

## A CONTROL MODEL: INTERPRETATION OF FITTS' LAW

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### ABSTRACT

Fitts' law has been universally cited as an index of difficulty or predictor of movement time (MT) for rapid aiming tasks since it was first published in 1954 (Fitts 1954). Many researchers report a remarkable correlation of Fitts' law and the observed movement times in aiming tasks. Other researchers report discrepancies, however, between observed movement time and the law, especially at low and high movement times, which correspond, respectively, to short movements to a large target, and long movements to a small target.

These discrepancies suggest that while the law predicts MT well for some human motions, the true basis for the law may not be known, and, as a consequence, that there may exist conditions where its application is appropriate and yet others where different laws should be used.

Fitts suggested the law as a model of the rate-limit of human information processing and movements. According to that view, the movement-problem is characterized by one half the target width (i.e., the target center-point is the aiming-point and  $1/2$  the target width is the error tolerance) and the movement amplitude. According to Fitts, the total movement amplitude ( $A$ ) can be regarded as  $N$  units, where each unit consists of  $1/2$  the target width, which are "processed" by the human at a maximum rate. Hence, as the target width ( $W$ ) is decreased or  $A$  is increased, the "difficulty" and MT of the task both increase. Further, if  $A$  is increased and the target width is also increased, making their ratio constant, the task difficulty and MT are constant. The remarkable ability of the law to predict these results suggests that its functional form is appropriate for at least some movement problems.

But this rate-limit model is not the only interpretation possible. Rapid movement of the hand to a target can be modeled from a different view-point: namely, as a control system. This paper gives the analytical results for several models: a first order model where it is assumed that the hand velocity can be directly controlled, and a second order model where it is assumed that the hand acceleration can be directly controlled. Two different types of control-laws are investigated. One is a linear function of the hand error and error rate; the other is the time-optimal control law.

The results show that the first and second order models with the linear control-law produce a MT function with the exact form of the Fitts' law. These models assume that the control-law aims for the center of the target, but that the motion is actually stopped when the edge of the target is reached. This corresponds to the situation in which the lateral hand movement is directed toward the center of the target and in which, if it were not for the vertical movement which causes the hand to hit the target at the target edge, the lateral movement would asymptotically approach the target center as time approaches infinity.

This control-law interpretation produces a formula for index of difficulty identical to Fitts' law, and yet it has nothing to do with information theory. It says, for instance, that the lateral hand motion is not (necessarily) a function of target width, but is instead a constant linear control function independent of target width. The control-law interpretation thus implies that the effect of target width on MT must be a result of the vertical motion which elevates the hand from the starting point and drops it on the target at the target edge. The control law interpretation further suggests that many movement time experiments may be inadequate because the end point conditions, such as the vertical and horizontal velocities, are not controlled but are allowed to vary.

The time optimal control law did not produce a movement-time formula similar to Fitts' law. However, the formula may be found to apply in yet other situations.

## INTRODUCTION

Fitts' law has been cited as a predictor of movement time or an index of difficulty for rapid aiming tasks as well as other selected tasks. In 1954, Fitts published a theory of task-difficulty in which the movement time (MT) for a hand-position task was given as:

$$MT = K \log \left( \frac{A}{W/2} \right) \quad A \geq W/2 \quad (1)$$

where the log is log base 2, A is the movement amplitude, and W is the target width.

The rationale Fitts presented for this formula developed an analogy between the rapid positioning task and Shannon's information theory. According to that rationale, one half the target width is the target error tolerance. The movement amplitude divided by this error tolerance gives the number of "tolerance units" that must be considered for the motion. The base 2 logarithm of the number of tolerance units is the number of bits i.e., the amount of information to be processed. Fitts reported that the correlation between the actual, measured MT and the formula was .99. While these early results were obtained for serial, self-paced tasks, Fitts later, in 1964 (Fitts and Peterson 1964), showed that the formula also applied to discrete tasks.

