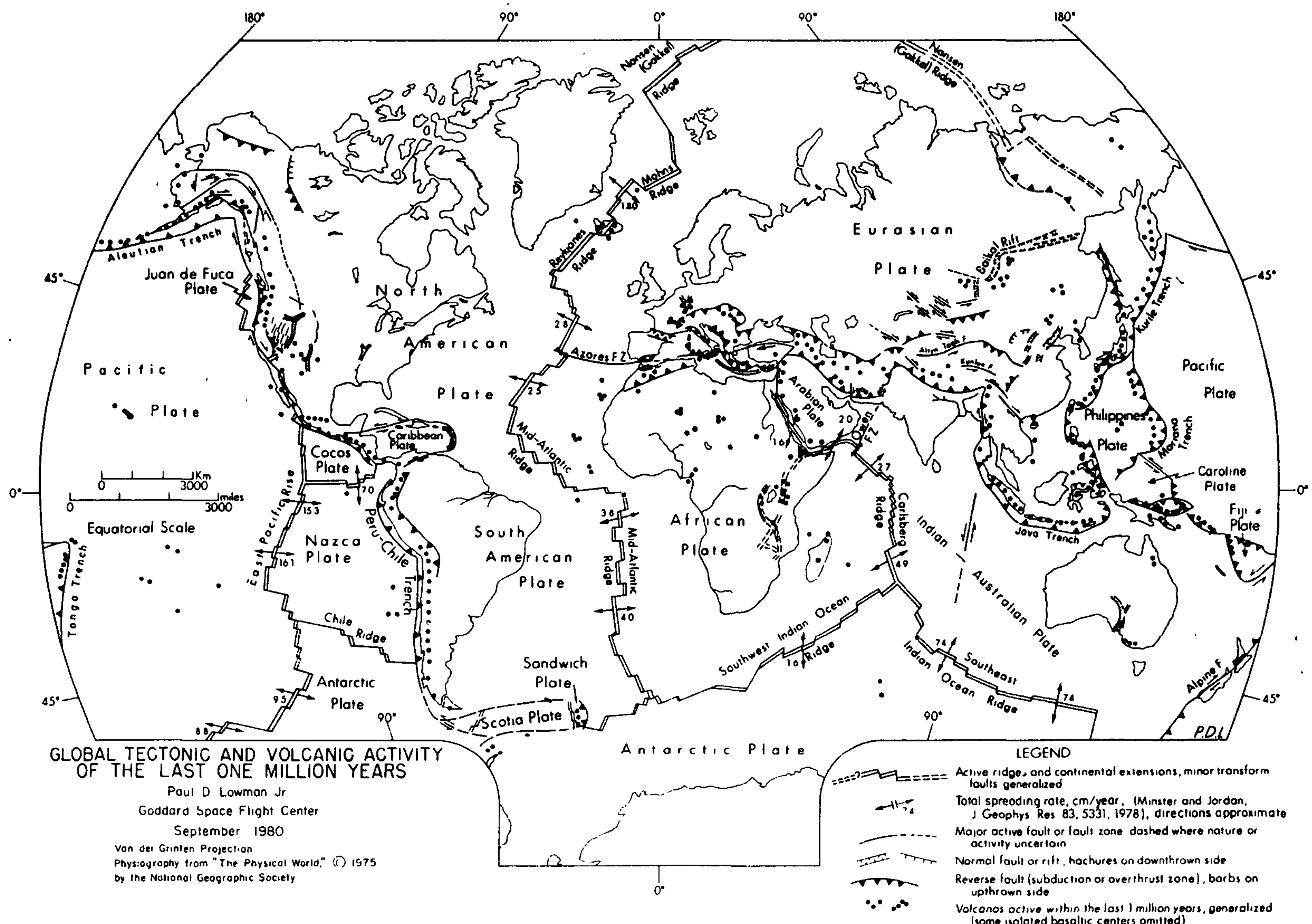


Fig. 4 Lowman's Global Plate Tectonic Map

Table 3 20-Station 8-Plate Simulation Network and AM1-2 Velocities

No.	Station	Latitude North (Deg Min)		Longitude East (Deg Min)		Plate	Velocity Vector (Cm/Year) (X*)	(Y*)	(Z*)
1	Johannesberg**	-29	10	28	02	AFRC	0.03	0.99	1.00
2	Cairo	30	3	31	15	AFRC	-0.49	-0.31	1.01
3	Lagos	6	27	3	28	AFRC	-0.09	0.34	0.62
4	Onsala**	57	0	13	0	EURA	-0.14	-0.32	0.14
5	Jodrell Bank	53	0	358	0	EURA	-0.13	-0.31	0.09
6	Shanghai	30	45	121	45	EURA	-0.09	-0.20	0.21
7	Fairbanks	64	50	212	10	NOAM	-1.38	-0.15	-0.59
8	Ft. Davis**	30	38	256	3	NOAM	-2.43	-0.07	-1.11
9	Westford	42	36	288	30	NOAM	-2.27	-1.29	-0.54
10	Maui**	20	30	203	45	PCFC	-1.80	8.35	4.59
11	Tahiti	-17	30	210	30	PCFC	-6.11	7.59	4.48
12	Marshall Isles	7	0	167	0	PCFC	2.73	9.24	4.76
13	Sao Paulo**	-23	33	313	21	SOAM	-2.26	-1.94	-0.33
14	Buenos Aires	-34	37	301	36	SOAM	-2.44	-1.30	-0.25
15	Caracas	10	35	293	40	SOAM	-2.75	-1.26	-0.26
16	Orroral**	-35	18	149	8	INDI	-3.63	1.70	5.63
17	Yaragadee	-29	3	115	21	INDI	-4.20	1.99	6.47
18	Bombay	18	56	72	51	INDI	-0.95	-1.25	4.30
19	Easter Isle**	-27	5	250	39	NAZC	6.04	-1.56	-1.04
20	Arabia**	24	39	46	46	ARAB	-1.42	-0.36	2.70

* Velocity Components (X towards 0° longitude, Y towards 90° longitude, Z towards earth's rotation pole, spherical earth approximation)

** 8-Station 8-Plate Network

for the sake of comparison with a positive semidefinite Q_X matrix. Next, the same set of experiments are performed for the 18-station network under realistic assumptions. Of course, the 18 station network is much stronger having more degrees of freedom. Using (3.3-60) the 8 station network has 10 degrees of freedom, the 18 station network, 105.

In order to study the effects of errors in the model matrix Q_X, first 3 cm and then 6 cm Gaussian noise is added to the expected deformations (4.2-16) computed from AM1-2 in order to construct a weak but somewhat realistic model matrix. The simulated baseline length changes are computed as described above using the "correct" model. Thus, the geodetic observations detect the "true" deformations within 3 cm observational noise but the geophysical model is rendered somewhat incorrect, and inconsistent with the geodetic data. The noise level on the deformations was chosen according to the uncertainties attached to the AM1-2 plate rotation vector parameters. The propagated deformations have standard errors on the order of several centimeters for $t - t_0 = 2$ years. This seems at first glance surprising considering the large uncertainties in the AM1-2 parameters. A closer examination indicates that the poles of the plate rotation vectors contain the largest uncertainties, particularly for the slower moving plates. These are not as critical as the plate rotation rates whose standard errors are comparatively smaller.

4.3.2 Results of the 8-Station 8-Plate Experiments

The results of the 8-station network experiments are listed in Table 4. It should be noted that all numbers are means over 3 runs

Table 4 Results of 8-Station 8-Plate Network Simulations (AM1-2 + vertical model, 3 cm baseline accuracy, 2 year time interval)

Property	BLIMBE (0)*	BLIMBE (3)*	BLIMBE (6)*	BLE (0)*	BLE (3)*	BLE (6)*	Bayesian (0)*	Bayesian (3)*	Bayesian (6)*	BLICUE (0)*	BLICUE (3)*	BLICUE (6)*
$\hat{V}^T P \hat{V}$ (unitless)	7.2	9.6	9.6	13.4	25.5	42.2	13.6 5.1	27.8 24.2	67.5 74.9	7.2	9.6	9.6
$\hat{X}^T M \hat{X}$ (unitless)	30.1	74.4	143.0	5.9	20.6	40.6	281.2	279.7	348.5	312.4	282.5	302.9
TR(Bias)*** (cm^2)		20.5 (0.9)**			36.4 (1.2)**			0			0	
TR(Cov(X))*** (cm^2)		76.2 (1.8)**			- -			50.9 (1.5)**			94.4 (2.0)**	
TR(MSE(X))*** (cm^2)		96.8 (2.0)**			54.3 (1.5)**			50.9 (1.5)**			94.4 (2.0)**	
$\hat{\sigma}_0^2$ (unitless)	0.7	1.0	1.0	0.7	1.7	3.0	0.7	2.0	5.5	0.7	1.0	1.0
$\hat{\tau}_0^2$ (unitless)	1.3	3.1	6.0	-	-	-	-	-	-	-	-	-
C-Measure (unitless)		10.3			6.8			4.6			10.0	
SMPE (Var) (cm^2)	4.4	6.1	11.3	0.9	5.0	11.8	0.8	5.3	19.4	4.3	5.5	13.2
SMPE (Bias) (cm^2)	0.0	0.1	0.1	0.0	0.1	0.2	0.0	0.2	0.8	0	0.2	0.7
RSMPE (cm)	2.1	2.5	3.4	1.0	2.3	3.5	0.9	2.3	4.4	2.1	2.4	3.7

* Model noise (cm)
** Numbers in parentheses are root mean square values (cm)
*** A Priori values

each with different noise in order to get a more representative sample. More "noise loops" were not considered necessary since the results from the 3 runs did not appear to differ significantly.

The first conclusion from these runs is that the results are in accordance with the optimal properties of each estimate as outlined in Chapter 3 and summarized in Table 1. The $\hat{V}^T P \hat{V}$'s for BLIMBE and BLICUE are equivalent and minimum, both being a generalization of the ordinary least squares estimate (BLUE). It is interesting to note that an error in the model matrix has a very small effect on $V^T PV$ for both 3 and 6 cm model noise. This error is reflected, as expected, in the norm $\hat{X}^T M \hat{X}$. This results from $\hat{V}^T P \hat{V}$ being defined in the observation space while $\hat{X}^T M \hat{X}$ is defined in the parameter space. As can be seen, the parameter space variance of unit weight $\hat{\tau}_0^2$ (3.3- 61) is a good indicator of the compatibility between the adopted model and the geodetic observations. The same holds for $\hat{\sigma}_0^2$ (3.3-189) for the BLE, where now both $\hat{V}^T P \hat{V}$ and $\hat{X}^T M \hat{X}$ are defined in the observation space. Furthermore, we recall that the BLE minimizes $\hat{V}^T P \hat{V} + \hat{X}^T M \hat{X}$, as is the case here. For the Bayesian estimate, the situation is similar to that of the BLE. $\hat{V}^T P \hat{V}$ is computed as in (3.4-36) and the variance of unit weight according to (3.4-34). Note that in Table 4 the contributions from both terms of (3.4-35) are listed.

For the Bayesian case, $\hat{X}^T M \hat{X}$ is not very informative. Recall that here M includes only the stochastic portion of AM1-2, as is the case for the BLICUE. In fact, for the latter none of the indicators in the table seem to reflect an error in the model matrix.

For the minimum bias property we compute an "average bias for (4.3-1)

$$\left[\frac{tr[(I-GA)Q_X(I-GA)^T]}{3p}\right]^{1/2} \tag{4.3-8}$$

We see that this is approximately 0.9 cm for BLIMBE and 1.2 cm for the BLE. Note that it does not depend on the actual observations so that it is an a priori indicator and a function of the network geometry (the design), the observational accuracy and the geophysical model. In any case, though BLIMBE is minimum bias, that of the BLE is not much larger. The BLIMBE minimizes the bias at the expense of variance compared to the BLE. In other words, BLIMBE minimizes the variance in the class of minimum bias estimators while the BLE does so in the class of all biased homogeneous estimators. Furthermore, the BLE minimizes the mean square error in this class. However, the heterogeneous Bayesian estimator has even lower variance and thus mean square error (there is no bias) than both biased homogeneous estimators.

The conditioning of the various solutions is reflected in the C-measure. In this example, the Bayesian estimator yields the most stable covariance matrix as can be seen from the correlation distribution given in Table 5. As can be seen the BLICUE seems to possess a structure that lies between the Bayesian type estimators (BLE and Bayesian) and the pseudoinverse one (BLIMBE) as was indicated in section 3.4.2.

