THE COLLAPSE OF THE LOCAL, SPITZER-HÄRM FORMULATION AND A GLOBAL-LOCAL GENERALIZATION FOR HEAT FLOW IN AN INHOMOGENEOUS, FULLY IONIZED PLASMA

Jack D. Scudder
NASA/Goddard Space Flight Center
Laboratory for Extraterrestrial Physics
Greenbelt, MD 20771

Stanislaw Olbert
Department of Physics and
Center for Space Research
Massachusetts Institute of Technology
Cambridge, MA 02139

ABSTRACT

This paper addresses the breakdown of the classical (CBES) field aligned transport relations for electrons in an inhomogeneous, fully ionized plasma as a mathematical issue of radius of convergence, defines for the first time the finite Knudsen number conditions when CBES results are accurate, and presents a global-local (GL) way to describe the results of Coulomb physics moderated conduction that is more nearly appropriate for astrophysical plasmas. This paper shows the relationship to and points of departure of the present work from the CBES approach. It also presents an analytical, more tractable, and economical way of evaluating the global aspects of the collisional physics than has been done numerically in the previous paper. The CBES heat law in current use is then shown to be an especially restrictive special case of the new, more general GL result. A preliminary evaluation of the dimensionless heat function discussed in the previous paper (Olbert, (1983)), but using analytic formulas, as presented below, shows that the dimensionless heat function profiles versus density of the type necessary for a conduction supported high speed solar wind appear possible.

Introduction

It has long been recognized that the Chapman (1916)-Chapman and Cowling (1939)-Braginskii (1965)-Enskog (1917)-Spitzer-Härm (1953)-Spitzer (1962) (CBES) transport results become unreliable when the Knudsen number, K, (a kinetic free path over macroscopic scale length) is not small. A serious outstanding issue in this regard is how small must K be in order that these transport formulas are accurate. This difficulty plagues the theoretical investigation of conduction supported, fully ionized stellar winds. The usual interpretation of this breakdown is that relevant physics has been omitted. Wave particle effects are often suggested to be important in these locales. However, this conclusion is not inescapable. Subtle prejudices have, until now, led to the insistence that the physics of these regions must remain local, even if wave particle effects must be invoked to do this. These assumptions are largely shaped by our everyday experience and are fortified by attendant mathematical convenience. The physics of Coulomb moderated transport is not amenable to this ad hoc approach as we have previously shown (Scudder and Olbert 1979a,b, SO I, II, Olbert (1983)).

Since electrons carry almost all the heat flux in a plasma the kinetic discussion that follows is oriented toward electron transport signatures. Because the bulk flows of astrophysics are usually smaller than characteristic electron thermal speeds certain dimensionless ratios are tailored for this electron situation. Unless specifically stated otherwise we will refer to electron kinetic properties for the remainder with the major exception of the plasma conservation equations where certain results pertain to the fluid as a whole. Because astrophysical plasmas are strongly "magnetized" we confine our attention to field aligned transport signatures.

1. The CBES Postulates

The CBES transport description for electrons proceeds under the interlocking postulates

1) that the microscopic electron velocity distribution in the proper frame, $f_e(\underline{w})$, that solves the kinetic equation is separable into a sum of two terms:

$$f_e \overset{\text{CBES}}{=} f_e^o + \delta f_e(K) \qquad (1)$$

such that all finite free path effects are solely contained in the Knudsen number, K, functional dependence of δf;

2) that the velocity dependence of $f_e^{(o)}$ is only through proper frame speed $w \equiv |\underline{w}|$;

3) that f_e^o is a local Maxwellian distribution:

$$f^o_{e(\underline{x})} \overset{\text{CBES}}{=} n_e(\underline{x}) \left(\frac{m_e}{\pi\, 2kT_e(\underline{x})} \right)^{3/2} \exp\left(- \frac{m_e w^2}{2kT_e(\underline{x})} \right) \qquad (2)$$

(where k is Boltzman's constant, n, T, and m are the local density, temperature and mass, respectively);

4) that δf_e is a perturbation of $f_e^{(o)}$ such that

$$\frac{\delta f_e(\underline{w})}{f_e^{o}(\underline{w})} = O(K), \qquad (3)$$

for a sufficiently broad range of speeds to control the transport signature sought; and

5) that the perturbation ordering parameter is the Knudsen number, K, which is assumed infinitesimal and defined as the ratio of a kinetic scale, λ, to a macroscopic scale, L_M. The macroscopic length L_M, defined by

$$L_M^{-1} = \left| \frac{d}{ds} \ \ell n \ M \right|, \qquad (4)$$

is the shortest scale length along the arc length s of the magnetic field among the macroscopic variables, M: density, n, temperature, T, magnetic field strength, B, or bulk speed, U. In the astrophysical context L_M is usually set by the density profile; we assume this in the remainder. The kinetic scale, λ, is strongly speed dependent in a plasma. It is usually chosen as that spatial length within which substantial isotropization and approach to a Maxwellian takes place. Those two physical time/length scales are commensurate (cf. Spitzer, 1962, p. 133) for particles with speeds $w_{T,e}$ equal that

of the root mean square speed $\sqrt{\frac{3kT_e}{m_e}}$. Spitzer (1962) defines the related

characteristic time, t_c, (eqtn 5-26) for this process for electrons to yield a Spitzer kinetic scale free path of the root mean square (rms) electron,

$$\lambda_{rms} \equiv w_{T,e}\, t_c\, (w_{T,e}) = \frac{(3kT_e)^2}{8(0.714)\, \pi n e^4 \, \ln \Lambda}, \tag{5}$$

where $\ln \Lambda$ is the Coulomb logarithm. Braginskii (1965) defines "the electron... collision time" with a formula (cf. 2.5e) that differs from Spitzer's by 3%. According to this practice "the Knudsen number" defined by

$$K \equiv \frac{\lambda_{rms}(w_{T,e})}{L_n} \tag{6}$$

appears as the dimensionless parameter for the plasma style Chapman-Enskog expansion.