Welford (1968) found that Fitts' law fits experimental data well except for near-zero movement times and except for the tendency of the data at the high end of the scale (i.e., for large movement times), where Fitts' law predicts a straight line function (i.e., a straight line on a log plot), to "curve gently upwards". Welford presents a number of alternative constructions of Fitts' law, including

$$MT = K \log \left( \frac{A}{W} + \frac{1}{2} \right), \quad A \geq \frac{W}{2} \quad (2)$$

in order to better fit the data.

Drury (1975), in studying foot pedal designs, found that both Fitts' law and the Welford formula provided a good fit to the data, with the correlation coefficient for either being of the order

of .98. Drury found that Welford's formula provided a somewhat better fit to his data, but also found a deviation for the higher movement times, where a "gentle upward curve" again appeared.

More recently, Buck (1983) proposed a modification of Fitts' law to include the effect of target location in addition to movement amplitude.

Wallace and Newell (1983) report results supporting the notion, corollary to the division of the movement amplitude into "tolerance units," that Fitts' law represents a discrete corrections model. This model assumes that the movement to the target consists of a series of discrete submovements each involving a visual error correction.

Jagacinski, Repperger, Ward, and Moran (1980) attempted to apply Fitts' law to the capture of moving targets. They found that target velocity interacts with the movement amplitude  $A$  and, consequently, that the law should be modified to include target velocity.

Sheridan and Ferrell (1974) discuss the development of Fitts' law and its information-theoretic basis. They recognize the empirical support for the law, but also state that the information-theoretic argument is "not entirely satisfying."

The researchers cited above are but a few of those who have systematically used Fitts' law in their work. Their conclusions are cited to illustrate a point: Although some researchers find Fitts' law to be highly correlated with a prescribed task MT, others find that the formula must be revised or that additional factors must be introduced.

These inconsistencies suggest that the true basis for the law may not be known, and, further, that there may exist conditions under which the law is valid and other conditions under which the law is simply not appropriate. Specification of the application-rules for the law would facilitate its correct use. Further, an investigation of the appropriate applications of the law may guide us to new laws or to a more general task-difficulty measure, representing difficulty or MT in cases where Fitts' law does not apply.

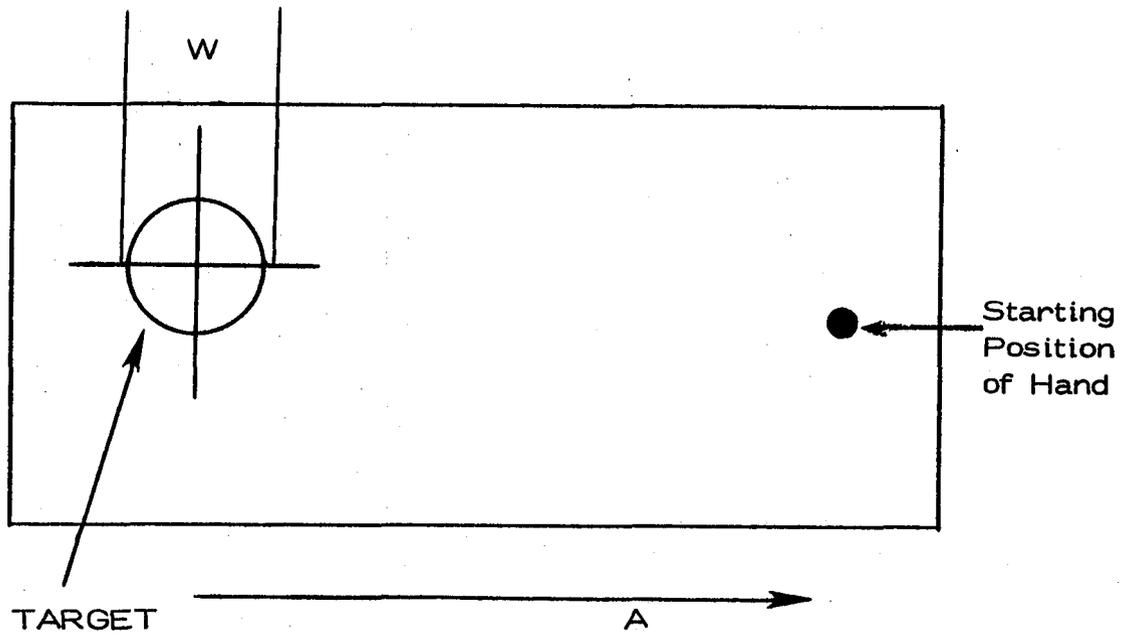
## The Control-Law Derivation of Fitts' Law