Finally, we examine what may be the most interesting indicator, the RSMPE of the estimated deformation compared to the "true"

Table 5 Correlation Distribution for Deformation Parameters, 8-Station 8-Plate Network

Correlations		BLIMBE		BLICUE		BLE		Bayesian	
		No.	%	No.	%	No.	%	No.	%
-1.0	-0.9	0	0.0	0	0.0	0	0.0	0	0.0
-0.9	-0.8	0	0.0	0	0.0	0	0.0	0	0.0
-0.8	-0.7	0	0.0	0	0.0	0	0.0	0	0.0
-0.7	-0.6	1	0.4	0	0.0	0	0.0	0	0.0
-0.6	-0.5	1	0.4	0	0.0	0	0.0	0	0.0
-0.5	-0.4	5	1.8	3	1.1	0	0.0	0	0.0
-0.4	-0.3	14	5.1	1	0.4	1	0.4	1	0.4
-0.3	-0.2	30	10.9	9	3.3	5	1.8	2	0.7
-0.2	-0.1	44	15.9	35	12.7	13	4.7	10	3.6
-0.1	0.0	69	25.0	84	30.4	78	28.3	73	26.4
0.0	0.1	52	18.8	98	35.5	129	46.7	150	54.3
0.1	0.2	36	13.0	39	14.1	38	13.8	32	11.6
0.2	0.3	12	4.3	6	2.2	11	4.0	8	2.9
0.3	0.4	12	4.3	1	0.4	1	0.4	0	0.0
0.4	0.5	0	0.0	0	0.0	0	0.0	0	0.0
0.5	0.6	0	0.0	0	0.0	0	0.0	0	0.0
0.6	0.7	0	0.0	0	0.0	0	0.0	0	0.0
0.7	0.8	0	0.0	0	0.0	0	0.0	0	0.0
0.8	0.9	0	0.0	0	0.0	0	0.0	0	0.0
0.9	1.0	0	0.0	0	0.0	0	0.0	0	0.0

Estimate: BLIMBE, BLICUE, BLE, Bayesian

deformation. The magnitudes of the "true" deformations (computed from the AM1-2 and the vertical motion model) for a two year period are listed in Table 6. Note the root mean square deformation of 6.3 cm. In all cases the bias term is small compared to the variance term. Of course, the Bayesian estimate yields the smallest RSMPE in the case of no model errors, i.e., when the assumption $E(\bar{X}) = X$ (3.4-1) holds. In this case, the adjustment just involves filtering out the observation errors after the correct deformations have been applied directly (3.4-20). Note that the weak Bayesian approach of the BLE yields comparable results even though the expected deformations are entered indirectly through the moment matrix Q_X. However, as the model errors increase, $\bar{X}$ moves away from X, the Bayesian estimate yields the largest RSMPE while the BLE is less affected. This will become more pronounced in the results of the next section. Here, the biased estimates and the BLICUE yield somewhat better results, i.e., they are less affected by model errors.

4.3.3 Results of the 18-Station 6-Plate Experiments

In this set of simulations, 18 stations are distributed, 3 per each of six major tectonic plates. The geophysical model is AM1-2, thus the model matrix is positive semidefinite. The same series of tests were performed as for the 8-station network. This simulation is more representative from the point of view of greater redundancy (105 degrees of function versus 10 for the 8-station simulation) and more interesting since only the AM1-2 model has been used as conceivably would be done in practice. The results are listed in Table 7.

Table 6 Sum of AM1-2 and Vertical Model Computed Deformations for 8-Station 8-Plate Network (2 year time interval)

No.	Station	Plate	Deformation Vector (cm) (X*)	(Y*)	(Z*)
1	Johannesberg	AFRC	1.64	2.83	1.13
4	Onsala	EURA	2.89	0.10	5.28
8	Ft. Davis	NOAM	-4.41	1.51	-3.23
10	Maui	PCFC	-6.99	15.09	10.50
13	Sao Paulo	SOAM	-3.23	-5.19	-1.46
16	Orroral	INDI	-10.00	5.03	8.87
19	Easter Isle	NAZC	12.06	-3.12	-2.08
20	Arabia	ARAB	-5.31	-3.37	3.70

Mean Deformation : 0.93 cm
Root Mean Square Deformation : 6.27 cm

* Deformation Components (X towards 0° longitude, Y towards 90° longitude, Z towards earth's rotation pole, spherical earth approximation)

Table 7 Results of 18-Station 6-Plate Network Simulations (AM1-2, 3 cm baseline accuracy, 2 year time interval)

	Estimate											
Property	MINOLESS			BLE			Bayesian			BLICUE		
	(0)*	(3)*	(6)*	(0)*	(3)*	(6)*	(0)*	(3)*	(6)*	(0)*	(3)*	(6)*
$\hat{V}^T P \hat{V}$ (unitless)	105.5	109.5	109.5	147.5	190.6	190.2	147.7 4.1	365.5 19.6	1043.8 73.6	105.5	109.5	109.5
$\hat{X}^T M \hat{X}$ (unitless)	12.8	38.8	50.3	5.0	43.2	79.4	250.9	262.6	341.7	257.7	258.2	240.4
TR (Bias) ** (cm^2)	93.4 (1.3)	55.2 (1.0)	56.3 (1.0) ***	28.7	27.3 (0.7) ***	27.1	0	0	0	0	0	0
TR (Cov (X)) ** (cm^2)	130.2 (1.6)	107.4 (1.4)	102.9 (1.4) ***	- -	- -	- -		44.0 (0.9) ***			124.3 (1.5) ***	
TR (MSE (X)) ** (cm^2)	223.6 (1.9)	162.6 (1.7)	159.2 (1.7) ***	48.3 (0.9)	46.4 (0.9)	45.9 (0.9) ***		44.0 (0.9) ***			124.3 (1.5) ***	
$\hat{\sigma}_0^2$ (unitless)	1.0	1.0	1.0	1.0	1.5	1.8	1.0	2.6	7.4	1.0	1.0	1.0
$\hat{\tau}_0^2$ (unitless)	0.7	2.2	2.8	-	-	-	-	-	-	-	-	-
C-Measure (unitless)	70.9	24.8	20.4	19.4	19.9	16.0		13.9			20.0	
SMPE (Var) (cm^2)	2.6	26.1	33.8	0.5	5.9	9.3	0.5	7.7	30.5	2.0	2.5	5.6
SMPE (Bias) (cm^2)	0.0	0.3	0.5	0.0	0.1	0.1	0.0	0.5	1.7	0	0.1	0.4
RSMPE (cm)	1.6	5.1	5.8	0.7	2.4	3.1	0.7	2.9	5.7	1.4	1.6	2.4

* Model noise (cm)
** A Priori values
*** Numbers in parentheses are root mean square values (cm)

The first immediate conclusion is that the estimator of section 3.3.1.4 for seminorms is no longer minimum bias. Recall that the proof of the minimum bias property assumed a positive definite $Q_{\bar{X}}$ matrix. Therefore, the first estimator in Table 7 is denoted as MINOLESS, in this case a minimum $Q_{\bar{X}}$ seminorm P least squares estimate. Recall that the minimum $Q_{\bar{X}}$ seminorm is conditional on P least squares, so again $\hat{X}^T M \hat{X}$ is smaller for BLE. Notice that this estimate is very sensitive to errors in the $Q_{\bar{X}}$ matrix as can be seen in the RSMPE it appears unsuitable for our purposes. The other estimators are not affected by the non-positive definiteness of the model matrix.

It should be noted here that a BLIMBE could be found for a positive semidefinite $Q_{\bar{X}}$ matrix. In fact it has the same equation as for the positive definite case (3.3-41) (B. Schaffrin, 1982, private communication). It was tested for this 18-station experiment but proved to be quite unsatisfactory. The trace of the bias matrix was extremely small ($\sim 2 \times 10^{-8}$ cm^2), so that one almost has an "unbiased" estimate. On the other hand, the standard errors of the parameter covariance matrix were unacceptably large with a magnitude of several meters and the deformations were estimated very poorly. This is a classic example of not choosing an estimator by only its seemingly optimal properties without checking it also by simulations. Recall that the BLIMBE is formally minimum variance, but only in the class of minimum bias estimators. This is an extreme case of minimizing bias at the expense of variance. Therefore, a warning is issued for all potential users of the BLIMBE with a $Q_{\bar{X}}$ seminorm.

The good RSMPE results of the BLICUE are quite misleading as will become clearer in the zero deformation tests of section 4.3.5. Recall that 3 cm and 6 cm Gaussian noise were added to the "true" deformations in order to construct an incorrect model matrix Q_X. The BLICUE constrains $C\hat{X} = C\bar{X}$ (section 3.4.2) so that the random noise has not significantly changed the overall six $C\bar{X}$ values. Therefore, this is not the best way to test the BLICUE estimate. Later results will show that the constraints $C\hat{X} = C\bar{X}$ will make the BLICUE estimate too sensitive to certain types of errors in the model matrix and therefore also unsuitable for our purposes.

We are left then with the strong and weak Bayesian approaches, i.e., Bayesian versus BLE. It seems clear that the weak approach, that is introducing the expected deformations through the model moment matrix Q_X, is preferable to the strong approach of direct use of the a priori information. When the model is "correct", the RSMPE's of both estimates are at the same level (approximately 0.7 cm). The AM1-2 computed magnitudes of deformations for the two year period are listed in Table 8. They have an RMS of 6.1 cm so that most of the deformation is being recovered. On the other hand, the BLE is less sensitive to errors in the geophysical model. Note that for both estimates, the variance of unit weight $\hat{\sigma}_0^2$ is close to unity for the "correct" model case. In the presence of model noise, $\hat{\sigma}_0^2$ indicates an imcompatibility between the geodetic observations and the geophysical model. One can then make a case that the Bayesian estimate may be more appropriate for testing a particular geophysical model, as indicated by the larger values for $\hat{\sigma}_0^2$.

Table 8 AM1-2 Model Computed Deformations for 18-Station 6-Plate Network (2 year time interval)

No.	Station	Plate	Deformation Vector (cm) (X*)	(Y*)	(Z*)
1	Johannesberg	AFRC	0.06	1.99	2.01
2	Cairo	AFRC	-0.99	-0.63	2.03
3	Lagos	AFRC	-0.18	0.68	1.24
4	Onsala	EURA	-0.28	-0.65	0.27
5	Jodrell Bank	EURA	-0.27	-0.61	0.18
6	Shanghai	EURA	-0.18	-0.40	0.42
7	Fairbanks	NOAM	-2.76	-0.30	-1.17
8	Ft. Davis	NOAM	-4.87	-0.15	-2.22
9	Westford	NOAM	-4.54	-2.57	-1.08
10	Maui	PCFC	-3.59	16.69	9.18
11	Tahiti	PCFC	-12.22	15.19	8.95
12	Marshall Isles	PCFC	5.47	18.48	9.52
13	Sao Paulo	SOAM	-4.52	-3.87	-0.66
14	Buenos Aires	SOAM	-4.88	-2.59	-0.50
15	Caracas	SOAM	-5.51	-2.52	-0.52
16	Orroral	INDI	-7.27	3.39	11.27
17	Yaragadee	INDI	-8.41	3.97	12.94
18	Bombay	INDI	-1.90	-2.50	8.61

Mean Deformation : 0.88 cm
Root Mean Square Deformation : 6.10 cm

* Deformation Components (X towards 0° longitude, Y towards 90° longitude, Z towards earth's rotation pole, spherical earth approximation)

4.3.4 MERIT-COTES Experiments

The planned MERIT 83/84 main campaign (Wilkins, 1981) may be the first opportunity to establish the frame for a future CTS, considering that approximately 20 globally distributed stations will be available with a combination of the best VLBI, SLR and LLR instrumentation (Mueller et al., 1982). The strength of the reference frame is actually an indication of how well the polyhedron samples the earth. For a finite number of stations, then we investigate how "optimal" are the possible MERIT-COTES networks from the point of view of monitoring deformations.

In (Mueller et al., 1982) it was assumed that no geophysical model is adopted and a free adjustment used for estimating deformations. The proposed MERIT-COTES (Working Group on the Establishment and Maintenance of a Conventional Terrestrial Reference System) networks (see Tables 9, 10 and Fig. 5) were compared to optimal network (polyhedra) designs for different numbers of vertices. These optimal polyhedra result from distributing p points on a sphere so that they are, in some sense, as far apart as possible from one another. It was stated there that an analogous optimality criterion is that the origin of the coordinate system defined by the p points is best determined (at least from the point of view of trace and determinant optimality defined below). This is proven in Appendices B-D using some recent results in optimal design theory. It was shown that to a good approximation the above criterion also provides the best configuration for analysis of polyhedron deformations. Obviously, the distribution of stations is constrained by various factors, foremost of which is the location of the land masses.