The CBES approach attempts a perturbation expansion for δf_e of the form

$$\delta f_e^{CBES} = \mathcal{F}(L_n/L_{Te},\ L_n/L_U,\ L_n/L_B,\ n,\ T_e,\ U,\ \underline{w})\ K + 0\ (K^2), \tag{7}$$

and the approximation is usually truncated with the first term in (7). Note that the structure of $\mathcal{F}$ depends only on the velocity space $\underline{w}$, on local macroscopic variables and ratios L_n/L_M of the scale length of the density to those of other macroscopic variables evaluated at the point $\underline{x}$. Consequently $\mathcal{F}$ is independent of K.

The CBES truncation (7) of δf_e translates immediately into a similar truncation of a Maclaurin series for $Q_\parallel$ since

$$Q_\parallel \equiv \iiint (f_e^{(o)} + \delta f_e)\ 1/2\ m_e\ w^5 \cos\theta \sin\theta\, d\theta\, d\phi\, dw. \tag{8a}$$

By postulate (2) $f_e^{(o)}$ is even in velocity space so that (8a) reduces to

$$Q_\parallel = 0 + \iiint (\delta f_e)\ 1/2\ m_e\ w^5 \cos\theta \sin\theta\, d\theta\, d\phi\, dw. \tag{8b}$$

Using equation (7) we obtain

$$Q_\parallel \overset{CBES}{\approx} K \iiint \mathcal{F}\ 1/2\ m_e\ w^5 \cos\theta \sin\theta\, d\theta\, d\phi\, dw. \tag{8c}$$

The remaining CBES postulates allow a reduction of (8c) to functional dependence on lower order moments; this permits a heat law for the fully ionized plasma, parallel to Fourier's law, that permits closure, viz:

$$Q_{\parallel}^{CBES} \simeq 0 + \mathcal{X}_o \, T_e^{5/2} \frac{dT_e}{ds} \qquad (9a)$$

which is easily rewritten as the infinitesimal heat flux

$$Q_{\parallel}^{CBES} \simeq \left[\frac{\mathcal{X}_o T_e^{7/2}}{\lambda_{rms}} \frac{L_n}{L_{Te}}\right] K, \qquad (9b)$$

where for a proton-electron plasma (Spitzer (1962))

$$\mathcal{X}_o = (0.419)\ (0.225)\ 20\ \left(\frac{2}{\pi}\right)^{3/2} \frac{k^{7/2}}{m_e^{1/2}\ e^4 \ln \Lambda}. \qquad (10)$$

Notice that the flow of heat in the CBES approach (8c) arises out of the non-zero perturbative correction to adiabatic ($Q_{\parallel} = 0$) behavior.

If the assumptions made to derive (9) are violated this marks the failure of a mathematical artifice: a simple, straightforward Maclaurin series perturbation scheme. It is, for example, well known that the Chapman-Enskog scheme does not provide access (even in principle) to the broadest class of solutions to the Boltzmann equation (Grad (1958)). Coulomb moderated transport effects are unavoidably present, but inaccessible within the CBES mathematical machinery. Extensions of the CBES approximation after the form of Burnett and others are analogous to attempts to enhance the radius of convergence of the Maclaurin series for the velocity distribution function, $f(\underline{w})$, and, therefore $Q_{\parallel}$ beyond those in equation (9) by retaining additional terms in the power series. These procedures are cumbersome. This is made clear by looking at a mathematical model of the perturbation issues at hand.

2. Model Perturbation Issues and their Resolution

Consider a system whose physical behavior is exactly described by the following function

$$g(K) = \tanh(K). \qquad K>0 \qquad (11)$$

[In the heat transport problem the exact $g(K) \propto Q_{\parallel}(K)$ is an unknown solution of an integro-partial differential equation]. If a description of the physical system were desired for the $K<<1$ regime a one term Maclaurin series expansion for tanh(K) would yield nearly indistinguishable results for g(K), viz

$$g(K) \sim g^{(1)}(K) \equiv K \qquad K<<1 \qquad (12)$$

(In our model problem $g^{(1)}(K)$ is proportional to $Q_{\parallel}^{CBES}$.) If equation (11) (the solution of the integro-differential equation) is the complete and accurate description of the physical system even for large K, the fact that $g^{(1)}(K)$ is inaccurate as K gets large implies nothing about the completeness of the physics embodied in equation (11).

To parallel the multi-term expansion of $f(\underline{w})$, and, therefore $Q_{\parallel}$, in the higher moment methods, consider additional truncated Taylor expansions $g^{(n)}(K)$ for $g(K)$

$$
\begin{aligned}
g^{(1)}(K) &= K, \\
g^{(2)}(K) &= K - K^3/3, \\
g^{(3)}(K) &= K - K^3/3 + 2K^5/15, \\
g^{(4)}(K) &= K - K^3/3 + 2K^5/15 - 17K^7/315.
\end{aligned} \qquad (13)
$$

Any finite truncation $g^{(n)}$ of g chosen in (13) is a polynomial in K which must diverge for large K in contradiction to the boundedness of $g(K)$ as $K \to \infty$. In any case the Maclaurin series for $g(K)$ of course diverges for $K > \pi/2$. This result is known because the full Maclaurin series of $g(K)$ is in hand. Clearly $g^{(\infty)}$ is a desperate way to approximate g - even for $K < \pi/2$. In constructing perturbation solutions that approximate $Q_{\parallel}$ the full expansion analogous to (13) is not available. The finite radius of applicability of the CBES expression for $Q_{\parallel}$ needs to be defined as we do below for the first time. It is quite common in non-linear, physical problems that boundedness of certain quantities such as total energy or the magnitude of the heat flow regardless of the size of K available impacts the forms of perturbation that are allowed and drive the physicist to seek more tractable approximations such as $g^{(p)}$ provides for g regardless of K described below. As such, this feature of this model is realistic.

The inapplicability of $Q_{\parallel}^{CBES}$ in the finite Knudsen number regime is essentially a failure of the above type: namely of a straightforward, local, one term, Maclaurin series expansion in K. (Recall that elementary discussions of heat flow e.g., Reif (1965) give intuitive derivations based on the energy flux imbalance across a mean free path element of the medium centered on the observer.)