The remarkably high correlation with observed data in some movement problems serves as a first clue. The log function suggests that the movement is described by an exponential solution i.e., by a function of time that exponentially approaches the steady state solution as time approaches infinity. Exponential solutions typically result from control policies where the hand velocity or acceleration is controlled as a smooth function of hand error (distance from the center of the target) and error rate. In contrast, however, to rapid aiming tasks, in which finite movement times are observed, exponential solutions require an infinite time to reach steady state.

In actual situations there is always a finite target tolerance: the motion does not need to proceed to the target center. It may stop at the target edge or anywhere in between the target edge and the target center. Such a situation, translated into mathematical terms, provides a log-solution time-function combined with a finite MT.

As an aid in presenting the mathematical development given below, consider the following aiming task. The task is to move the hand rapidly from a starting position on a table to a target, which is also on the table (see Fig. 1). The control strategy for the LATERAL portion of the hand movement can take several forms, which are described subsequently, but is assumed, in all forms, to be a linear function of error alone, or of error and error rate. Error is the instantaneous distance from the hand to the center of the target. The target center is the "aiming" point of the lateral motion i.e., the lateral hand motion is such that, if not disturbed by the vertical hand motion hitting the target, the lateral hand motion would come to rest at the target center. The vertical motion directs the hand upwards and then downwards so that the hand or a hand-held pen actually hits the edge of the target, causing the hand to stop.

In order to illustrate the mathematical development for a simple control law, assume that the lateral-movement control-law is such that the error rate ( $\dot{X}$ ) is a linear function of the error ( $X$ ):



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Figure 1. Hand-to-Hand Motion in Perspective

$$\dot{X} = -KX, \quad K > 0 \quad (3)$$

The solution to this equation is:

$$X(t) = X(0)e^{-Kt} \quad (4)$$

Taking the log (base 2) of both sides yields

$$\log (X(t)/X(0)) = -Kt \log(e) \quad (5)$$

Solving for t gives

$$t = c \log (X(0)/X(t)), \text{ where } c = 1/K \log(e) \quad (6)$$

Now, we recognize that  $X(0)$  is really the movement amplitude  $A$  and  $X(MT)$  is really the "error" at movement time  $MT$ , when the hand is stopped at the edge of the target.

That is,

$$\begin{aligned} X(0) &= A \\ X(MT) &= W/2 \end{aligned} \quad (7)$$

Thus,

$$MT = c \log (2A/W), \quad (8)$$

which is the same equation as Fitts law.

As shown in the Appendix B, the same equation is obtained, except for an additive constant, when a second-order model is used with a linear control law.

Appendix C gives the  $MT$  for a "time-optimal" control-law where maximum force is applied laterally until the hand is stopped at the edge of the target. The  $MT$  equation then has the form:

$$MT = \sqrt{\frac{2 |A - W/2|}{F}} \quad (9)$$

where  $F$  is the maximum force that can be applied to the hand and  $||$  indicates absolute value.

Thus, even though the task is described as a "rapid" movement task, the control strategy actually used is apparently not a time optimal (i.e., a minimum time) strategy.

### Theoretical Consequences

Now considering that the first and second order models using linear control laws produce MT functions that are similar or identical to Fitts' law, there exists a control-law interpretation of MT for rapid motions. There are, of course, numerous models and control laws, both linear and non-linear, that can be formulated. The key model and control-law feature may be that the lateral hand movement is governed by a smooth function of error and error rate i.e., by a control law that will tend to bring the hand error and error rate to zero simultaneously at the target center. This provides the log function for MT.