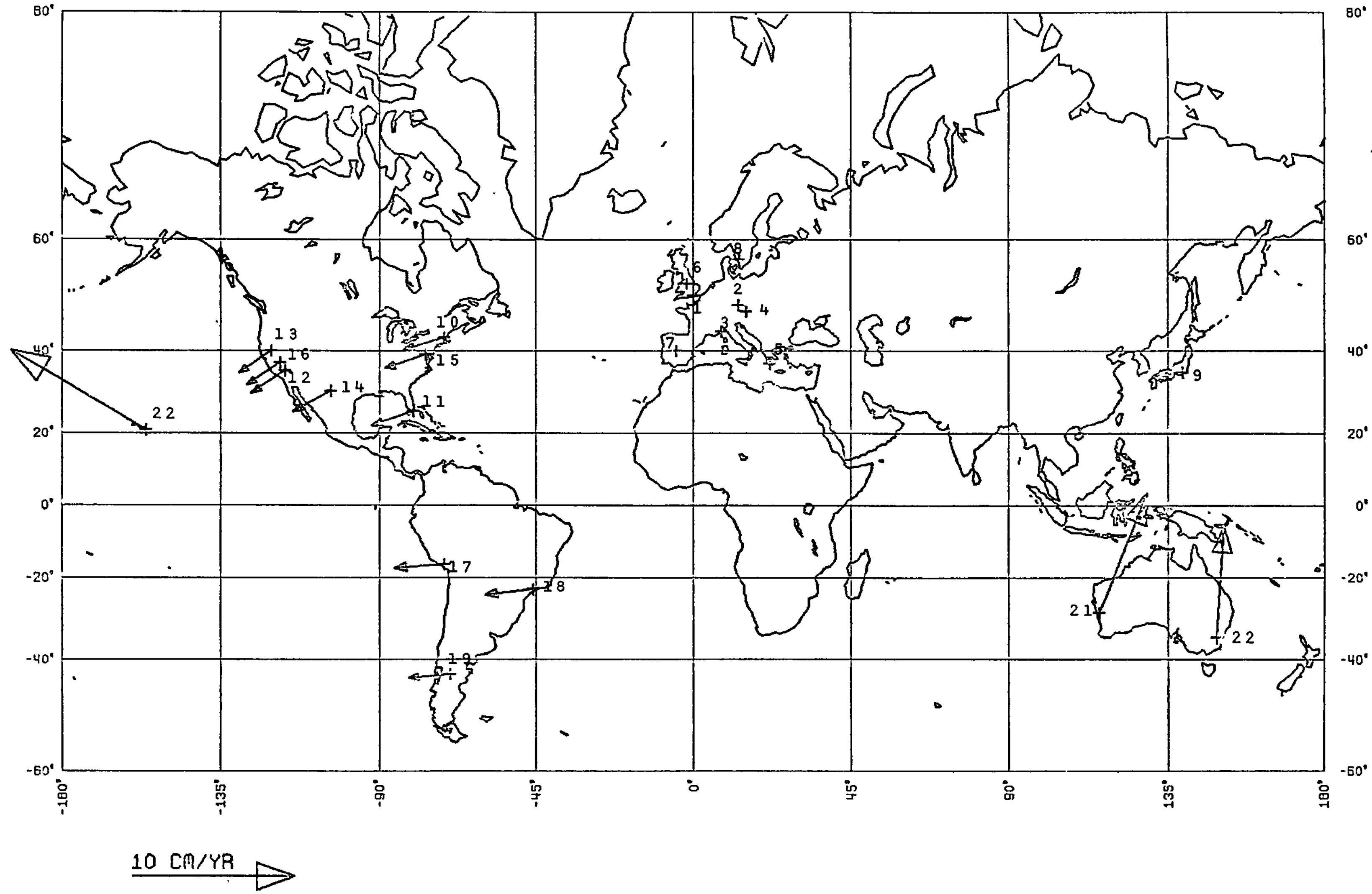


Fig. 5 MERIT-COTES Stations and AM1-2 Computed Velocities (see Table 10)

Table 9 MERIT-COTES Global Networks Directory

Proposed Stations 1983/84	Status(1981)			Single Expt.							Combined Experiments									
				7	10	11	12	13	14	12	18A	B	C	D	20A	B	C	D	24A	B
Operational																				
*Ft. Davis	M3	G3	LL	V	V	V	V	L	L	VL	VL	VL	VL	VL	VL	VL	VL	VL	VL	VL
*Wetzell	M3	G3	LL	V	V	V	V	L	L	VL	VL	VL	VL	VL	VL	VL	VL	VL	VL	VL
Maui		G3	LL					L	L	AL	AL	L	L	L	AL	L	L	L	AL	L
Greenbelt		G3						L	L	AL	AL	L	AL	L	AL	L	AL	L	AL	L
Westford	M3			V	V	V	V			VT	VT	V	VT	VT	VT	V	VT	VT	VT	V
Onsala	M3			V	V	V	V			VT	VT	V	VT	VT	VT	V	VT	VT	VT	V
Owens Valley	M3			V	V	V	V			VT	VT	V	V	V	VT	V	V	V	VT	V
Crimea			LL																	
Arequipa		G2						L	L						AL	L	L	AL	AL	L
Grasse		G2	LL					L	L						AL	L	L	L	AL	L
Kootwijk		G2																		
Construction																				
Herstmonceux		G3						L	L	L	AL	L	L	L	AL	L	L	L	AL	L
Graz	M3	G3						L	L	L	AL	L	L	L	AL	L	L	L	AL	L
Simosato		G3						L	L	L	AL	L	L	AL	AL	L	L	AL	AL	L
Richmond	M3			V	V	V	V			VT	VT	V	VT	V	VT	V	VT	VT	VT	V
Upgrading																				
Quincy		G3						L	L		AL	L	AL	L	AL	L	AL	L	AL	L
Yaragadee		G3						L	L		AL	L	L	AL	AL	L	L	L	AL	L
Jodrell Bank	M3			V	V	V	V				VT	V	V	V	VT	V	V	V	VT	V
*Canberra	M3	G3	LL		V	V	V	L	L	VL	VL	VL	VL	VL	VL	VL	VL	VL	VL	VL
Madrid	M3				V	V	V				VT	V	VT	VT	VT	V	VT	V	VT	V
Goldstone	M3				V	V	V													
Dionysos		G3						L	L		AL	L	L	AL	AL	L	L	L	AL	L
Tahiti-Hoblas6		G3																		
Others																				
Sao Paulo						V	V				VT	V	V	VT	VT	V	V	VT	VT	V
Santiago							V		T										AT	AT

* Primary Collocation Sites

Experiment Codes :
V - VLBI
L - SLR,LLR or both
A - VLBI Mobile (2)
T - Laser Mobile (4)

Status Codes :
M3 - MARK III VLBI
G2 - 2nd Generation Laser
G3 - 3rd Generation Laser
LL - Lunar Laser

Table 10 MERIT-COTES Stations and AM1-2 Velocities

No.	Station	Latitude North (Deg	Min)	Longitude East (Deg	Min)	Plate	Velocity Vector (X*)	(Y*) (Cm/Year)	(Z*)
1	Herstmonceux	51	0	0	0	EURA	-0.13	-0.30	0.10
2	Wettzel	49	9	12	53	EURA	-0.13	-0.29	0.16
3	Grasse	44	0	8	0	EURA	-0.12	-0.26	0.16
4	Graz	48	0	15	0	EURA	-0.12	-0.29	0.17
5	Dionysos	37	0	22	0	EURA	-0.10	-0.23	0.24
6	Jodrell Bank	53	0	358	0	EURA	-0.13	-0.31	0.09
7	Madrid	40	0	355	0	EURA	-0.11	-0.25	0.10
8	Onsala	57	0	13	0	EURA	-0.14	-0.32	0.14
9	Simosato	35	0	142	0	EURA	-0.10	-0.23	0.09
10	Westford	42	36	288	30	NOAM	-2.27	-1.29	-0.54
11	Richmond	25	36	279	37	NOAM	-2.49	-0.83	-0.83
12	Owens Valley	37	13	241	43	NOAM	-2.21	0.22	-1.12
13	Quincy	39	58	239	4	NOAM	-2.14	0.22	-1.09
14	Ft. Davis	30	38	256	3	NOAM	-2.43	-0.07	-1.11
15	Greenbelt	39	1	283	10	NOAM	-2.36	-1.10	-0.66
16	Goldstone	35	26	243	7	NOAM	-2.25	0.23	-1.14
17	Arequipa	-16	28	288	30	SOAM	-2.98	-0.93	-0.22
18	Sao Paulo	-23	0	314	0	SOAM	-2.24	-1.96	-0.33
19	Santiago	-43	00	290	0	SOAM	-2.25	0.26	-1.14
20	Canberra	-35	18	149	8	INDI	-3.63	1.70	5.63
21	Yaragadee	-29	3	115	21	INDI	-4.20	1.99	6.47
22	Maui	20	42	203	44	PCFC	-1.78	8.34	4.58

* Velocity components (X towards 0° longitude, Y towards 90° longitude, Z towards earth's rotation pole, spherical earth approximation)

The purpose of those simulations was to examine how close the possible networks could come to the ideal case of being able to locate the stations anywhere on earth. In other words, how close can one come to constructing an optimal polyhedra on the available land areas.

In the case when a geophysical model for the station deformations is adopted, this must be a factor in the design of an optimal network. Unlike in the previous case, the optimal polyhedra are not so apparent. Here, we use as a basis for comparison the 18 station network distributed over the six major tectonic plates as described in ection 4.3.1. In addition, one station is added to both the smaller Nazca and Arabian plates to form a 20 station network. In constructing these "optimal networks" we were guided by the considerations outlined in Section 4.3.1. The MERIT-COTES networks of (Mueller et al., 1982), now reanalyzed using the AM1-2 model, are compared to these two networks. We can then compare the different design measures under the two assumptions, no plate model or AM1-2 model, to test if the conclusions in (Mueller et al., 1982) differ in any way.

In (Mueller et al., 1982) the corresponding covariance matrices for the deformation estimate (3.3-55) and (3.3-83) are given assuming $M = Q_X^{-1} = I$ by

$$\Sigma_{\hat{X}} = \sigma_0^2 \, N^+ \tag{4.3-9}$$

$$= \sigma_0^2 \, [(N + C^T C)^{-1} - C^T (CC^T CC^T)^{-1} C] \tag{4.3-10}$$

respectively, where $\sigma_0^2 = 1$. Here we use the BLE for the comparisons based on the positive assessment of that estimator resulting from the simulations of the previous two sections, specifically the mean square error matrix

$$MSE(\hat{X}) = Q_X - Q_X A^T (AQ_X A^T + P^{-1})^{-1} AQ_X \tag{3.3-147}$$

A canonical notion of design optimality is not available, and we will briefly describe several common design criteria. All of these can be expressed in terms of the reduced eigenspace (the non-zero eigenvalues and their corresponding principal eigenvectors) of $MSE(\hat{X})$.

A-optimality is defined as minimizing the average variance (the A-measure), in this case the average mean square error, or equivalently the spectral mean, i.e.,

$$\min \frac{1}{3p} \text{ trace } [MSE(\hat{X})] = \min \frac{1}{3p} \sum_{i=1}^{3p} \sigma_i^2 \tag{4.3-11}$$

$$\min \frac{1}{3p-6} \sum_{i=1}^{3p-6} \lambda_i \tag{4.3-12}$$

where λ_i are the non-zero eigenvalues and σ_i^2 the diagonal elements of $MSE(\hat{X})$.

D-optimality is defined as minimizing the determinant of $MSE(\hat{X})$ raised to the $1/(3p-6)$ power (the D-measure) or equivalently

$$\min \left[\prod_{i=1}^{3p-6} \lambda_i \right]^{\frac{1}{3p-6}} \tag{4.3-13}$$

E-optimality is defined as

$$\min_{\lambda_i} \lambda_{max} \tag{4.3-14}$$

We shall refer to the maximum eigenvalue as the E-measure.

Another criterion, though used usually for determining the condition of a matrix rather than for optimal design, can be termed C-optimality, and is defined by

$$\min_{\lambda_i} \frac{\lambda_{max}}{\lambda_{min}} \tag{4.3-15}$$

the ratio of maximum and minimum eigenvalues. This is useful, since it is unitless and independent of scale factor, i.e., baseline precision.

It should be noted that all of these criteria are rotationally and translationally invariant (isotropic and homogeneous) (Grafarend, 1974). That is, only the relative distribution of the stations affects the optimal design.

Besides these four optimal measures, we compute the "average" bias, (4.3-8) squared to be in the same units (cm^2) as the first three measures.