Different kinds of perturbation expansions (c.f., e.g., Baker (1975)) are possible that attempt to bridge asymptotic regimes in a way that truncated Taylor series cannot. An example of this for the model problem is to use a Padé approximant (which is by no means unique!) for $g(K)$ given by

$$g^{P}(K) = \frac{K + cK^2 + dK^3}{1 + aK + bK^2 + dK^3}, \quad K > 0, \qquad (14)$$

with a, b, c, and d, chosen to optimize the approximation for intermediate K such as by matching the Maclaurin series of (14) to that of g as far as the free constants allow. Notice that for small K, i.e., $K << 1/a$

$$g^{P}(K) \sim K \qquad K << 1$$

but that for $K >> (1/d)^{1/2}$

$$\lim_{K \to \infty} g^{P}(K) = \lim_{K \to \infty} g(K) = 1$$

Also note that $g^{P}(K)$ is not a polynomial in K. In a practical sense (14) represents a physically viable, if approximate, re-summed form of the infinite

Maclaurin series for g(K) which is well behaved in either the small or large K regime as well as for $K > \pi/2$, the radius of convergence of the Maclaurin series for g. Obviously, the exact solution (11) is preferable to (14), if available. In most realistic physical problems the exact solution is not available in closed form. This implies that physical arguments as for example the boundedness of the heat flux even for large K can guide the ansatz.

This paper reports on an initial attempt to provide the analogue of equation (14) for Coulomb physics controlled heat flow $Q_{\parallel}(K)$ for finite K, whose leading order Maclaurin series reduces for $K <<< 1$ to $Q_{\parallel}^{CBES}$.

3. Established Kinetic Facts vs. CBES Postulates

At the Solar Wind 4 Meeting in 1978 at Burghausen, we reported on the properties of velocity distributions shaped by the Coulomb cross section without making gradient expansions (cf. SO I, II; a portion of the previous paper in this monograph reports on extensions of that work). We have demonstrated that non-local aspects of the steady state distributions of material along the tubes of force play a vital role in shaping the speed dependence in the trans, $(E > kT_c)$, and extra-thermal, $(E > 7kT_c)$, electrons; the latter group, by observations, controls the magnitude of the net skew of $f_e^{obs}(\underline{w})$ and, therefore, the magnitude of the heat flux (eqtn 8a,b).

In the presence of general finite gradients, Maclaurin series perturbation expansions for $f_e^{obs}(w)$ about a local Maxwellian are doomed because the strong speed dependence of the Coulomb cross section allows a global communication for electrons $E > 7kT_c$ along the magnetic tube of force (cf. SO I). The information that f_e (corona) $>> f_e$ (observer) manifests itself in the over populated, skewed, tails of the local distribution which observationally exceed a local Maxwellian population by several orders of magnitude above 100 eV at 1 AU. cf. SO II Figure 1. These are Knudsen number effects of the system's distribution of material in the even part of $f_e^{(o)}$. The speed dependence of $f_e^{(o)}$ has been observationally and theoretically shown to be non-Maxwellian as a result of Knudsen number effects. This violates CBES postulate 1 and 3. In this important trans and extrathermal regime $\delta f/f_{Maxwellian}$ is not even the same order of K local, which at 1 AU for example is $\sim$ 1/2, much less infinitesimal as assumed by CBES.

4) Conservation Laws vs. CBES Implications

Provided $Q_{\parallel}$ is not zero, conservation laws for the plasma as a whole neglecting viscosity and ohmic entropy production (cf. eg., Rossi and Olbert, 1970) in steady, field aligned flow imply that

$$(3PV)/(2Q_{\parallel}) \frac{d}{ds} \ln (P/\rho^{5/3}) = -\frac{d}{ds} \ln W. \qquad (15)$$

In the above P is the total gas pressure of electron and ions, $P \equiv P_e + \Sigma P_i$, ρ is the total mass density, V the magnitude of the field aligned flow velocity, $Q_{\parallel}$ the parallel heat flux, and W is the dimensionless heat function defined by

$$W = Q_{\parallel}/(\rho V U_{\infty}^2) = Q_{\parallel}/(\alpha B U_{\infty}^2), \tag{16}$$

where U_{∞} is the asymptotic flow speed and α is the flux tube stream constant defined by the ratio of mass to magnetic flux:

$$\rho \underline{V} \equiv \alpha \underline{B}. \tag{17}$$

As discussed by Olbert (1981) using current notation (and others) in steady state there exists an energy streamline constant defined by

$$U \equiv \frac{1}{2} V^2 + \frac{5}{2} \frac{P}{\rho} + \psi + \frac{Q_{\parallel}}{\rho V},$$

where ψ is the gravitational potential. Dividing this expression by the square of the asymptotic radial wind speed U_{∞}^2 and using (17) yields an equivalent streamline constant where the dimensionless heat function is involved:

$$(U/U_{\infty}^2) \equiv \frac{1}{2} (V/U_{\infty})^2 + \frac{5}{2} \frac{P}{\rho U_{\infty}^2} + \frac{\psi}{U_{\infty}^2} + W.$$

At 1 AU the leading term is the largest and dominant term. At the base of the corona the last term is the dominant term. The viability of the coronal expansion via conduction reduces to showing that

$$\frac{1}{2} \left(\frac{V(1\ AU)}{U_{\infty}}\right)^2 \backsim W\ (1.03\ R_{\odot})$$

In this sense the size of the dimensionless heat function W is crucial to the energetic feasibility of a conduction supported solar wind expansion (cf. discussion below).

Equation (15) may be rewritten without further approximations as

$$(PV/Q_{\parallel}) = (2\ L_D)/(3\ L_W) \tag{18}$$

where the spatial scale for departures from adiabatic expansion, the "diabatic" scale, is implied by

$$L_D^{-1} = \left| - \frac{d}{ds} \ln\ (\rho^{5/3}/P) \right| \tag{19}$$

and the spatial scale for heat divergence, L_W is implied by

$$L_W^{-1} = \left| - \frac{d}{ds} \ln W \right| \tag{20}$$

In the CBES perturbation scheme, gradients of all macroscopic variables, M, are assumed to be weak in the sense that

$$\lambda_{rms}/L_M = O(K) \tag{21}$$

is sufficiently small to ensure the validity of all the postulates. For a given experimental regime of K the practical question arises if it is small enough to assure quantitative precision of the CBES formulas. We proceed now to delineate what implied conditions on other macroscopic variables are sufficient to guarantee the validity of the CBES results.