Evaluation of the control-law interpretation can be accomplished by examining data revealing the lateral and vertical position of the hand as a function of time and by computing the control-law employed. If the control-law has constant coefficients (see equation 3), a simple control-law interpretation of MT will then exist. If the computed control-law has varying coefficients along the trajectory, then another model -- perhaps a non-linear model accounting for a non-linear muscle function, or a higher order model -- must be investigated.

The control-law model says that MT is determined by the LATERAL hand motion, since it is the lateral motion that determines where the hand will be as a function of time -- for instance, when the hand will be at the target edge. The accuracy of the hand's final resting position is governed by the VERTICAL motion, which might be a ballistic response for short MT, where ballistic parameters are fixed early in the movement, or a scheduled response for longer MT, in which vertical hand movement is coordinated with lateral hand-position error via feedback.

The control-law model also says that the LATERAL hand response path as a function of time (see equation 4) is actually independent of the target width. Yet, for a fixed movement amplitude A (i.e., a fixed distance from the initial hand position to the center of the target), a smaller target width requires a longer MT (see equation 8) because the hand has a greater actual

traveling distance. This suggests that the term "index of difficulty" is misapplied since the same response path as a function of time is used for a constant amplitude  $A$  but varying target width  $W$ . Since the hand is moving with an ever decreasing velocity as the target center is approached, the time per unit distance is increasing. Consequently, small changes in target width result in large changes in MT.

Further, the control-law model says that one system differential equation explains the lateral hand movement for all amplitudes  $A$ . Different initial positions, corresponding to various amplitudes, result in different paths as a function of time; but, once a differential equation is accepted as a model for the task, it represents the hand movement for values of  $A$  and  $W$ .

The observations presented above lead naturally to the concept that the LATERAL-movement differential equation may be that suggested by spring-mass theory. As explained by spring-mass theory, muscle parameters determining final hand position are preset prior to actual movement. According to this theory, the "springs" are set so that the target center is the "final position" for lateral movement (i.e., the final position of the hand, if it were not stopped at the target edge by the vertical hand motion). Thus, there is a direct correspondence between spring-mass theory and the control-law interpretation of MT for lateral hand motion.

A further observation resulting from the control-law is that different constants are expected as multipliers of the log term as different parts (systems) of the body are used to move the hand or hand-held pointer. Thus, for short  $A$ , when only the fingers are used, one constant value is appropriate. When the wrist, and/or arm, and/or shoulder, and/or torso are used, other constants are appropriate. When a consistent set of these systems is used an appropriate set of representative constants can be determined.

But how is the constant adjusted as various systems or system combinations are used to perform a task? This problem may be the reason that Fitts' law often fails for short and long MT. For it would seem appropriate that the scaling of the amplitude factor  $A$  would be a function of all the systems used to perform the task, but that the scaling of the target width  $\left(\frac{W}{2}\right)$  would be a function of only the system (or systems) used during the terminal portion of the task.

In conclusion, an alternative interpretation of Fitts' law has been identified in the control-law model. Its advantage over the information theoretic approach to Fitts' law is that its application-rules can be easily established, and, further, that the formula arising from it can be easily modified as different types of motions or combinations of types of motion are considered.

## APPENDIX A

### 1st Order Model: Linear Control Law

**Assumption:** Operator can control the lateral velocity of the hand directly\* and moves laterally toward the center of the target, but stops when the edge of the target is reached. The hand is stopped instantaneously because the hand or hand-held pen hits the target edge.

\*Direct control of the hand's lateral velocity assumes that any acceleration required (even an infinite acceleration) can be provided to establish the desired velocity.

Equation of Motion:

$$\dot{X}_1 = -KX_1, \quad (1)$$

where  $X_1$  is the lateral error, i.e., the distance of the hand from the target, and

$K$  is a constant.