In Table 11 the optimal measures of the MERIT-COTES 18A and 20A networks are compared to their corresponding "optimal" 18 and 20 station nets. Recall that each baseline is observed twice, once by VLBI and once by laser, or equivalently the baseline lengths are observed with $3/\sqrt{2}$ cm accuracy. Note that both the 18A and 20A nets cover only 5 tectonic plates. As expected, the "optimal networks"

Table 11 Comparison of MERIT-COTES Networks Design Measures with 18-Station 6-Plate and 20-Station 8-Plate Networks

Experiment		Design Measures								
No.	Description	A-Measure A1*	A-Measure A2**	D-Measure D1*	D-Measure D2**	E-Measure E1*	E-Measure E2**	C-Measure C1*	C-Measure C2**	Bias **
		(cm^2)		(cm^2)		(cm^2)		(unitless)		(cm^2)
	MERIT-COTES with Single Type of Instrument****									
	VLBI :									
7	Extended Polaris	16.5	3.0	6.2	1.5	153.9	5.7	120	479	2.1
10	Exp. 7 + DSN	3.6	2.3	3.2	1.4	15.3	5.4	16	148	1.7
11	Exp.10 + Sao Paulo	2.9	1.8	2.8	1.4	10.1	4.8	12	116	1.1
12	Exp.11 + Santiago	2.6	1.5	2.5	1.3	8.6	4.6	11	108	1.0
	Laser :									
13	Status in 1983	2.5	1.3	2.3	1.3	10.1	4.4	15	38	0.9
14	Exp.13 + Santiago	2.2	1.2	2.1	1.3	7.1	4.4	11	63	0.8
	MERIT-COTES with Combined Lasers and VLBI'S****									
12	Operational (9)***	2.5	1.2	1.9	1.3	12.3	4.7	28	51	1.0
18A	VLBI+SLR+LLR (all)	1.0	0.9	0.9	1.1	4.1	4.2	16	38	0.6
18B	Primary (3)	4.9	1.1	3.2	1.3	39.7	4.8	75	35	0.8
18C	Continental (9)	2.3	1.1	1.7	1.2	16.3	4.6	43	33	0.7
18D	Global (9)	1.9	1.0	1.6	1.2	6.8	4.5	19	33	0.7
20A	Exp. 18+G2SLR (all)	0.8	0.8	0.8	1.1	3.3	4.4	15	111	0.6
20B	Primary (3)	3.7	1.1	2.7	1.2	23.2	4.9	48	115	0.7
20C	Continental (9)	2.1	1.0	1.6	1.2	12.9	4.7	36	110	0.7
20D	Global (9)	1.8	1.0	1.5	1.2	7.7	4.7	22	117	0.7
	Comparison Networks									
18	6-PLATE (ALL)	0.8	0.6	0.8	1.0	1.5	2.8	9	19	0.4
20	8-PLATE (ALL)	0.8	0.6	0.7	1.0	1.4	2.9	9	19	0.4

* No plate model, N=I (3 cm baseline accuracy)
** AM1-2 Model (3 cm baseline accuracy -- 2 year interval)
*** Numbers in parentheses indicate number of collocated sites
**** For station locations see Table 9 and Figure 5

yield better results as reflected particularly in the E and C measures implying that the MERIT-COTES networks could be substantially improved. The large C-measure for 20A indicates that the two European stations added to 18A degrade the structure of the MSE matrix since these stations are added in a small region of a plate that is already densely covered in that particular area. Note that the bias term for the "optimal nets" is also smaller.

It should be mentioned that the comparison of the no model and AM1-2 model experiments also shown side by side in Table 11 needs some qualification. First, two different estimators have been used. Second, the no model case is independent of the time interval between observations, while the AM1-2 Q_X matrix is a function of time. As can be seen by examination of the results, the conclusions in (Mueller, et al, 1982) change in degree. That is, there are much smaller differences between the various collocation schemes, indicating that the absolute motion model provides an underlying frame of reference for the monitoring of deformations.

Note that only the C-measures increase in the AM1-2 case. This is due to the positive semidefiniteness of Q_X as compared to the positive definite $M = I$ case.

The conclusions for the single typed instrument experiments also do not change. Note though that the geophysical model improves the weaker VLBI experiment 7. This points to the advantages of a geophysical model. It is possible to perform a 3-dimensional deformation analysis in a network of less than global extent. For example, in the VLBI experiment 7 stations are distributed over the North American plate and only the westernmost portion of the Eurasian plate. On the other hand, the free adjustment (M = I) requires a more global distribution for a meaningful 3-dimensional analysis. Furthermore, consider the polyhedron as wrapped completely about the earth. If the network is not closed, that is if there is a gap due to certain baselines not being measured, the free adjustment will have added to it further rank deficiencies of a geometrical nature besides the six that result from the coordinate system definition problem. This kind of singular situation could occur for example in a limited VLBI network where problems of mutual radio source visibility make certain baselines nonobservable. A realistic geophysical model could be very helpful in overcoming such problems particularly using the BLE.

The 12 station VLBI and the 13 (14) station laser networks are again of basically the same quality. The structure of the laser net is stronger as reflected in the C-measure. This is most likely due to the distribution of stations over 5 plates instead of 4 in the VLBI case as well as the better coverage of the plates on which the stations are located.

Finally, note that the MERIT-COTES stations may not be in their optimal locations from the point of view of geophysically stable (on the intraplate level) sites, for example, the Southern European and Japanese stations. Furthermore, it would be prudent, as mentioned before, to increase the number of stations per major plate to four or five to increase the reliability of these subnets for interplate motion detection. Intraplate and local motions could be monitored by filling in the subnets (within each plate) by, for example, GPS interferometric observations (Counselman and Shapiro, 1978).

4.3.5 Zero Deformation Tests

In this section we test the situation in which the deformations are assumed to behave according to some model but actually there is no motion. This means that the difference between the baseline lengths at two epochs are due solely to observational noise. In these tests we use the Q_X matrix computed from AM1-2 but only 3 cm Gaussian noise is added to the initial baseline lengths to simulate the baseline observations. We use the 18-station network.

The results are listed in Table 12. As is apparent from the RSMPE, the biased estimates are less affected by the introduction of the faulty model, particularly the BLE. This is explained, as was noted before, from the deformations being introduced in a weak way in the biased estimates (MINOLESS and BLE) adjustments through the moment matrix. This same information is applied in a strong way as corrections to the station coordinates in the conditionally unbiased adjustments (Bayesian and BLICUE).

The results from this test point again to the BLE as the best deformation estimator. Recall that for the BLE estimation model, the deformation parameter vector X was assumed random and the model derived expected deformation $\bar{X}$ deterministic, while for the other three estimation models the opposite assumption was made. The BLE approach then is to filter the deformations (signals) from the

Table 12 Zero Deformation Simulations (AM1-2, 3 cm baseline accuracy, 2 year time time interval, 18-station 8-plate network)

	Estimate			
Property	MINOLESS	BLE	Bayesian	BLICUE
$\hat{V}^T P \hat{V}$ (unitless)	105.5	147.5	181.6 104.6	105.5
$\hat{X}^T M \hat{X}$ (unitless)	11.7	4.0	81.0	142.3
TR(Bias)* (cm^2)	93.4 (1.3)**	28.7 (0.7)	0	0
TR(Cov(X))* (cm^2)	130.2 (1.6)**	- -	44.0 (0.9)**	124.3 (1.5)**
TR(MSE(X))* (cm^2)	223.6 (1.9)**	48.3 (0.9)**	44.0 (0.9)**	124.3 (1.5)**
$\hat{\sigma}_0^2$ (unitless)	1.0	1.0	1.9	1.0
$\hat{\tau}_0^2$ (unitless)	0.7	-	-	-
C-Measure (unitless)	70.9	19.4	13.9	20.0
SMPE (Var) (cm^2)	2.6	0.5	11.7	23.2
SMPE (Bias) (cm^2)	0.0	0.0	0.0	0.8
RSMPE (cm)	1.6	0.7	3.4	4.9

* A Priori values
** Numbers in parentheses are root mean square values (cm)

observational noise. This explains its success in this test since only noise was present in the simulated observations. The fact that the model matrix Q_X indicated expected deformations (with an RMS of about 6 cm) did not alter the filtered deformations significantly. On the other hand, the a posteriori variance of unit weight ($\hat{\sigma}_0^2 = 1$) did not indicate anything peculiar although that there were no statistically significant deformations could have been determined from examining the deformation estimates. The Bayesian a posteriori variance of unit weight ($\hat{\sigma}_0^2 = 1.9$) indicates that baseline observations and the geophysical model were incompatible. This is seen again, then, to be the main positive point of the Bayesian estimator.

4.3.6 No Model Tests

In this section we consider the case where the AM1-2 model is correct but no model weight matrix is used, i.e., $M = I$. Alternatively, this could mean that we expect no secular deformations with a particular uncertainty, or $M = k^2 I$. Here we assume this to be at the 10 cm level. In this case, as described in Chapter 3, the BLICUE reduces to the BLIMBE, the Bayesian to the BLE which approximates the BLIMBE to a degree depending on the scale factor k^2. Only the BLIMBE should be used in this case since the normal matrices can be quite ill conditioned for the other estimators as indicated by the C-measure in Table 13, yielding unstable covariance (or mean square error) matrices.

The important result from this test is that under the assumptions made (3 cm baseline noise, AM1-2 motions and a 2 year re-observation period) the RSMPE is at about the 5 cm level.

Table 13 No Plate Model Simulations (M=I) (3 cm baseline accuracy, 2 year time interval, 18-station 6-plate network)

	Estimate			
Property	BLIMBE	BLE	Bayesian	BLICUE
$\hat{V}^T P \hat{V}$ (unitless)	118.8	119.0	119.0 8.9	118.8
$\hat{X}^T M \hat{X}$ (unitless)	9.2	8.9	8.9	9.2
TR (Bias) * (cm^2)	600.0 (3.3) **	601.7 (3.3) **	0	0
TR (Cov (X)) * (cm^2)	84.5 (1.3) **	- -	682.7 (3.6) **	684.5 (3.6) **
TR (MSE (X)) * (cm^2)	684.5 (3.6) **	682.7 (3.6) **	682.7 (3.6) **	684.5 (3.6) **
$\hat{\sigma}_0^2$ (unitless)	1.0	1.1	1.1	1.1
$\hat{\tau}_0^2$ (unitless)	0.2	-	-	-
C-Measure (unitless)	8.7	201.0	201.0	200.0
SMPE (Var) (cm^2)	22.7	22.6	22.6	22.7
SMPE (Bias) (cm^2)	0.8	0.8	0.8	0.8
RSMPE (cm)	4.8	4.8	4.8	4.8

* A Priori values
** Numbers in parentheses are root mean square values (cm)

Comparing this to the previous tests in which somewhat erroneous models were used, it can be concluded that it is preferable to apply a somewhat incorrect model than no model at all. However, this is contingent on the plate tectonic theory being essentially correct, though the precise motions may not be well known.

4.4 The Treatment of Errors in X_0

In the simulations of this chapter, we assumed a linear model for the computation of expected deformations from the station velocities given by the plate tectonic model. Furthermore, we neglected the effect of uncertainties in the fundamental CTS coordinates X_0. That is, we treated X_0 as errorless even though it will have been estimated from the observations of different measurement systems as described in section 2.2. Whether or not to consider the associated covariance matrix of X_0, Σ_{X_0} when computing deformations is a matter of philosophy. In any case, we seek a well defined datum to which deformations of the polyhedron are referred. It is clear that since the deformation problem is dynamic, the X_0 estimate can never be improved except by a redefinition of the reference frame initial epoch at a later time when improved geodetic observational accuracy would warrant it. However, since we are primarily interested in the changes in the fundamental coordinates $(X - X_0)$, the effect of errors in X_0 on these quantities should diminish with time.

In this section, we add an offset term to the linear model (4.2-16) and indicate how errors in X_0 can be incorporated into the estimation by means of the model matrix Q_X. This is done using a state transition matrix approach which is almost trivial for this case but can be generalized to deal with more complicated models. The motions of the polyhedron are referred to an initial epoch, i.e., to the fundamental polyhedron. If there were no deformations in the initial polyhedron, it would continue to rotate with the earth in its initial configuration. However, the earth is deforming so that perturbations are present which must be monitored. These perturbations may be periodic or secular.