Since D and W are combinations of macroscopic variables, CBES ordering (21) implies that the associated inverse scales, L_D^{-1} and L_W^{-1} are also O(K) in the sense of (21). Within the CBES framework the general conservation of energy, (18), can only be consistently satisfied if the enthalpy and parallel heat flux are of the same order:

$$PV^{CBES} / Q_{\parallel}^{CBES} = O(1). \tag{22a}$$

This constitutes a restriction on the physical problems for which CBES is internally consistent since (22a) together with (9b) imply that the enthalpy flux must be proportional to K for small K:

$$PV^{CBES} = K\,\delta \tag{22b}$$

Except for the consequences of the CBES postulates (21) there is no general ab initio argument that all physical systems will satisfy conservation of energy in such a way that the enthalpy flux is comparable to or less than the CBES heat flux in the Knudsen regime (K <<< 1) where $Q_{\parallel}^{CBES}$ calculation is rigorous. By explicitly determining δ and, therefore, the limits on V^{CBES} in the CBES regime, we argue that $V_{solar\ wind} > V^{CBES}$ throughout the entire solar wind expansion.

It is worth noting that CBES transport is usually discussed (cf. for example Cohen et al. (1950)) for plasmas at rest so that the speed, V, vanishes identically. Correspondingly (22) is satisfied in the CBES approach as a result of a postulate ($V \equiv 0$) which is generally not true in astrophysical contexts. For the remainder we proceed on the assumption that V does not vanish throughout the system.

The CBES heat flux may be rewritten as

$$Q_{\parallel} = P_e\, w_o K, \tag{23}$$

where

$$w_o = 3.3\,(L_n/L_{Te})\, w_{T,e}\,. \tag{24}$$

It then follows that the CBES regime refers for a flowing medium to physical systems such that the inverse diabatic scale, L_D^{-1}, is second order small in the formal expansion parameter, K, (Rossi and Olbert, 1970, p. 436), since equations (18, 21-24) imply

$$\lambda_{rms}/L_D \overset{CBES}{\lesssim} G\,K^2, \quad (25a)$$

where

$$G = 1.1\,(2P_e/P)\,(\tau_{p,n}/\tau_{e,T_e})\,(L_n/L_W). \quad (25b)$$

In the above the rms electron transit time of its temperature scale, τ_{e,T_e} is given by

$$\tau_{e,T_e} = + L_{T_e}/w_{T,e},$$

and the fluid transit time of the electron density scale height, $\tau_{p,n}$, is given by

$$\tau_{p,n} = L_n/V.$$

For naturally occurring systems L_{T_e} is usually larger than L_n; however $w_{T,e}$ is usually larger than V and the ratio of the two times is more nearly commensurate than either the ratio of scales or speeds separately. On this basis G is not an enormous number being clearly $\lesssim$ 1 for CBES ordering. Using the solar corona as a prototype for evaluating G we obtain G $\lesssim$ 1/6 beyond the sonic point, where we have used the relationship that $d\ell n T_e/d\ell n\, n_e \lesssim 1/6$ (Sittler and Scudder (1980)) and that $d\ell n\, W/d\ell n\, n \lesssim 1/50$ (Olbert (1983)). These considerations reinforce the important conclusion that indeed

$$\lambda_{rms}/L_D \overset{CBES}{\lesssim} K^2, \quad (26)$$

which is a more restrictive, but, nevertheless, consistent ordering of scales with the initial CBES ordering (21).

Apart from trivial constants, the inverse scale length, $L_{f^o}^{-1}$, associated with the change of the electron velocity distribution function at zero proper frame velocity is the same as that of the diabatic expansion (26). Therefore, (26) may be restated in microscopic language to establish contact with SO I, II: the CBES description is internally consistent with conservation of energy for systems (such as thermodynamic equilibrium (!)) with L_{f^o} that is much larger than any of the driving gradient scales of the macroscopic variables, L_M. Rewriting in symbols equation (26) the inverse scales of f^o must be one order weaker than those of the macroscopic variables.

$$L_{f^o}^{-1} \lesssim (L_M^{-1})\,K.$$

This is precisely what is not realized in astrophysical plasmas and leads, via the Coulomb cross section "window", to global lowest order signatures in the distribution functions and correspondingly in the transport SO I, II.

5. Enforcing CBES Postulate (3) in an Inhomogeneous Plasma

In order that the lowest order $f_e^{(o)}$ be Maxwellian in an inhomogeneous, fully ionized plasma (CBES postulate 3), it is no longer sufficient to only require that the rms free path, λ_{rms}, be small compared to the scale height. When K is finite in a fully ionized plasma then the rms free path of the thermal and suprathermal electrons are no longer infinitesimal, being related to the rms free path by the formula

$$\lambda(u) = u^4 \lambda_{rms}, \tag{26a}$$

where $u = w/w_{T_e}$. By contrast collisions between neutral molecules have a free path with essentially no speed dependence for speeds above $\sqrt{3kT/m}$, Meyer (1899).

If, as is usual (cf. Parker (1964)), one assumes $K << 1$ is sufficient for CBES heat transport results to ensue then (26a) implies that there always exists a finite u_Q so that the following ordering of scales is possible:

$$\lambda_{rms} <<< L_n < \lambda(u_Q). \tag{26b}$$

Unlike the situation for collisions between neutrals, equation (26a) implies that no matter what the size of the Knudsen number (so long as it is finite), there always exists a speed domain ($u > u_Q$) of the velocity space for which the (shortest) macroscopic scale length L_n is shorter than the free path of members of the velocity distribution which have speeds in this domain. This domain need not be the sole province of the cosmic rays!