Solution as a function of time:

$$X_1(t) = X_1(0)e^{-Kt} \quad (2)$$

Taking log (base 2) yields:

$$\log \left( \frac{X_1(t)}{X_1(0)} \right) = -Kt \log(e) \quad (3)$$

Solving for  $t$  results in:

$$t = \frac{1}{K \log(e)} \log \left( \frac{X_1(0)}{X_1(t)} \right) \quad (4)$$

Since

$$x_1(0) = A$$

and

$$x_1(MT) = W/2, \quad (5)$$

substitution yields

$$MT = \left( \frac{1}{K \log(e)} \right) \log \left( \frac{2A}{W} \right) \quad (6)$$

or

$$MT = C \log (2A/W), \quad (7)$$

where

$$C = 1/K \log(e).$$

## APPENDIX B

### 2nd Order Model: Linear Control Rule

#### Assumptions:

1. Operator can control the acceleration of the hand directly (i.e., can apply any force required to establish the desired acceleration.)
2. Operator uses a control rule which is a linear function of error and error rate.
3. Hand is stopped instantaneously at edge of target because hand or hand held pen hits the target edge.

#### Second order equation

$$\ddot{X}_1 = -2ZN\dot{X}_1 - N^2X_1 \quad (1)$$

or

$$\dot{X}_1 = X_2 \quad (2)$$

$$\dot{X}_2 = -2ZNX_2 - N^2X_1 \quad (3)$$

where  $X_1$  is the error (displacement from center of target)

$$\dot{X}_1 = X_2 \text{ is the error rate}$$

Z is damping ratio

N is natural frequency

There are two types of solutions to these equations: One solution, represented by Z less than 1, corresponds to the case where, if the vertical hand motion did not hit the target edge thus stopping the hand, the lateral hand motion would overshoot the target center line before returning to oscillate about the target center line with an asymptotically decreasing oscillation. This response is shown in Figure B1. Assuming that the hand is initially at rest i.e.,