Consider that this motion could be described by a set of simultaneous first-order differential non-linear differential equations (Liebelt, 1967)

$$\frac{dX}{dt} = H(X(t), d(t), t) \tag{4.4-1}$$

where d(t) is a set of specified forcing functions. The integration of these equations of motion using the initial condition (state) $X(t_0) = X_0$, results in the deformed state (the state vector) X(t) at some later epoch. Since we invariably have to linearize our problems, we can simplify (4.4-1) by assuming a linear system

$$\frac{dX}{dt} = F(t)X + G(t)u(t) \tag{4.4-2}$$

where u is a set of specified forcing functions which are related to the time rate of change of X by the matrix G which in our case would be computed from adopted earth models. In our application, we assume a homogeneous linear system so that $u(t) = 0$ and

$$\frac{dX}{dt} = F(t)X \tag{4.4-3}$$

This has a solution of the form

$$X(t) = S(t,t_0)X(t_0) \tag{4.4-4}$$

where

$$\dot{S}(t,t_0) = F(t)S(t,t_0) \tag{4.4-5}$$

and

$$S(t_0,t_0) = I \tag{4.4-6}$$

S is called the state transition matrix familiar in satellite orbit determination. If we can determine this matrix, then the deformed state of the polyhedron can be computed at any epoch by operating S on the initial state X_0. In our case, we assume (compare to (4.2-16)

$$X(t) = X_0 + \dot{X}_0(t - t_0) = X_0 + V(t - t_0) \tag{4.4-7}$$

where X_0 denotes the fundamental coordinates and $\dot{X}_0$ the change in these coordinates computed from the AM1-2 plate tectonic model. The equations of motion are easily written as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{4.4-8}$$

Their solution is given in terms of S , X_0 and $\dot{X}_0$ by

$$\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t-t_0 & 0 & 0 \\ 0 & 1 & 0 & 0 & t-t_0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t-t_0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \tag{4.4-9}$$

$$X = S(t_1 t_0) \cdot \begin{bmatrix} X_0(t) \\ \dot{X}_0(t) \end{bmatrix}$$

Of course, this is just equivalent to (4.4-7). To compute the moment matrix Q_X

$$\Sigma_X = S\Sigma_{X_0,\dot{X}_0} S^T = \Sigma_{X_0} + (t-t_0)^2 \Sigma_{\dot{X}_0} \tag{4.4-10}$$

and

$$\begin{aligned} Q_X &= E\{XX^T\} \\ &= E\{(X_0 + \dot{X}_0 t)(X_0 + \dot{X}_0 t)^T\} \\ &= E\{X_0 X_0^T\} + (t-t_0)^2 E\{\dot{X}_0 \dot{X}_0^T\} \end{aligned} \tag{4.4-11}$$

where we assume that

$$E\{X_0 \dot{X}_0^T\} = E\{\dot{X}_0 X_0^T\} = 0 \tag{4.4-12}$$

It follows that (compare to 4.2-17,18)

$$Q_X = \Sigma_{X_0} + (t-t_0)^2 (\Sigma_{\dot{X}_0} + \dot{X}_0 \dot{X}_0^T) \tag{4.4-13}$$

where the $X_0 X_0^T$ term is already taken care of in the linearization of the deformation mathematical model (3.2-1).

In order to add the effects of errors in X_0, then, to the deformation estimation, it is necessary to add its covariance matrix Σ_{X_0} to the model matrix Q_X. Thus, the linearization point X_0 of (3.2-2) is seen to be stochastic and therefore so is L_0, the initial baseline lengths of the polyhedron and the zero order term of the Taylor's expansion. It follows that for (3.2-6) and (3.2-7)

$$D[V] = Q_V = \sigma_0^2(P^{-1} + P_0^{-1}) = D[L] \tag{4.4-14}$$

where $\sigma_0^2 P_0^{-1}$ is the covariance matrix of the initial distances L_0 propagated from Σ_{X_0}. The same dispersion matrix for V is used for all the four estimators of Chapter 3.

Although this model is simple, it does provide a general method whereby a more complex model matrix Q_X could be derived. It could include complicating factors such as interplate and local motions, tidal effects, etc. This would require differential equations of the form (4.4-2). A finite element method may be most applicable to solving this system and determining the state transition matrix. In this case, the original state (X_0) of the polyhedron would serve as initial conditions and there could also be appropriate boundary conditions. In this way, too, the deformations of non-CTS stations could be determined by densifying the finite element mesh.

5 CONCLUSIONS AND RECOMMENDATIONS

5.1 Alternatives for Reference System Maintenance

The frame of the future CTS is to be defined at an initial epoch by an adopted set of spatial coordinates of a global network of observatories mainly VLBI, SLR and LLR stations. In order to insure that the earth orientation parameters are referred to the same set of axes, the deformations of the polyhedron need to be estimated periodically, i.e., a new set of CTS coordinates. This is what is referred to as maintaining the reference system so that the frame is accessible to the user by the earth orientation (and translation) parameters in a consistent and accurate manner.

In order to maintain the reference system on the deformable earth, either some constraints must be applied or geophysical models adopted, or both. One body of opinion holds that no geophysical model should be adopted at least not in the initial stages of the new CTS operations. During its early stages, one of the functions of the CTS could be to test if the geodetic observations are indicating motions compatible with those predicted by plate tectonic theory. Drewes (1982) presents an estimation procedure to estimate plate rotation parameters from a combination of geodetic and geophysical data, i.e., the inverse problem of instantaneous plate kinematics (Minster and Jordan, 1974). Once the plate parameters are estimated, the forward problem yields expected deformations of the stations that could be used to improve CTS deformation analysis.

If no geophysical model is adopted, all the four estimators of Chapter 3 reduce to the I-norm BLIMBE, or to the familiar free adjustment. Therefore, one is limited essentially to applying the set of inner constraints $C\hat{X} = 0$ which, without any real physical justification, impose no net rotation nor translation for the estimated deformations $\hat{X}$. As indicated by the simulations of Chapter 4, if there are secular deformations of the CTS stations at the level predicted by the absolute motion plate models, the constraints $C\hat{X} = 0$ are inappropriate. The reference system will nevertheless be maintained in a well defined manner. However, the CTS will then have fairly high sensitivity to changes in the distribution of the observing stations, and moreover to the actual locations of the chosen sites contradicting one of the requirements of Chapter 1. With time, distortions may accumulate in the system and the deformation estimates of the free adjustment may less and less resemble the physical deformations.

Another body of opinion maintains that a geophysical model should be adopted from the initial stages of CTS operations. After all, one of the primary reasons for the establishment of a new CTS is the general acceptance of plate tectonic theory. This approach does not necessarily contradict the requirement of avoiding as much as possible dependence on geophysical hypotheses. The CTS frame is still defined by the coordinates of the fundamental polyhedron and is invariant with respect to an adopted geophysical model. The geophysical model does affect the periodically updated CTS coordinates though hopefully improving the estimation of station deformations as compared to the free adjustment. In order to reduce the dependence on geophysical hypotheses, any deformation estimator should not be highly sensitive to the adopted model. Furthermore, if the model is incorrect, a good estimator will alert the investigator to this.

Three alternatives have been suggested for maintaining the reference frame when a geophysical model is adopted and four estimators proposed. The first alternative uses the constraints $CM\hat{X} = 0$ and its corresponding estimator is the BLIMBE, i.e., the weighted free adjustment. As shown in Chapter 2, the resulting reference frame axes are a discrete Tisserand's mean axes of crust. In the case of a positive semi-definite Q_X matrix (associated with any absolute motion plate model) the minimum bias estimator is unsatisfactory as mentioned in Chapter 4. The MINOLESS proposed in section 3.3.1.4 is found to be quite sensitive to errors in the Q_X matrix and, therefore, also unsatisfactory for deformation analysis.

The second alternative is to combine the baseline measurements and the expected deformations, computed from an absolute motion plate model, without any constraints. This implicitely fixes the CTS frame axes in the mantle (and to the mantle and crust which rotate together in a mean sense). Two estimators presented follow this approach, the BLE and Bayesian estimators. The BLE introduces the geophysical model through a moment matrix in a weak Bayesian manner, while the Bayesian estimator does so directly in a strong Bayesian manner. When the model is correct, both approaches provide good estimates. On the other hand, the BLE is less sensitive to the errors in the geophysical model. In the case of applying the AM1-2 model when there are in reality zero deformations (only observational noise) the BLE does not indicate any significant deformations. This follows from its being a filter rather than a true estimator. Note that this approach is less sensitive to the distribution of the observing stations and in the frequency of observations since the geophysical model is good for any station location (although unstable areas should be avoided). It would seem that conversely the dependence on the geophysical hypothesis should increase. This is so for the Bayesian estimate but not for the BLE as described above. Furthermore, an extension of BLE (least squares collocation) can be used to predict deformations in case of trouble at a number of CTS stations, as well as to estimate the coordinates of new stations.

The third alternative is to use constraints of the form $C\hat{X} = C\bar{X}$ (instead of $C\hat{X} = 0$) where $\bar{X}$ is computed from the geophysical model as explained in Chapter 4. It has been seen from examination of the corresponding BLICUE estimator that this approach is a hybrid of the two previous ones. It has been shown to be quite sensitive to errors in the geophysical model in certain cases and does not give an indication of the presence of a poor model. This can be explained by its weighted constraint interpretation.

Summarizing, on the basis of the simulations of Chapter 4 it seems preferable to adopt even a weak but realistic absolute plate motion model for CTS operations than none at all. In this case, the BLE seems most suitable for deformation analysis. If the model is of poor quality, the estimation algorithm should indicate this. If no model is adopted, the I-norm BLIMBE can insure a well defined reference system but of questionable physical significance.

5.2 MERIT-COTES Networks and the 18-Station 6-Plate Network

Several conclusions can be drawn from the comparison of the MERIT-COTES networks to the 18-station six-plate network of section 4.3.4 other than the ones already mentioned. The planned MERIT-COTES networks for the MERIT main campaign provides a good starting point for the eventual establishment of a new CTS. However, the distribution of stations over the major plates should be improved. Furthermore, the number of stations per plate should be at least four from the point of view of reliability as well as geometrical strength (for which three seems to be adequate though). Nevertheless, it is strongly recommended that over the duration of the MERIT main campaign two or three short sessions be devoted to observations from all stations. The first such session at the start of the campaign could be used to establish an initial CTS frame. Subsequent sessions could monitor deformations using the recommended BLE estimator. For testing the compatibility of the baseline measurements with geophysical models, particularly the plate motion hypotheses, the Bayesian estimator is more appropriate.

Under the following assumptions,

1. an 18-station six-plate network
2. 3 cm baseline length accuracy (this implies removal of the larger systematic errors)

3. a fairly reliable absolute motion model

4. no significant anomalous deformations

it can be concluded from the simulations of Chapter 4 that station deformations could be estimated at the 1 cm level from periodic reobservation of baseline lengths every one or two years. This will essentially remove the effects of interplate motions from the error budget of the required 5 cm accuracy short term variations in polar motion and earth rotation. The resulting reference frame will provide an accurate, well defined and consistent zero-order global network for geodetic and geophysical studies.

A potentially major problem in this optimistic scenario is in assumption 4 above. This could be due to site stability problems, intraplate motions and unmodeled tidal effects such as ocean loading. Therefore, the sites must be chosen very carefully by geophysical surveys. Once they are chosen, local effects could be monitored by on-site observations such as gravity observations and local geodetic surveys. Intraplate motions and some local effects could be monitored by GPS derived baselines, particularly in the interferometric mode. Further investigations are recommended to study optimal ways to incorporate these observations in a well defined manner into CTS operations.

5.3 A Final Comment on Estimation

The examination of the four estimators in Chapter 3 and the simulations of Chapter 4 have provided insight into the meaning of biased and unbiased estimation. An estimate of a parameter vector is unbiased if and only if there are no restrictions on the estimation.

In order words, the estimate can conceivably take on any value in the parameter space.

If the design matrix A is singular, there exist relationships among the parameters, restricting the resulting estimate. In the presence of a priori information on the parameters, it is still possible to construct an unbiased estimate (strong Bayesian) with the assumption that the expectation of the a priori parameter vector is equal to the true one. (Under this assumption, it is unbiased also when A is of full rank.) Formally, we have constructed an unbiased estimate but we have "biased" the estimate in the direction of the a priori value. Strictly, this estimate can be only considered conditionally unbiased. If the assumption is correct (within its uncertainties), the result will be an estimate with lower variance than the BLUE (when A is full rank) or BLIMBE (when A is rank deficient). If incorrect, the estimate will suffer accordingly. This is the basic danger in applying the strong Bayesian estimate.

A better approach, in the case of a priori information and whether or not A is of full rank, is to use a biased estimate (BLE), by constructing an appropriate moment matrix Q_X. This reduces the effects of an incorrect assumption for X, which follows from its filtering interpretation (treating X as a random variable). Its more general prediction capabilities are also very useful.

