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ON HYBRID AND MIXED FINITE ELEMENT METHODS

Th. H. H. Pian

Massachusetts Institute of Technology  
Cambridge, Massachusetts, U.S.A.



SUMMARY

Three versions of the assumed stress hybrid model in finite element methods and the corresponding variational principles for the formulation are presented. Examples of rank deficiency for stiffness matrices by the hybrid stress model are given and their corresponding kinematic deformation modes are identified. A discussion of the derivation of general semi-Loof elements for plates and shells by the hybrid stress method is given. It is shown that the equilibrium model by Fraeijns de Veubeke can be derived by the approach of the hybrid stress model as a special case of semi-Loof elements.

1. INTRODUCTION

It is generally realized that an assumed stress hybrid element is based on the complementary energy principle using equilibrating stress field in the interior of the element and independent displacements along the element boundary. However, when a compatible displacement field can be constructed it is also possible

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to use the Hellinger-Reissner principle for the formulation. Indeed, it may not be generally known<sup>[1]</sup> that the original derivation of the assumed stress hybrid element was made by using the Hellinger-Reissner principle. In that derivation the assumed stresses happened to satisfy the equilibrium equations and the resulting element stiffness matrix is identical to that by the complementary energy principle, i. e. by a model which was later named hybrid model. Hybrid and mixed models, thus, are not mutually exclusive.

In this paper we discuss several problems associated with this hybrid/mixed method:

- 1) Different formulations of the assumed stress hybrid/mixed elements.
- 2) Rank deficiency in the assumed stress hybrid elements.
- 3) Semi-Loof elements for plates and shells by the hybrid formulation and formulation of the equilibrium element by the procedure of the hybrid stress model.

## 2. FORMULATIONS OF THE HYBRID/MIXED ELEMENTS

Three different versions of the hybrid element formulation are presented here. The first one is by the Hellinger-Reissner principle which can be expressed as:

$$\begin{aligned} \pi_R &= \sum_n \left[ \int_{V_n} \left\{ -\frac{1}{2} \underline{\sigma}^T \underline{S} \underline{\sigma} + \underline{\sigma}^T (\underline{D} \underline{u}) \right\} dV - \int_{S_{\sigma n}} \bar{\underline{T}} \underline{u} dS \right] \\ &= \text{Stationary} \end{aligned} \quad (1)$$

where

$\underline{\sigma}$  = stresses

$\underline{u}$  = displacements

- $\underline{S}$  = elastic compliance matrix  
 $\underline{D}$  = matrix of differential operators that defines  
the strain displacement relations  $\underline{\epsilon} = \underline{D} \underline{u}$   
 $\underline{T}$  = vector of surface tractions  
 $V_n$  = volume of the nth element  
 $S_{\sigma n}$  = boundary of the nth element over which trac-  
tions  $\underline{T}$  are prescribed

Here  $\sum_n$  denotes summation over all elements, and for simplicity in the present illustration, body forces are considered absent and displacements  $\underline{u}$  are assumed to satisfy the prescribed boundary conditions.

When the state of stress  $\underline{g}$  is in equilibrium, i.e.

$$\underline{D}^T \underline{g} = 0 \text{ in } V_n; \text{ and } \underline{v}^T \underline{g} = \underline{T} \text{ on } \partial V_n \quad (2)$$

where  $\underline{D}^T$  represents matrix of differential operators and  $\underline{v}$  represents the directional cosine of the surface normal, then by the divergence theorem the variational functional becomes

$$\Pi = -\sum_n \left[ \frac{1}{2} \int_{V_n} \underline{g}^T \underline{S} \underline{g} dV - \int_{\partial V_n} \underline{T}^T \underline{u} dS + \int_{S_{\sigma n}} \underline{T}^T \underline{u} dS \right] = -\Pi_{mc} \quad (3)$$

where  $\partial V_n$  is the entire boundary of the nth element, and the displacements along the boundary are now devoted by  $\underline{u}$ . It is noted that the above expression would be identical to  $-\pi_{mc}$  where  $\pi_{mc}$  is the functional corresponding to the modified complementary energy principle associated with the assumed stress hybrid model<sup>[2]</sup>. When the body force term is included, a mixed variational principle results. This also suggests that body force can be distributed rationally based on independently assumed displacement functions.