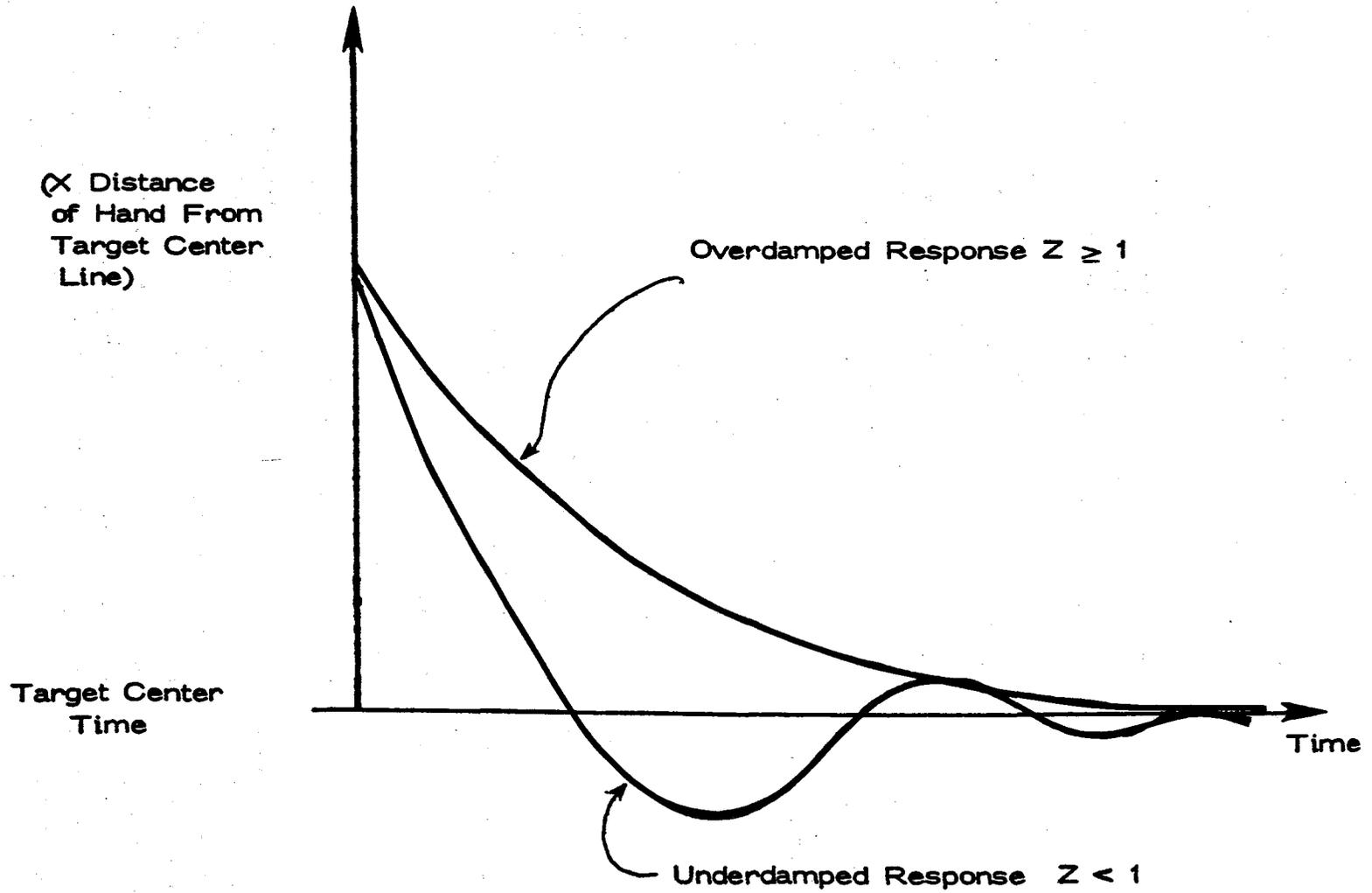


Figure B1. Two Types of Hand Movement

$$\dot{X}_1(0) = 0, \quad (4)$$

the solution to equation 1 or 2 & 3 is

$$X_1(t) = X(0)e^{-ZNt} \sin(Nt \sqrt{1-Z^2} + \psi) \quad (5)$$

where  $\psi$  is a constant.

Since our interest is computing the time when  $X_1$  is less than the 1/2 the target width and remains within the target inspite of overshoots, we can replace the sin function by its largest value namely: a "1" which yields

$$X_1(t) = X_1(0)e^{-ZNt} \quad (6)$$

then taking the log of both sides

$$\log\left(\frac{X_1(t)}{X_1(0)}\right) = -ZNt \log(e) \quad (7)$$

Since the initial position of the hand is A units from the target center line and at  $t = MT$ , the hand is stopped at the edge of the target,

$$X_1(0) = A$$

$$X_1(MT) = W/2$$

thus with the substitutions:

$$MT = C \log\left(\frac{A}{W/2}\right) \quad (8)$$

where

$$C = \frac{1}{ZN \log(e)}$$

The second type of solution referred to above, represented by Z equal to or greater than 1, corresponds to the case where, if the vertical hand motion did not hit the target edge thus stopping the hand, the lateral hand motion would asymptotically approach the target center line without overshoots. This response is also shown in Figure B1.

When  $Z \geq 1$ , it is convenient to transform the equations with the following.

$$T_1 = 1/T_2 N^2 \quad (9)$$

$$T_2 = \left( Z \pm \sqrt{Z^2 - 1} \right) / N \quad (10)$$

providing:

$$\ddot{X}_1 = -(\tau_1 + \tau_2) \dot{X}_1 / \tau_1 \tau_2 - X_1 / \tau_1 \tau_2 \quad (11)$$

or

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = -(\tau_1 + \tau_2) X_2 / \tau_1 \tau_2 - X_1 / \tau_1 \tau_2 \quad (12)$$