Alternatively, in the case of singular A, the BLIMBE provides a minimum bias estimate (in contrast to minimum mean square error for BLE) that might be preferable depending on the application. For non-singular A , the BLIMBE reduces to the BLUE invariant with respect to a chosen weighted norm in the parameter space.

Summarizing, if even weak but essentially realistic a priori information on the parameters is available, it makes little sense to ignore it. For non-singular A , this is what we do though when the the BLUE is used. However, if we do not choose to ignore this information, why try to construct a seemingly unbiased estimate (the Bayesian estimate). Perhaps the term unbiasedness (and the related estimability) is conveying incorrect connotations of approbation that are difficult to shed. In the case when the BLUE does not exist (as in the deformation problem) we may be tempted to apply a conditionally unbiased estimate (Bayesian again or BLICUE) in order to seemingly make up for the biasedness (or non-estimability) of the problem. We have shown that (at least for analyzing deformations) it is better to stay within the class of biased estimators in the application of a priori information. Perhaps this conclusion holds for problems that are non-singular to begin with. That is, in the presence of a priori information, it would be a better practice to move into the class of biased estimators. This seems to conform with the current mode of thought and investigations of the broader statistical community. (See (Trenkler, 1981) for a good review.)

APPENDIX A

THE WEIGHTED PSEUDOINVERSE

A.1 Conditions for a Weighted Pseudoinverse

Consider an inconsistent set of linear equations $Y = AX$, for a rank deficient A matrix. A is of dimension $n \times u$ and maps X in E^u into Y in E^n. A weighted (ellipsoidal) norm is defined in E^u

$$\| X \|_M = (X^T M X)^{1/2}$$

and in E^n

$$\| L \|_P = (L^T P L)^{1/2}$$

We assume that P and M are positive definite matrices. The following theorem from (Rao and Mitra, 1971) provides the conditions for a solution $X = GY$ to be minimum M-norm P least squares.

Theorem. Let there exist a matrix G such that GY is a minimum M-norm P least squares solution of $AX = Y$. Then it is necessary and sufficient that the following conditions hold

$$AGA = A \tag{A-1}$$

$$GAG = G \tag{A-2}$$

$$(GA)^T M = MGA \tag{A-3}$$

$$(AG)^T P = PAG \tag{A-4}$$

where P and M are positive definite matrices.

A matrix G that fulfills these conditions is denoted by A^{+}_{PM}, a minimum M-norm P least squares g-inverse or a weighted pseudoinverse (Bouilion and Odell, 1971). In the case M and P = I this reduces to A^{+}, a minimum norm least squares g-inverse, or simply the familiar pseudoinverse.

Consider conditions (A-1) - (A-4) when M = P = I. If a matrix fulfills (A-1), it is called a generalized inverse A ; (A-1) and (A-2), a reflexive inverse A ; (A-1) - (A-3), a left weak inverse A ; (A-1), (A-2) and (A-4), a right weak inverse A ; and all four conditions, a pseudoinverse. Only a pseudoinverse is unique. These relationships are illustrated in Fig. 6.

Conditions (A-1) and (A-4) are equivalent to

$$A^TPAG = A^TP \tag{A-5}$$

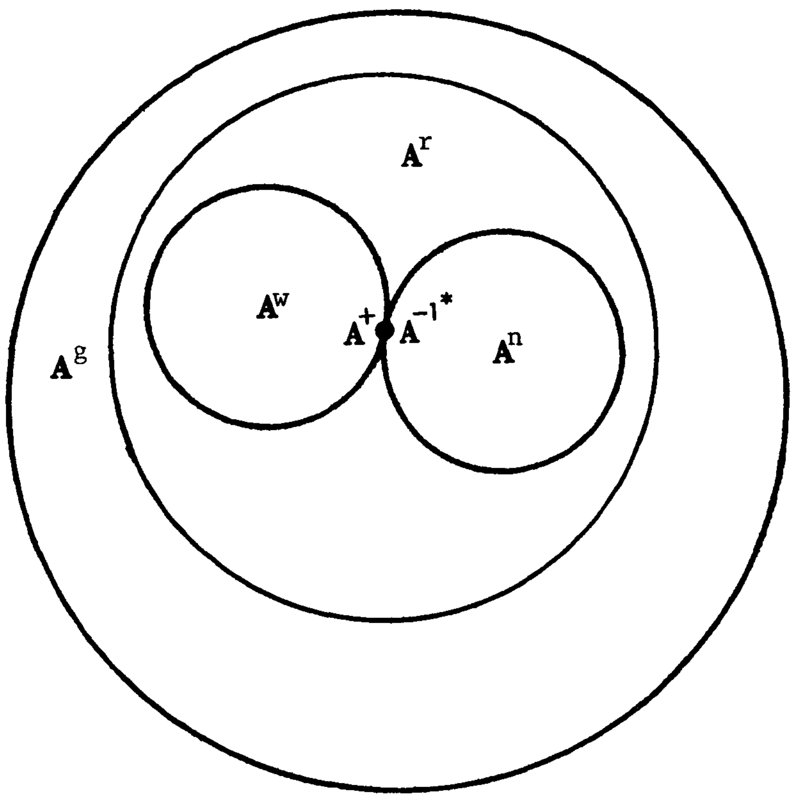
which can be proven as follows. Assume that (A-5) is fulfilled. Then,

$$\begin{aligned}(AG)^TP &= G^TA^TP \\ &= G^TA^TPAG = [G^TA^TPAG]^T \\ &= [G^TA^TP]^T = PAG\end{aligned}$$

which is just condition (A-4). Using this result

$$\begin{aligned}AGA &= P^{-1}PAGA \\ &= P^{-1}(AG)^TPA = P^{-1}G^TA^TPA \\ &= [A^TPAGP^{-1}]^T \\ &= [A^TPP^{-1}]^T \\ &= A\end{aligned}$$

which is condition (A-1).



* $\mathbf{A}$ full rank

Fig. 6 Classes of Generalized Inverses

Conversely, given (A-1) and (A-4)

$$
\begin{aligned}
A^T PAG &= A^T (AG)^T P \\
&= A^T G^T A^T P \\
&= (PAGA)^T \\
&= (PA)^T \\
&= A^T P
\end{aligned}
$$

which completes the proof.

Condition (A-2) and (A-3) are equivalent to

$$G^T MGA = G^T M \tag{A-6}$$

The proof is as follows. Given (A-6)

$$
\begin{aligned}
(GA)^T M &= A^T G^T M \\
&= A^T G^T MGA \\
&= (A^T G^T MGA)^T \\
&= (A^T G^T M)^T \\
&= MGA
\end{aligned}
$$

which is condition (A-3). Using this result

$$
\begin{aligned}
GAG &= M^{-1} MGAG \\
&= M^{-1} (GA)^T MG = M^{-1} A^T G^T MG \\
&= [G^T MGAM^{-1}]^T \\
&= [G^T MM^{-1}]^T \\
&= G
\end{aligned}
$$

which is condition (A-2). Given (A-2) and (A-3)

$$\begin{aligned} G^TMGA &= G^T(GA)^TM \\ &= [MGAG]^T \\ &= [MG]^T \\ &= G^TM \end{aligned}$$

which completes the proof.

A.2 A Proof for A^+_{PM} (M Positive Definite)

Here we prove that $G = M^{-1}N(NM^{-1}N)^gA^TP$ of section 3.3.1.1 and 3.3.1.2 is A^+_{PM}. For this we will need the results (Rao and Mitra, 1971, p. 22)

(a) One choice of $(A^T)^g$ is $(A^g)^T$

(b) $A(A^TPA)^gA^TPA = A$ and $(A^TPA)(A^TPA)^gA^T = A^T$ for any matrix P such that $R(A^TPA) = R(A)$ which automatically holds if P is positive definite.

First, we show that G fulfills condition (A-5),

$$\begin{aligned} A^TPAG &= NG \\ &= NM^{-1}N(NM^{-1}N)^gA^TP \\ &= NM^{-1}N(NM^{-1}N)^gNN^gA^TP \\ &= NN^gA^TP \\ &= A^TP \end{aligned}$$

For condition (A-6)

$$G^TMGA = PA[(NM^{-1}N)^g]^T NM^{-1}MM^{-1}N(NMN)^g N$$

$$= PA[NM^{-1}N]^g NM^{-1}N[NMN]^g N$$

$$= PA[NM^{-1}N]^g N$$

$$= PA[NM^{-1}N]^g NM^{-1}M$$

$$= G^TM$$

Therefore, it follows that G is a minimum M-norm P least squares g-inverse.

A.3 An Interesting Relationship

Here we show that $G = M^{-1}N(N^{-1}MN)^g$ is an N^+_{IM} for the system of consistent normal equations NX = U. In order to prove this we show that conditions (A-1) - (A-4) are fulfilled (in this case P does not appear in the conditions, being contained in $N = A^TPA$) using again (a) and (b) of A.2.

(1) $NGN = NM^{-1}N(NM^{-1}N)^g N = N$

(2) $GNG = M^{-1}N(NM^{-1}N)^g NM^{-1}N(NM^{-1}N)^g$

$$= M^{-1}N(NM^{-1}N)^g = G$$

$$(3) \quad (GN)^T M = (MGN)^T$$

$$= [MM^{-1}N(NM^{-1}N)^g N]^T$$

$$= [N(NM^{-1}N)^g N]^T$$

$$= N[NM^{-1}N)^g N$$

$$= MM^{-1}N(NM^{-1}N)^g N$$

$$= MGN$$

$$(4) \quad (NG)^T = [NM^{-1}N(NM^{-1}N)^g]^T$$

$$= NM^{-1}N(NM^{-1}N)^g$$

$$= NG$$

proving the assertion for G . This proves that

$$A^+_{PM} = N^+_{IM} A^T P \tag{A-7}$$

A.4 <u>A Proof for A^+_{PM} (M Positive Semidefinite)</u>

Here we prove that

$$G = (N+M)^{-1}N[N(N+M)^{-1}N]^g A^T P$$

of (3.3-119) fulfills the four conditions of section 3.3.1.4. It follows that $G = A^+_{PM}$ is, in this case, a minimum M-seminorm P-least squares solution for $AX = Y$. We assume that P is positive definite and use (a) and (b) of A.2. Then,

(1) $PAGA = PAN^{g}NGA$

$$= PAN^{g}N(N+M)^{-1}N[N(N+M)^{-1}N]^{g}N$$

$$= PAN^{g}NA$$

$$= PA$$

(2) $MGAG = M(N+M)^{-1}N[N(N+M)^{-1}N\ ]^{g}N(N+M)^{-1}N[N(N+M)^{-1}N]^{g}A^{T}P$

$$= M(N+M)^{-1}N[N(N+M)^{-1}N]^{g}A^{T}P$$

$$= MG$$

(3) $MGA = (N+M)GA - NGA$

$$= N[N(N+M)^{-1}N]^{g}N - N(N+M)^{-1}N[N(N+M)^{-1}N]^{g}N$$

$$= N[N(N+M)^{-1}N]^{g}N - N \quad \text{(symmetric)}$$

$$= [N(N+M)^{-1}N]^{g}N - N]^{T}$$

$$= (MGA)^{T} = (GA)^{T}M$$

(4) $PAG = PA(N+M)^{-1}N[N(N+M)^{-1}N]^{g}A^{T}P$

$$= PA(N+M)^{-1}N[N(N+M)^{-1}N]^{g}NN^{g}A^{T}P$$

$$= PAN^{g}NN^{g}A^{T}P$$

$$= PAN^{g}A^{T}P \quad \text{(symmetric)}$$

$$= [PAN^{g}A^{T}P]^{T}$$

$$= (PAG)^{T} = (AG)^{T}P$$

which completes the proof.

APPENDIX B

APPROXIMATE THEORY FOR OPTIMAL DESIGN

Consider the Gauss-Markoff model $(L, AX, \sigma_0^2 P^{-1})$ for the observation equations

$$L = AX + V \tag{B-1}$$

where the rank of A is full. Each row of A is a u-dimensional vector in the design space, a subspace of E^u. The normal matrix N is known as the Fisher information matrix (Federov, 1972). Actually the definition of Fisher information is much more general (see for example Silvey, 1980) but simplifies to N for the linear estimation problem given above. The design problem is choosing n-vectors $A_1, \ldots, A_n$ such that the covariance matrix (without loss of generality we assume $P = I$)

$$\Sigma_X = [\sum_{i=1}^{n} A_i^T A_i]^{-1} \tag{B-2}$$

is minimized in some sense. Alternatively, we wish to minimize the information matrix N . Several design measures have been presented in section 4.3.4. For the linear model, the design matrix is independent of X . However, for the linearized model, it is dependent on the approximate values of X. In the case of deformation analysis, this value is given by X_0, the fundamental coordinates. Since the expected deformations are small, several cm/year over baselines of intercontinental extent, the theory developed for linear models is applicable here, too.