Having established (26b) it is important to assess the potential impact it may have on the kinetic determination of transport in the system. If K be so small that u_Q is a relativistic cosmic ray and the mean energy of the gas is ultra non-relativistic, the practical impact of (26b) on the CBES regime is not severe. However, u_Q needs to be well above the dominant proper frame speeds where the perturbations to $f_e^{(o)}$ kinetically determine the heat flux (equation 8), the highest moment in the Onsager relations. The precise size of u_Q so as to assure that non-local effects are collisionally destroyed and do not dominate the transport becomes the prerequisite for allowing postulate 3 of CBES to be reasonable: find u_Q such that

$$\lambda_{rms} < \lambda(u_Q) << L_n. \tag{26c}$$

Based on the theoretical properties of the Coulomb cross section SO II argued that significant non-local suprathermal particles will be present above the dimensionless speed $u = 2.16$. The observed solar wind electron velocity distribution function was shown to develop strongly overpopulated skewed suprathermal tails (the halo) above this speed. As an empirical fact the *in situ* heat flux is determined by the skew of the electron distribution near but above this transition. We suggest that

$$u_Q \gtrsim 3.5 \tag{26d}$$

is appropriate for the inequality in (26c). Using this result in (26c) and noting (26a), we can restate the sufficient local condition that $f^{(o)}$ be locally

Maxwellian, in an inhomogenous, fully ionized plasma without postulating special spatial distributions of the plasma in the adjoining volume, viz

$$150\ K^{CBES} \ll 1. \qquad (27)$$

Notice that this result is considerably more restrictive than the usually stated (and implied sufficient) condition in the astrophysical context for CBES validity: $K \ll 1$. Equation (27) defines the regime for finite K that is consistent with use of CBES results.

From the internal energy equation, (25), in the consistent Knudsen number regime K^{CBES} is algebraically equal to

$$K^{CBES} \equiv \frac{3}{2} (L_W/L_D)\ (P/P_e)\ (V/w_o); \qquad (28)$$

however, as the prerequisite for fulfilling CBES postulate (3) equation (27) must be satisfied. This yields the compatibility condition

$$M_S^A \overset{CBES}{\lll} \mathcal{M}, \qquad (29)$$

where the adiabatic Mach number is defined as

$$M_S^A = V/(5/3\ P/\rho\ )^{1/2} \qquad (29b)$$

and

$$\mathcal{M} = 0.73\ (2P_e/P)^{1/2}\ (L_D/L_W) \qquad (29c)$$

which follows from (28) using (27):

$$K \overset{CBES}{\lll} 6.6 \times 10^{-3} \qquad (30)$$

Equations (29) and (30) are not physical constraints but are corollaries of the mathematical postulates made for the CBES approach which render the (electron) transport problem to be "local".

Unless a fully ionized plasma is prepared in a contrived way (as described below) the contradiction of either (29, 30) implies that CBES is no longer appropriate for electron signatures. Depending on the observational technique(s) one or the other of these constraints may be more accessible for a distant observer of astrophysical phenomena.

For specially prepared equilibria (as discussed by CBES) such as an isothermal flux tube that has slight temperature and density variation but still "superadiabatic" in the sense of (26) and, therefore possessing very low contrast in $f^{(o)}$ between its extremities (a very difficult idealization to achieve in an astrophysical setting) condition 30 can be relaxed by several orders of magnitude to

$$K_{isothermal}(s) \ll 1,$$

provided this condition is met everywhere along the magnetic tube of force connected to the point of interest. This is the system to which CBES refers. The relaxation of (30) results from a very special preparation of the inhomogeneous system. For this circumstance individual trans and extrathermal particles still have a global range over arc lengths large compared to λ_{rms}; however, the probability function f^{o} refers to the pattern of phase space occupation and it is not modified in lowest order in the presence of finite K<<1 provided the system is prepared in this special way. Alternatively, in the presence of an absolutely uniform T_e and a macroscopic density gradient the individual trans and extrathermal electrons give up sufficient energy to the polarization electric field that the lowest order pattern for $f^{(o)}$ can still be self-similar along B even in the presence of strong density gradients.

We submit that these special systems are so contrived that they are unlikely prototypes of plasmas in the astrophysical or other contexts and that conditions (29, 30) are more generally applicable. Gravity destroys the weak density gradient regimes; T_e can not be artificially assumed constant throughout a given astrophysical flux tube and the lowest order $f(\underline{w})$ becomes global in character.

For the solar wind it is clear by direct observation that at 1 AU that $M_S^A \gg 1$, that $K \sim 1/2$, that both the inequalities (29 and 30) are strongly violated, and that the observed $f_e(\underline{w})$ is not deformed as predicted by CBES theory (Montgomery, 1972). What is forcefully driven home by (30) is that the CBES description of thermal conduction, for example, is invalid at the very base of the coronal expansion. Given observed coronal velocities in the range of 10's km/s

and an implied base sonic Mach number at $10^{6\circ}$K of approximately .08 and the determination in the previous paper that $L_n \ll L_w$ in approximately the ratio of 1:50 implies (if $P_e = P_i$) that

$$\mathcal{M}_\odot = 5 \times 10^{-3}$$

and that

$$M^A_{S_{1.03r_\odot}} \sim .08 > \mathcal{M}_\odot$$

and that the CBES determination of supply of conduction flux to the expansion is in error from the base of the corona. This conclusion should be contrasted with Parker's (1964, eqtn 76) original argument that CBES should be appropriate for the initial portion of expansion possibly as far as 1 AU on the basis of an argument like

$$\text{"CBES Flux"} = \chi_o T_e^{5/2} \frac{dT_e}{dr} \lll \frac{3}{2} NkT_e \left(\frac{3kT_e}{m_e}\right)^{1/2} = \text{"Saturation Flux"}$$

which is equivalent (using (6, 9b, 10, 24) to

$$K^{\text{Parker}} \ll .45 \left(\frac{L_T}{L_N}\right) \sim 2.5 \qquad (32)$$

which we now recognize as an overly generous upper bound for K when contrasted with (27).

Use of the CBES heat transport description within the conservation equations for a flowing medium with M_S^A > << 1 (eqtn 29) is clearly inconsistent with the entire set of postulates necessary to derive equation (9). The failure of such a kinetically inconsistent model to predict the observed solar wind properties implies nothing about the importance of conduction in driving winds. This point is especially crucial since the properties of the conduction flux near the supersonic transition are extremely critical for the determination of the essentially asymptotic wind properties that are accessible to spacecraft observations.

6. The Global-Local Formulation of the Transport Problems

In the present restricted monograph we only outline the philosophy and present the results of the procedure used to obtain the heat flux form quoted below.