Assuming that the hand is initially at rest i.e.,

$$\dot{X}_1(0) = 0 \quad (13)$$

Solving for  $X_1(t)$  yields:

$$X_1(t) = X_1(0) \left\{ \begin{array}{l} \frac{(\tau_1 - a)}{\tau_1(\tau_1 - \tau_2)} e^{-t/\tau_1} - \frac{(\tau_2 - a)}{\tau_2(\tau_1 - \tau_2)} e^{-t/\tau_2} \\ (\tau_1 + \tau_2) \end{array} \right. \quad (14)$$

$$\text{where } a = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}$$

The second order system has two functions of time as indicated by the two exponential terms. Normally all terms in the equation would be used to calculate the value of  $X$  as a function of time. It is possible, however, to calculate an upper and a lower bound of  $X$  as follows:

$$\frac{X_1(t)}{X_1(0)} = \frac{T_1+T_2}{T_1-T_2} \left\{ \frac{T_1-a}{T_1} - \frac{T_2-a}{T_2} e^{-C_3 t} \right\} e^{-C_1 t} \quad (15)$$

where

$$\begin{aligned} C_1 &= 1/T_1 \\ C_2 &= 1/T_2 \\ C_3 &\geq 0, C_2 > C_1 \end{aligned} \quad (16)$$

Now  $e^{-C_3 t}$  has a maximum value of 1 and a minimum value of 0.

Thus an upper bound for  $X$  is

$$\begin{aligned} \frac{X_1(t)}{X_1(0)} &= \frac{T_1+T_2}{T_1-T_2} \left\{ \frac{T_1-a}{T_1} - \frac{T_2-a}{T_2} \right\} e^{-C_1 t} \\ &= K_1 e^{-C_1 t} \end{aligned} \quad (17)$$

and a lower bound is

$$\frac{X_1(t)}{X_1(0)} = \frac{T_1+T_2}{T_1-T_2} \left\{ \frac{T_1-a}{T_1} \right\} e^{-C_1 t} \quad (18)$$

$$\frac{X_1(t)}{X_1(0)} = K_2 e^{-C_1 t} \quad (19)$$

Thus taking the log (base 2) yields:

$$t = K_3 \log \left( \frac{X_1(0)}{X_1(t)} \right) + K_3 \log K \quad (20)$$

$$\text{where } K_3 = \frac{1}{C_1 \log(e)}$$

$K = K_1$  or  $K_2$  for upper, lower bound

respectively.

Substituting as before:

$$x_1(0) = A$$

$$x_1(MT) = W/2$$

Yields:

$$MT = K_3 \log \left( \frac{2A}{W} \right) + K_3 \log K \quad (21)$$

## APPENDIX C

### Second Order: Time Optimal Control

#### Assumptions:

1. Operator applies and maintains a constant maximum lateral force to accelerate the hand toward the target.
2. When the edge of the target is reached the hand is instantaneously stopped because the hand or hand held pen hits the edge of the target.

Note: A description of this response is plotted in a phase plane shown in the Figure C1. Also shown in the figure is an alternative trajectory resulting from an alternative strategy. These trajectories show that considerable variation in the control strategy and, consequently, in response time is possible within the task specification because both the lateral and vertical terminal velocities are not limited by the experiment design.

#### Equation of Motion:

$$\ddot{X} = +u \quad (1)$$

where

$\ddot{X}$  is the second derivative of  $X$

and

$u$  is the applied force

According to assumption 1,  $u$  is limited such that  $|u| = F$ , where  $| |$  indicates the absolute value and  $F$  is the maximum force available.