Optimal design criteria, in general, consist of different functions of the information matrix

$$\phi(N(d)) \tag{B-3}$$

where d denotes the A_i vectors that make up a particular design. An optimal design, denoted by d^*, will maximize ϕ. Since d consists of a discrete number of vectors A_i, it is not practical to apply standard optimization techniques to maximize ϕ. A way to generalize this problem in terms of continuous functions has been developed by (Kiefer, 1974) which he has termed approximate theory. We outline this approach as presented by (Silvey, 1980).

Consider a design space $\mathcal{A}$, a compact subset of E^u. Only certain collections of $A_i \in \mathcal{A}$ can be considered as valid designs (for example, the number of vectors must exceed the number of parameters). The collections or events can have probabilities assigned to them. The class of possible events forms a field, particularly a Borel field since any combination of these events is also an event and belongs to the field. Let H denote the class of probability distributions over the Borel field of possible events. That is, each possible design has a probability associated with it. Then $\eta \in H$ can be thought of as a design measure. For every η define its information matrix

$$N(\eta) = E\{\tilde{A}_i^T \tilde{A}_i\} \tag{B-4}$$

where $\tilde{A}_i \in \mathcal{A}$ is a random vector with distribution η. The set of all possible information matrices is given by

$$\mathcal{N} = \{N(\eta) : \eta \in H\} \tag{B-5}$$

Define ϕ as a real-valued function over the set of $u \times u$ symmetric non-negative definite matrices. The design problem can then be stated as determining η^* that maximizes $\phi\{N(\eta)\}$ over H. Such a design is called ϕ-optimal. The properties of ϕ, are outlined in (Federov, 1972; Silvey, 1980). The above definitions, then, allow us to consider continuous optimal designs.

Two directional derivatives are defined. The Gateaux derivative of ϕ at N_1, in the direction of N_2 is

$$G_\phi(N_1,N_2) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\{\phi(N_1 + N_2) - \phi(N_1)\} \tag{B-6}$$

If ϕ is differentiable at N_1,

$$G_\phi(N_1,\Sigma a_i N_i) = \Sigma a_i G_\phi(N_1,N_i) \tag{B-7}$$

a property that simplfies the design problem considerably. The Frechet derivative of ϕ at N_1, in the direction of N_2 is

$$F_\phi(N_1,N_2) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\{\phi[(1-\varepsilon)N_1 + \varepsilon N_2] - \phi(N_1)\} \tag{B-8}$$

By definition,

$$F_\phi(N_1,N_2) = G_\phi(N_1,N_2-N_1) \tag{B-9}$$

Two theorems are important in deciding whether a particular design is optimum.

<u>Theorem 1</u>. When ϕ is concave on $\mathcal{N}$, η^* is ϕ-optimal if and only if

$$F_\phi\{N(\eta^*)\ ,\ N(\eta)\} \leq 0 \qquad \text{for all } \eta \in H$$

<u>Theorem 2</u>. If ϕ is concave on $\mathcal{N}$ and differentiable at $N(\eta^*)$ then η^* is ϕ-optimal if and only if

$$F_\phi\{N(\eta^*),\ A_iA_i^T\} \leq 0 \qquad \text{for all } A_i \in \mathcal{A}$$

For similar theorems treating the more general case of a rank deficient A matrix, see (Silvey, 1980). Theorem 2 will be useful in the study of optimal polyhedra on a sphere (Appendix D). In order to apply this theorem to this problem it will be necessary to compute the Frechet derivative for each ϕ-optimality criterion. We do this for D- and A-optimality.

For D-optimality we define

$$\phi = \log[\det N] \tag{B-10}$$

to ensure concavity. Then

$$\begin{aligned} J &= \log \det\ (N_1 + \varepsilon N_2) - \log \det N_1 \\ &= \log\ [\det(N_1 + \varepsilon N_2)N_1^{-1}] \\ &= \log\ [\det(I + \varepsilon N_2 N_1^{-1})] \end{aligned}$$

Expanding in a McLauren series

$$J = \log \{\det I + \frac{d}{d\varepsilon} [\det (I + \varepsilon N_2 N_1^{-1})]\}_{\varepsilon=0} + \dot{o}(\varepsilon^2)$$

but

$$\frac{d}{dX} (\det A) = \det A \ \mathrm{tr}\ (A^{-1} \frac{dA}{dX})$$

(Bodewig, 1959). Therefore,

$$J = \log\{\det I + \varepsilon [\det(I + \varepsilon N_2 N_1^{-1})$$

$$\cdot \mathrm{tr}((I + \varepsilon N_2 N_1^{-1})^{-1} \frac{d}{d\varepsilon} (I + \varepsilon N_2 N_1^{-1}))]\}_{\varepsilon=0}$$

$$= \log\{1 + \varepsilon[\det(I + \varepsilon N_2 N_1^{-1}) \ \mathrm{tr}((I + \varepsilon N_2 N_1^{-1})^{-1} N_2 N_1^{-1})]\}_{\varepsilon=0}$$

$$= \log\{1 + \varepsilon[\det I \ \mathrm{tr}(N_2 N_1^{-1})]\} + o(\varepsilon^2)$$

$$= \log\{1 + \varepsilon \ \mathrm{tr}(N_2 N_1^{-1})\} + o(\varepsilon^2)$$

Since

$$\log(1 + X) = X - \frac{1}{2} X^2 + \ldots$$

$$J = \varepsilon \ \mathrm{tr}(N_2 N_1^{-1}) + o(\varepsilon^2)$$

and we can compute (B-6) for D-optimality

$$G (N_1, N_2) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \varepsilon \mathrm{tr}(N_2 N_1^{-1})$$

$$= \mathrm{tr}(N_2 N_1^{-1}) \qquad \text{(B-11)}$$

From (B-9)

$$F_\phi(N_1,N_2) = tr(N_2-N_1)N_1^{-1}$$
$$= tr[N_2N_1^{-1} - {}_uI_u] \qquad \text{(B-12)}$$
$$= tr(N_2N_1^{-1}) - u$$

and

$$F_\phi(N_1,A_i^TA_i) = tr(A_i^TA_iN_1^{-1}) - u$$
$$= tr(A_iN_1^{-1}A_i^T) - u \qquad \text{(B-13)}$$
$$= A_iN_1^{-1}A_i^T - u$$

where A_i is a row of the design matrix A. Then according to the second theorem above, η^* is D-optimal if and only if

$$A_iN^{-1}(\eta^*)A_i^T \leq u \qquad \text{for all } A_i \in \mathcal{A} \qquad \text{(B-14)}$$

For A-optimality we consider the concave function

$$\phi(N) = -tr(N^{-1})$$

Then,

$$K = \phi(N_1 + \varepsilon N_2) - \phi(N_1)$$
$$= -tr[(N_1 + \varepsilon N_2)^{-1}] + trN_1^{-1}$$
$$= trN_1^{-1} - \varepsilon\frac{\partial}{\partial\varepsilon}[tr(N_1 + \varepsilon N_2)^{-1}]_{\varepsilon=0} + trN_1^{-1} + o(\varepsilon^2)$$

Now,

$$\frac{\partial}{\partial\varepsilon} trA = tr\frac{\partial}{\partial\varepsilon} A$$

and

$$\frac{\partial}{\partial \varepsilon} A^{-1} = -A^{-1} \frac{\partial A}{\partial \varepsilon} A^{-1}$$

(Federov, 1972). Then

$$K = \varepsilon \, tr[(N_1 + \varepsilon N_2)^{-1} N_2 (N_1 + \varepsilon N_2)^{-1}]_{\varepsilon=0} + o(\varepsilon^2)$$

$$= tr[N_1^{-1} N_2 N_1^{-1}] + o(\varepsilon^2)$$

from which (B-6) for A-optimality is

$$\begin{aligned} G_\phi(N_1,N_2) &= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \varepsilon \, tr(N_1^{-1} N_2 N_1^{-1}) \\ &= tr(N_1^{-1} N_2 N_1^{-1}) \end{aligned} \qquad \text{(B-15)}$$

and from (B-9)

$$\begin{aligned} F_\phi(N_1, N_2) &= tr[N_1^{-1}(N_2 - N_1)N_1^{-1}] \\ &= tr[N_1^{-1} N_2 N_1^{-1} - N_1^{-1}] \\ &= G_\phi(N_1,N_2) - trN_1^{-1} \end{aligned} \qquad \text{(B-16)}$$

It follows,

$$\begin{aligned} F_\phi(N_1, A_i^T A) &= tr(N_1^{-1} A_i^T A_i N_1^{-1}) - tr(N_1^{-1}) \\ &= tr(A_i^T A_i N_1^{-1} N_1^{-1}) - tr(N_1^{-1}) \\ &= tr(A_i N_1^{-1} N_1^{-1} A_i) - tr(N_1^{-1}) \\ &= A_i N_1^{-1} N_1^{-1} A_i^T - trN_1^{-1} \end{aligned} \qquad \text{(B-17)}$$

Therefore, a necessary and sufficient condition that a design η^* is ϕ-optimal is that

$$A_i N^{-1}(\eta^*) N^{-1}(\eta^*) A_i^T \leq trN^{-1}(\eta^*) \qquad \text{for all } A_i \in \mathcal{A} \qquad \text{(B-18)}$$

APPENDIX C

POLYHEDRA

C.1 Some Polyhedra Definitions and Classifications

A polyhedra is a finite set of polygons arranged in space in such a way that every side of each polygon belongs to just one further polygon, with the restriction that no subset has the same property. The polygons are called faces, the sides edges, and the juncture of several edges, vertices. In order to define the regularity of a polyhedron, the concept of vertex figure is needed. A vertex figure of a polygon is the segment joining the midpoints of any two adjacent edges. The vertex figure of a polyhedron is the polygon whose edges are the vertex figures of all the faces that surround a vertex, i.e. the polygon formed by joining the midpoints of the edges which meet at a common vertex. Generally, this is a skew polygon. A regular polyhedron is one whose faces and vertex figures are all regular. Such polyhedra have faces of all of one kind of congruent regular polygons. (See (Coxeter, 1963; Fejes Toth, 1964) for more details.)

The following scheme is used to classify regular polyhedra

$$\{p,q\} \tag{C-1}$$

where p is the number of edges on each face and q, the number of vertex figures (also, the number of edges meeting at each vertex)(Coxeter, 1963).

The only possible (p,q)'s are for the five Platonic solids

(3,3) - tetrahedron

(3,4) - octahedron

(4,3) - cube

(3,5) - icosahedron

(5,3) - dodecahedron

See Fig. 7. All (p,q)'s can be inscribed in a sphere. They are also reciprocal (dual). This means that if we join the centers of adjacent edges by segments, we obtain the reciprocal polyhedron (q,p). This fact was used to construct near-optimal polyhedra (called dual polyhedra) in (Mueller, et al, 1982).