The philosophy of this approach is that, like in eqtns 11-13, a simple Maclaurin series of either 1 or n terms about a Maxwellian is hopeless because such a procedure neglects the global distribution of phase density along the tube of force. The tails of $f(\underline{w})$ and their skew cannot be accurately predicted by a one term Maclaurin perturbation expansion of f about a local Maxwellian. It is the boundary conditions at the origin and at infinity in the mathematical example and at the sun and infinity in the physical example, as well as the strong speed dependence of the Coulomb cross section that force a more useful approximation scheme. Both from considerations of the conservation equations and from integral formulations for the local form of $f(\underline{w})$ (SO I, II and Olbert (1983)) global correction must be incorporated in the lowest order ansatz for $f^{(o)}(\underline{w},\underline{x})$ before attempting to determine the skew and, therefore, the net heat flux $(\overline{8b})$. In mathematical language this represents an attempt to renormalize the perturbation expansion of CBES.

In the face of significant non-local corrections to local gaussian behavior we have adopted a lowest order functional ansatz for $f^o(\underline{w}, \underline{x})$ that while still only speed dependent contains an additional shape parameter, κ, beyond the shape factor in a gaussian identified with the temperature. This is a significant departure from CBES postulate (3). This ansatz is called the kappa distribution:

$$f_\kappa^{(o)}(\underline{w},\underline{x}) = f_\kappa(\kappa(\underline{x}),\underline{w}_c^2(\underline{x}),\, n(\underline{x}),\, \underline{w}^2) = \frac{A_\kappa n}{(\sqrt{\pi} w_c)^3}\left(1 + w^2/(\kappa w_c^2)\right)^{-(\kappa+1)} \qquad (33)$$

originated by Olbert et al. (1967) and Olbert (1969). This lowest order distribution contains extrathermal tails which can support the global Coulomb physics, but asymptotes to a gaussian/Maxwellian distribution as $\kappa \to \infty$. In every respect the kappa function represents a mathematical "extension" of the Maxwellian function. The physical meaning of n is the usual number density, w_c is the speed of the most probably occurring member of the speed distribution, w is the proper frame speed, and $\kappa(\underline{x})$ controls the local strength of the extrathermal tails: small κ implies strong tails, κ going toward infinity implies increasingly weaker tails. $A_\kappa(\kappa(\underline{x}))$ is a normalization constant which ranges between 0.75 to 1. Paralleling in concept the Padé approximant (14) for g by choosing a, b, c and d appropriately, $\kappa(x)$ must be determined. This is done self-consistently within the perturbation scheme described below.

Clearly, as in any perturbation theory, the better the ansatz, the better the approximation becomes. In our choice of ansatz we have been influenced by economy of free variables, the energy dependence of the observed $f(\underline{w})$, the global aspects of the Coulomb physics (as outlined in SO I, II and previous paper) and that a member of the functional ansatz should be a gaussian.

We have used a speed dependent Krook collision term as in SO I and II and have successfully imposed conservation of electrons through a mandated $\kappa(s)$. The imposition of this collisional invariant constraint is well known; the equation of constraint relates κ to other state variables (cf. (34) below). In the presence of a speed dependent Krook collision operator this equation of constraint relates the variation of the non-thermal tail index, $d\kappa$, to the variation of the normalization of phase density $d\ \ln\ (n_e/w_c^3)$ along the tube of force. This equation has the form

$$d\kappa(s) = -\ \ell(\kappa(s),\kappa_o(s_o))\ d\ \ln\ (n_e(s)/w_c^3(s)). \qquad (34a)$$

For reasonable interplanetary profiles κ is smaller near the sun than at 1 AU. This implies that the isotropic part of $f(\underline{w})$ near the sun has more of a non-thermal tail than at 1 AU.

Equation (34a) may be recast to show its non-local character

$$\nu(s)/\nu(s_o) = \exp\left[-\int_{\kappa_o(s_o)}^{\kappa(s)} \frac{d\kappa'}{\ell(\kappa',\kappa_o)}\right], \qquad (34b)$$

where $\nu(s) \equiv t_c^{-1}(s)$ is proportional to n_e/w_c^3. Notice that equation (34b) (implicitly) prescribes $\kappa(s)$ as a function of the contrast of collision frequency which equals the contrast in the phase space density (at zero proper frame energy) between s and some reference point, s_o, where $\kappa=\kappa_o$ is known. Mathematically we have succeeded in instilling sufficient information in (renormalizing) the lowest order $f^{(o)}$ so that the global, velocity dependent, Coulomb moderated mobility of the electrons does not violate charge neutrality. This lowest order failure of the CBES, Maclaurin series approach to obtaining extrathermal tails on $f(w)$ is now removed.

It is crucial to appreciate that it is the skewness of the increasingly non-local trans and extra-thermals which determine the heat flux (8) for astrophysical plasmas. The "resummation" of non-local effects embedded in our ansatz (33) as constrained by (34b) heavily impacts the dominant speed, w^*, where subsequently obtained skews of f make the principal contribution to $Q_\parallel$ (8). This effect is exacerbated by the high power of w involved in determining Q and the fact that the perturbation δf_e about $f_e^{(o)}$ itself has significant speed dependence.

We then obtain a gradient expansion about the local, isotropic $f_\kappa^{(o)}$ to obtain the skew correction term that represents the flow of heat. The analogue of (9a) has the global-local (GL) structure (if there exists no current in the proper frame which determines the proper frame parallel electric field):

$$Q_{\parallel}^{GL}(s) = -CTS(\kappa(s), \kappa_o(s_o)) \frac{d}{ds} \ln T_c - CSS(\kappa(s), \kappa_o(s_o)) \frac{d}{ds} \ln (n/T_c^{3/2}) \quad (35)$$

$$CTS \gtrsim CSS > 0; \; kT_c = \frac{1}{2} m w_c^2,$$

where CTS and CSS are global functions of system parameters, which sense, through equation 34b, the collision frequency contrast of the physical system connected to the observer, while the gradients sense local changes in the state variables. The functional dependence of the global-local form of $Q_{\parallel}^{GL}$ (35) should be contrasted with the completely local form of CBES in equation 9.