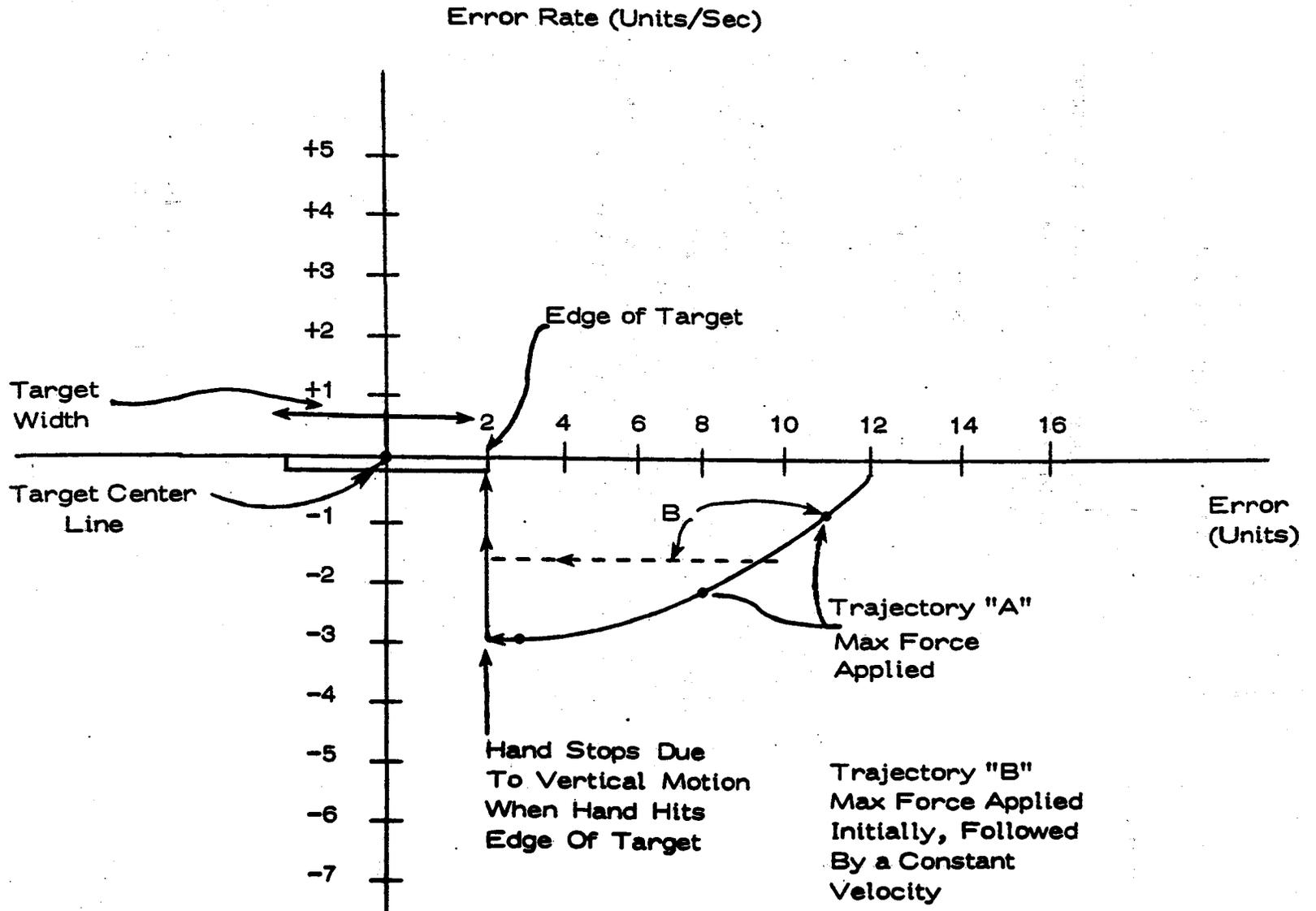


Figure C. Phase Plane Trajectories

According to optimal control theory (Elgerd 1967) most rapid motion for the motion system given above, occurs when

$$u = \pm F \quad (2)$$

The solution for any trajectory when F is constant is,

$$X(t) = \frac{u}{2} t^2 + X(0) + \dot{X}(0)t \quad (3)$$

Solving for t yields:

$$t = \frac{-\dot{X}(0) \pm \sqrt{\dot{X}(0)^2 - 2u (X(0) - X(t))}}{u}$$

If the hand is initially at rest then:

$$\dot{X}(0) = 0$$

and

$$t = \pm \sqrt{\frac{-2u(X_1(0) - X_1(t))}{u}}$$

where

$$u = -F \operatorname{sgn} (X(0) - X(t))$$

But, as in the analyses given in Appendices A, B:

$$\begin{aligned} X(0) &= A \\ X(MT) &= W/2 \end{aligned}$$

Thus,

$$MT = \sqrt{\frac{2|A - W/2|}{F}}$$

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