A semi-regular polyhedron has regular polygons as faces, but the faces are not all of the same kind. There are 13 such polyhedra called after Archimedes. These all denoted symbolically by the number of the edges about one vertex. For example, for a polyhedron of 24 vertices we have (using the classification of Fejes Toth (1964))

(3,3,3,3,4) - snub cube (snub cuboctahedron)

i.e., about each vertex we have 4 triangles and one square (Fig. 7) and

(3,8,8) - truncated cube

i.e., around each vertex there is 1 triangle and 2 octagons. For 8 vertices, the antiprism (Fig. 7) is represented by

(3,3,3,4)

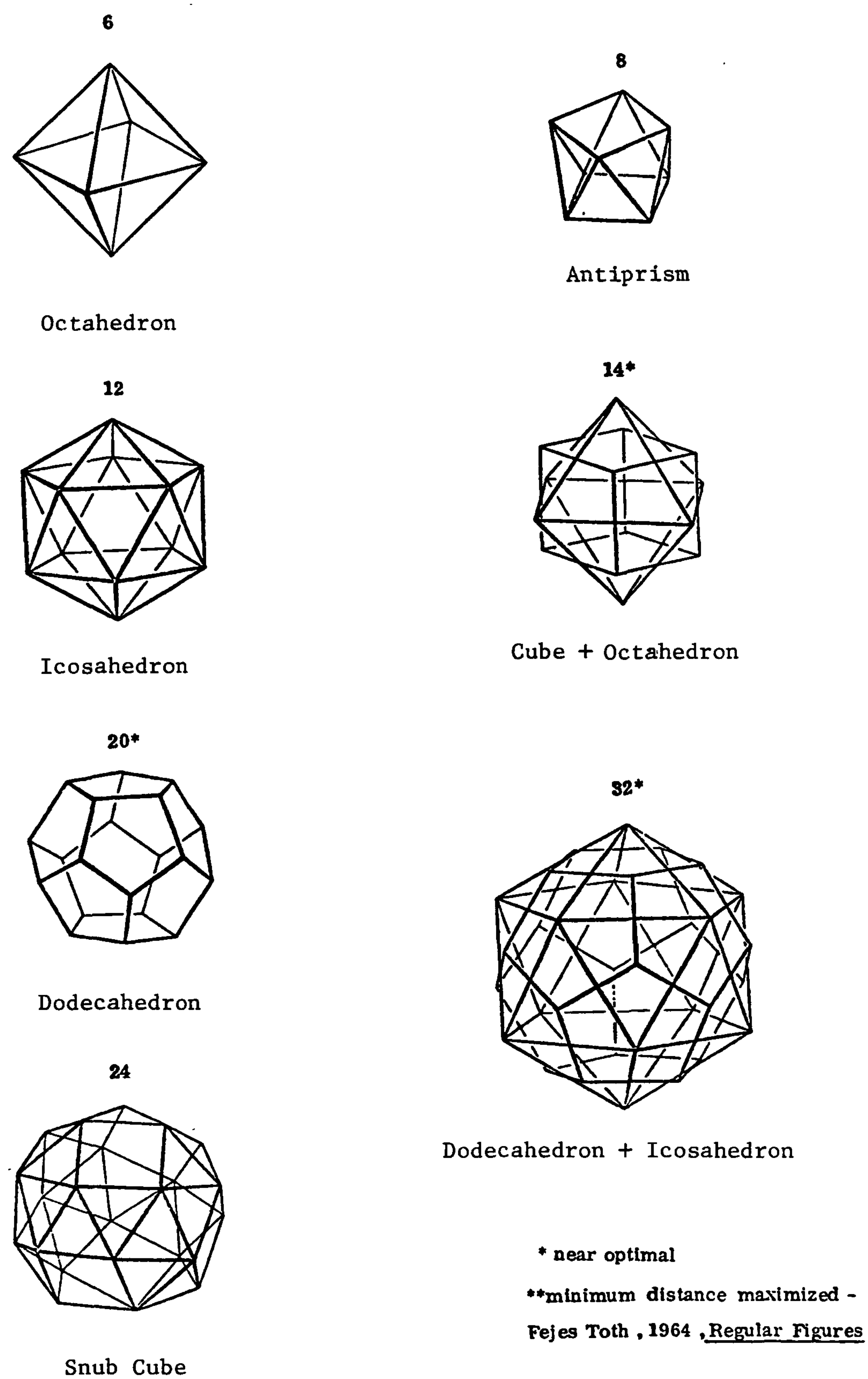


Fig. 7 Examples of Optimal** Polyhedra

i.e., about each vertex we have three triangles and one square. All three of these semiregular polyhedra can be inscribed in a sphere.

C.2 The Golden Proportion (Section) and Polyhedra Coordinates

The pentagram is a five-pointed star constructed by extending the edges of a regular pentagon. The 10 outside edges of a pentagram is τ times the length of the edges of the original pentagon, where

$$\tau = \frac{\sqrt{5} + 1}{2} \tag{C-2}$$

is called the golden proportion. It is a root of

$$X^2 - X - 1 = 0 \tag{C-3}$$

Many interesting relationships are attributed to τ (Pugh, 1976). Some of these relate to polyhedra. Dividing each edge of an octahedron by $\tau : 1$ yields an icosahedron. If a diagonal is drawn across each pentagonal face of a dodecahedron, a cube is formed. The edges of the cube will be τ times the edges of the dodecahedron.

The Cartesian coordinates for the regular and semi-regular polyhedron described in the previous section are often expressed in terms of τ. For the tetrahedron (p = 4), a set of coordinates is

$$(1,1,1) \; , \; (1,-1,1) \; , \; (-1,1,-1) \; , \; (-1,-1,1) \tag{C-4}$$

those of the octahedron (p = 6)

$$(\pm 1,0,0) \; , \; (0,\pm 1,0) \; , \; (0,0,\pm 1) \tag{C-5}$$

and those of the cube (p = 8)

$$(\pm 1, \pm 1, \pm 1) \tag{C-6}$$

or

$$(\pm a, 0, \pm h) \ , \ (0, \pm a, \pm h)$$
$$a = \sqrt{\frac{2}{3}} \ ; \ h = \frac{1}{\sqrt{3}} \tag{C-7}$$

For the icosahedron (p = 12)

$$(0, \pm\tau, \pm 1) \ , \ (\pm 1, 0, \pm\tau) \ , \ (\pm\tau, \pm 1, 0) \tag{C-8}$$

and the dodecahedron (p = 20)

$$(0, \pm\tau^{-1}, \pm\tau) \ , \ (\pm\tau, 0, \pm\tau^{-1}) \ , \ (\pm\tau^{-1}, \pm\tau, 0) \ , \ (\pm 1, \pm 1, \pm 1) \tag{C-9}$$

(Coxeter, 1963). For the semi-regular polyhedra, the antiprism (p = 8) inscribed in the unit sphere has coordinates

$$(\pm a, 0, h) \ , \ (0, \pm a, h) \ , \ (\pm\frac{a}{\sqrt{2}}, \pm\frac{a}{\sqrt{2}}, -h) \tag{C-10}$$

where

$$a = \frac{2}{\sqrt{4+\sqrt{2}}} \ ; \ h = \frac{1}{\sqrt{1+2\sqrt{2}}} \tag{C-11}$$

The snub cube has coordinates (W. McWorter, 1982, private communication)

$$(\pm\alpha, \pm 1, \pm\alpha^2) \ , \ (\pm 1, \pm\alpha, \pm\alpha^2) \ , \ (\pm\alpha^2, \pm\alpha, \pm 1) \tag{C-12}$$

where $=.5436890127\ldots$ is a root of

$$X^3 + X^2 + X - 1 = 0 \tag{C-13}$$

and those of the truncated cube

$$(\pm a, \pm b, \pm b) \ , \ (\pm b, \pm a, \pm b) \ , \ (\pm b, \pm b, \pm a) \tag{C-14}$$

where

$$a = \frac{1}{\sqrt{7+4\sqrt{2}}} \ ; \ b = \frac{1}{\sqrt{5-2\sqrt{2}}} \tag{C-15}$$

APPENDIX D

POLYHEDRA DESIGN

"On a planet, say, ten inimical dictators govern. How must the residences of these gentlemen be placed in order to be as far as possible from one another?" (Fejes Toth, 1964).

Consider the following variation of the reference frame problem. A total of p polyhedron stations can be distributed over the earth's surface. Assume a spherical earth and that the stations can be located anywhere on the surface. We define the optimal reference frame as the one that best defines the center of the sphere, i.e. the origin. Assume that the "observations" are the radii R to each polyhedron station. Our mathematical model is then

$$R_i = [X_i - X_0)^2 + (Y_i - Y_0)^2 + (Z_i - Z_0)^2]^{1/2} \qquad \text{(D-1)}$$

The parameters are the center of the sphere (X_0,Y_0,Z_0). The optimal design is then the choice of p stations, i.e. $(X_i,Y_i,Z_i, i=1, p)$. This is analogous to the problem of how to distribute the stations that will best monitor the change in the size and shape (deformation) of the polyhedron. In both problems the orientation of the reference frame axes is arbitrary.

Linearizing the above model yields the elements of one row of A of the design matrix

$$A_i = \left[\frac{X_i - X_0}{R} \quad \frac{Y_i - Y_0}{R} \quad \frac{Z_i - Z_0}{R}\right] \tag{D-2}$$

Assuming a unit sphere and that the origin is at (0,0,0)

$$A_i = [X_i \quad Y_i \quad Z_i] \tag{D-3}$$

The information matrix N for this problem for a particular design, η, and assigning equal probabilities λ_i to each point, is

$$N(\eta) = \sum_{i=1}^{p} \lambda_i A_i A_i^T = \frac{1}{p} \begin{bmatrix} \Sigma X_i^2 & \Sigma X_i Y_i & \Sigma X_i Z_i \\ & \Sigma Y_i^2 & \Sigma Y_i X_i \\ \text{Sym} & & \Sigma Z_i^2 \end{bmatrix} \tag{D-4}$$

As an example, consider the case of p = 8 and check the cube and anti-prism configurations for D-optimality and A-optimality. One set of coordinates for the cube or the unit sphere is given by the 8 combinations (C-6)

$$\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$$

We take this as a possible design η_0 whereby

$$N(\eta_0) = \frac{1}{24} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \frac{1}{3} I$$

Now, for η_0 to be D-optimal (B-14)

$$A_i N^{-1}(\eta_0) A_i^T \leq 3$$

for all $A_i \in \mathcal{A}$. But A_i is restricted to points on the unit sphere. So

$$A_i N^{-1} A = [X_i \;\; Y_i \;\; Z_i] 3I \begin{bmatrix} X_i^T \\ Y_i^T \\ Z_i^T \end{bmatrix}$$

$$= 3(X_i^2 + Y_i^2 + Z_i^2) \leq 3$$

indicating that the cube distribution is D-optimal.

For A-optimality (B-18)

$$A_i N^{-1}(\eta_0) N^{-1}(\eta_0) A_i \leq \text{tr } (N^{-1}(\eta)) \qquad \text{for all } A_i \in \mathcal{A}$$

We have

$$[X_i \;\; Y_i \;\; Z_i] 9I \begin{bmatrix} X_i^T \\ Y_i^T \\ Z_i^T \end{bmatrix} = 9(X_i^2 + Y_i^2 + Z_i^2) \leq 9$$

so that the cube distribution is also A-optimal.

The antiprism can be shown to maximize the shortest distance between any two vertices of the polyhedron (Fejes Toth, 1964). We test whether this criterion passes the D- or A-optimality test. One possible set of coordinates of the antiprism is given above by (C-10). Then

$$N(\eta_0) = \begin{bmatrix} \frac{a^2}{2} & 0 & 0 \\ 0 & \frac{a^2}{2} & 0 \\ 0 & 0 & h^2 \end{bmatrix}$$

$$N^{-1}(\eta_0) = \begin{bmatrix} \frac{2}{a^2} & 0 & 0 \\ 0 & \frac{2}{a^2} & 0 \\ 0 & 0 & \frac{1}{h^2} \end{bmatrix}$$

$$A_i^T N^{-1}(\eta_0) A_i = \frac{2}{a^2} X_i^2 + \frac{2}{a^2} Y_i^2 + \frac{1}{h^2} Z_i^2$$

Is this less than 3 for any point on the unit sphere? Try the point (0,0,1). In this case

$$\frac{1}{h^2} = 3.82 > 3$$

which shows that the antiprism is not D-optimal. A similar calculation shows that it is not A-optimal, either. These two results are verified in the experiments of (Mueller, et al, 1982). However, the antiprism gave a lower E- and C-measure (Section 4.3.4) than the cube.

Consider the dodecahedron distribution for p = 20. The information matrix is given by

$$N(\eta_0) = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2} & 0 \\ 0 & 0 & \tau^2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} \tau^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\tau^2} \end{bmatrix}$$

$$+\frac{1}{5}\begin{bmatrix} \frac{1}{\tau^2} & 0 & 0 \\ 0 & \tau^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} \tau^2+\frac{1}{\tau^2}+2 & 0 & 0 \\ 0 & \tau^2+\frac{1}{\tau^2}+2 & 0 \\ 0 & 0 & \tau^2+\frac{1}{\tau^2}+2 \end{bmatrix}$$

$$= \frac{1}{5}\,(\tau^2 + \frac{1}{\tau^2} + 2)I$$

for $R = \sqrt{3}$ and therefore,

$$N(\eta_0) = \frac{1}{15}\,(\tau^2 + \frac{1}{\tau^2} + 2)I$$

for the unit sphere. But

$$\tau^2 + \frac{1}{\tau^2} = 3$$

so

$$N(\eta_0) = \frac{1}{3}I$$

and

$$N^{-1}(\eta_0) = 3\,I$$

which proves that this distribution is D-optimal. In fact, it has the same information matrix as for the cube. Therefore, A-optimality holds, too. It can be shown that all regular polyhedra (i.e. the Platonic solids) have the same information matrix and therefore all are A- and D-optimal. This also holds for the dual combinations

$p = 14$: cube and octahedron

$p = 32$: icosahedron and dodecahedron

(see Fig. 7)

and for

$p = 18$: octahedron and icosahedron

Thus, regularity (or semi-regularity) is the criteria for A- and D-optimality for the origin definition problem.

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