7. The CBES Limit of Global-Local

We recall that a condition via equations (25, 26) for internal consistency of the CBES regime in the presence of flows ($K \to 0$) is that

$$\lambda \frac{d}{ds} \ln(n/T^{3/2}) = O(K^2).$$

This implies that the second term of (35) is quadratic while the first is linear in the expansion parameter K. [As an aside CTS $\backsim$ CSS]. The imposition of near adiabaticity implies large collision frequencies, $\nu = O(1/K)$, which implies progress toward Maxwell-Boltzmann distribution. Therefore, from the equation of constraint (34) since $\ell(\kappa, \kappa_o)$ is bounded and $O(1)$

$$\kappa - \kappa(s_o) = O(K^2),$$

which implies that the CBES regime is characterized by a constant

$$\kappa \backsim \kappa_o + O(K^2) \to \infty,$$

which by construction of our ansatz is functionally possible.

Since $\lim_{\kappa \to \infty} T_c = T_e$

we may now form the CBES limit of $Q_{\parallel}^{GL}$ (eqtn 11) as

$$\lim_{\substack{\to CBES \\ (K \to 0)}} Q_{\parallel}^{GL} = - \lim_{\substack{\kappa \to \infty \\ \kappa_o \to \infty}} [CTS(\kappa, \kappa_o) \frac{d}{ds} \ln T_c + O(K^2)] = Q_{\parallel}^{CBES} + O(K^2),$$

where the quadratic terms throughout the CBES approach have already been neglected. We reemphasize here the extremely narrow classes of astrophysical problems (delimited by equation (29, 30)) for which CBES results are quantitative.

8. The Global-Local Evaluation of the W Function for High Speed Wind Conditions

As a final result we illustrate in Figure 1 the dimensionless heat function $W(\rho)$ defined by (16) and central to the previous paper by Olbert (1983), but here evaluated with $Q_{\parallel}^{GL}$ from equation (35). We have used the same coronal hole density and magnetic profiles as used in Olbert (1983) for the high speed solar wind (U_{∞} = 700 km/s) and a composite electron temperature profile that has a coronal hole value at 1.03 r_o (8×10^5 °K), has the correct $d\ell nT/d\ell nn$ characteristics near 1 AU (Sittler and Scudder (1980)), and an electron temperature characteristic of the high speed solar wind at 1 AU of 8×10^4 °K. We emphasize that these results are preliminary as discussed below, since all aspects of the calculation are not fully self-consistent, such as picking a density and temperature profile that are actually compatible with the postulate of steady state. In that respect the results of Figure 1 represent a feasibility study for investigating qualitative regimes of $W(\rho)$ that are possible without doing the quadruple numerical integrals as have been painstakingly done in the previous paper. This matter of complete self-consistency is an issue in both these papers and is currently under investigation.

We note the properties of W^{GL} that the previous paper (Olbert (1983)) reports as essential for a conduction supported high speed wind: 1) the initial plateau at large densities; 2) the plateau magnitude (this implies sufficient random energy available to be converted to directed energy for 700 km/s wind) and 3) the sharp decrease near the critical point. The second plateau at lower densities has little effect on the dynamics of the solar wind expansion. Figure 2 illustrates the relative contributions of the two terms of equation (35) to the final $Q_{\parallel}^{GL}$; note that different density regimes reflect different relative importance of the two terms and neither term is negligible and that the second term of (35) (neglected by CBES) determines a larger share of the heat than the conventional grad T term at the base of the coronal expansion.

Figure 1 represents a realization of $W(\rho)$ using the hybrid-global-local approach instead of the quadruple numerical integrals required in the integral formulation of the previous paper. In the preceding paper (Olbert (1983)) all gradations of global Coulomb effects are accounted for in the presence of a leading term Ohm's law electric field. In this paper the electric field has been set by the zero current condition, particles are conserved and there is a hybrid global-local approach to the tail/skew problem. Both approaches attempt, with different levels of precision, to assess the importance of the strong velocity dependence of the Coulomb cross section which invalidates the CBES approach.

The $W(\rho)$ function is very sensitive to the profiles of n and T_e. For a given density and magnetic tube geometry the electron temperature profile is not free to be prescribed, but is determined by the energy equation (15). In future this internal self-consistency will be addressed. For the present these results represent significant progress toward affirming the existence of stellar winds with large asymptotic velocities supported by Coulomb mediated heat conduction, with the solar wind as a prime example.

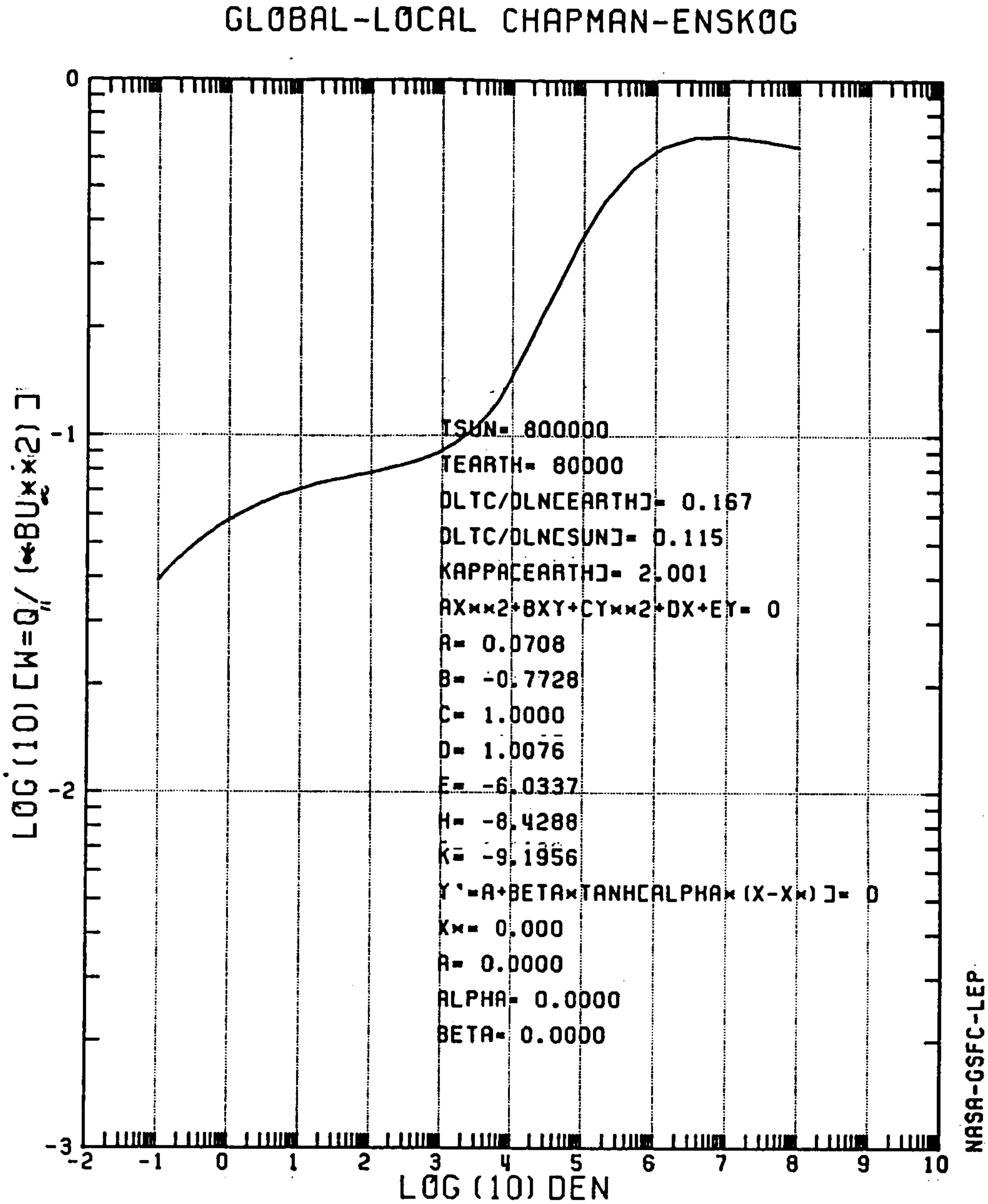

Fig. 1 Profile of the dimensionless heat function $W = Q_{\parallel}/(\alpha BU^2)$ for coronal hole magnetic field and density profile used by Olbert$^{\infty}$(1983), but evaluating $Q_{\parallel}$ via the global local approach outlined in text. Abscissa is electron density.

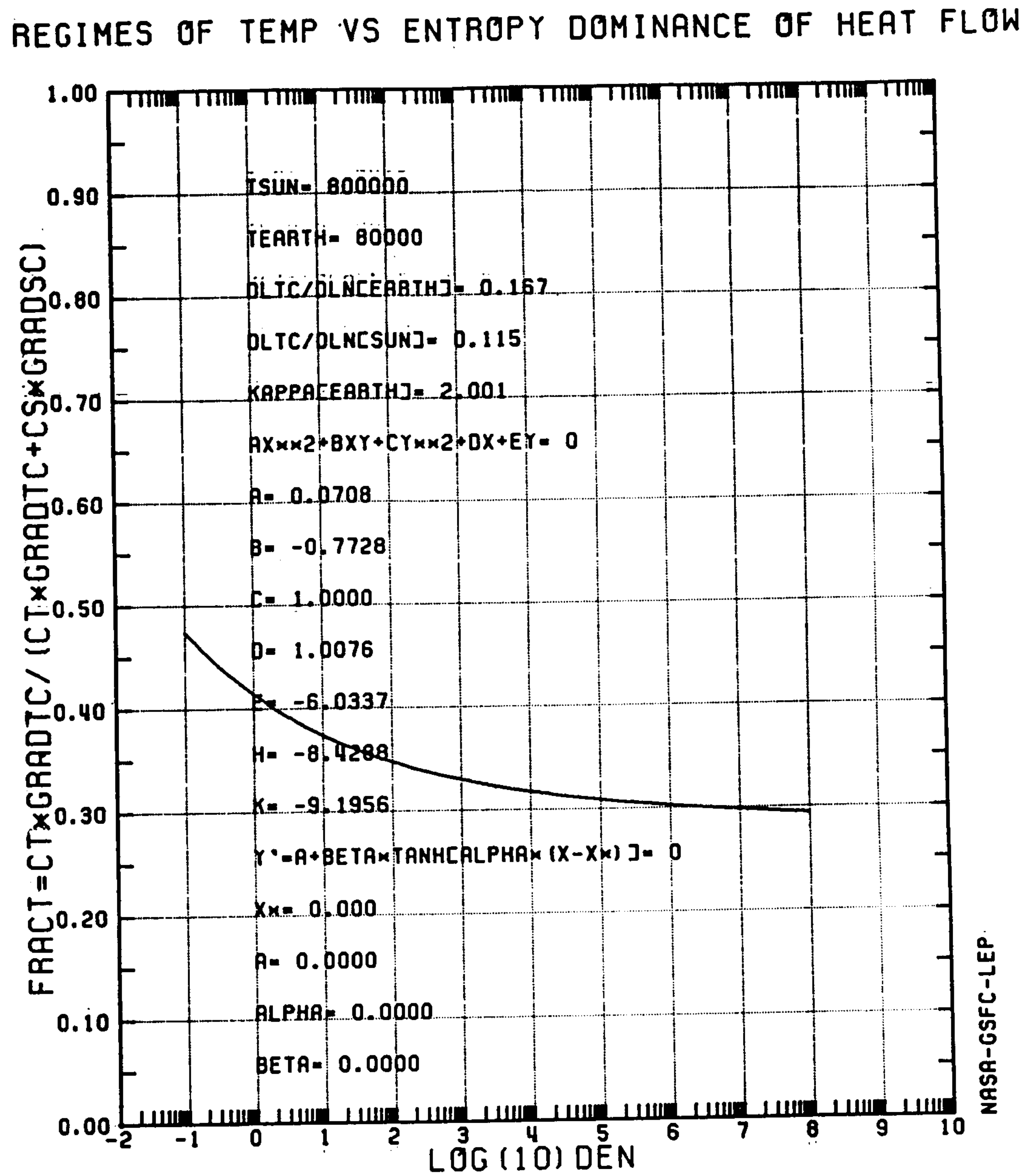

Fig. 2 The fraction of the suggested heat flux determined by the first term of the full heat expression of equation 35 for the heat flow profile of Figure 1. Note that there exists varying domains of relative importance with the second term of (35), not included in the CBES expression of equation (9), playing the dominant role near base of coronal expansion.

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