In case that both the equilibrium and compatibility conditions are satisfied both stresses and displacements can be expressed in terms of the same functions. The volume integral in Eq. (3) can be reduced into a surface integral along the boundary and another modified complementary energy principle can be written as follows<sup>(3)</sup>.

$$\pi_{mc}^* = \sum_n \left[ \int_{\partial V_n} \left( \frac{1}{2} \underline{T}^T \underline{u} - \underline{T}^T \underline{\tilde{u}} \right) dS + \int_{S_{\sigma n}} \underline{T}^T \underline{\tilde{u}} dS \right]$$

=stationary (4)

In the above expression the tractions  $\underline{T}$  and displacements  $\underline{u}$  arrive from the same stresses  $\underline{g}$

The steps to be taken in formulating the assumed stress hybrid elements by these three variational principles are indicated in Table 1. Here the approximation of the stresses in the interior of the element is a common step for all methods, and although many terms in these variational functional are different, the resulting  $\pi$ -functions are of the same form, i.e. in terms of stress parameters  $\underline{\rho}$  and nodal 'displacements  $\underline{q}$ . Since the stresses are independent for different elements, one can take variation with respect to the  $\rho$ 's in the element level and obtain expressions of  $\underline{\rho}$  in terms of  $\underline{q}$ . The final expression for  $\pi$  is now in terms of only nodal displacements  $\underline{q}$  as unknowns. The matrix  $\underline{k}$  is the element stiffness matrix given by

$$\underline{k} = \underline{G}^T \underline{H}^{-1} \underline{G} \quad (5)$$

Matrices  $\underline{k}$  can be assembled to form the stiffness matrix  $\underline{K}$  for the global system. Taking variation of  $\pi$  with respect to unrestrained nodal displacements leads to

the system of matrix equations of the finite element method:

$$\underline{K} \underline{q} = \underline{Q} \quad (6)$$

An investigation of the relative efficiency between the formulations of the element stiffness matrices by  $\pi_R$  and  $\pi_{mc}$  for rectangular block elements with 8-nodes was conducted<sup>(4)</sup> and the results indicated clearly that the computing time required for the  $\pi_R$  formulation is shorter than that by the  $\pi_{mc}$  formulation. The reason for this is that in the evaluation of each element of the matrix  $\underline{G}$  in Table 1, the formulation by  $\pi_R$  involves a single volume integral while for the formulation of  $\pi_{mc}$  the corresponding surface integral must be divided into six separate ones for the six individual faces.

The formulation of the assumed stress hybrid element by  $\pi_R$  has also been extended to the general 8-node hexahedron elements which have straight edges but may have non-flat faces. In formulating such a finite element by  $\pi_R$ , it is required to introduce over the element a set of curvilinear coordinates  $\xi, \eta, \zeta$  in the same way as the isoparametric element in the conventional assumed displacement method. It is well known that for conventional isoparametric elements, it is, in general, not possible to evaluate, in closed-form, the integrals required in the element stiffness matrix, hence numerical integration procedures are required. However, in the formulation of such elements by the assumed stress hybrid models the matrix  $\underline{H}$  can be evaluated in closed form. By recognizing identical elements in the  $\underline{H}$  matrix, it is possible to optimize the computing effort and as a result, the computer time needed

to evaluate the stiffness matrix by  $\pi_R$  is only a few percent higher than that for the conventional isoparametric element [5]. Discussions on the derivation of hybrid stress elements by  $\pi_R$  have also been given in Refs [6-11].

A typical application of the variational function  $\pi_{mc}^*$  is in two-dimensional linear fracture mechanics. Standard technique of complex stress functions in plane elasticity problems permits a proper approximate solution which satisfies the equilibrium and compatibility equations as well as the stress free boundary conditions at the surface of the crack. A super-element which contains an embedded crack can thus be derived to be used jointly with conventional finite elements for the analysis of elastic stress intensity factors [3].

### 3. RANK DEFICIENCY IN ASSUMED STRESS HYBRID ELEMENTS

In Table 1 the relation between the stress parameters  $\underline{\beta}$  and the modal displacement  $\underline{q}$  is governed by

$$\underline{K} \underline{\beta} = \underline{G} \underline{q} \quad (7)$$

It has been pointed out [2,12] that if  $m$  is the number of stress parameters and  $n$  is the number of generalized displacements of which  $r$  nodal displacements must be restrained to prevent rigid body motion, then when  $m$  is smaller than  $(n-r)$ , kinematic deformation modes will appear and the rank of the stiffness matrix will be less than  $(n-r)$ . It is well known that the condition;

$$m \geq (n - r) \quad (8)$$

is only a necessary condition for preventing any kinematic deformation modes, and an element may still be rank

Table 1 Formulation of assumed stress hybrid element

$\pi_R$	$\pi_{MC}$	$\pi_{MC}^*$
$\bar{q} = \bar{p} \bar{\beta}$		
$\left. \begin{aligned} \bar{u} &= A \bar{q} \\ \bar{D} \bar{u} &= \bar{B} \bar{q} \end{aligned} \right\} \text{in } V_n$	$\left. \begin{aligned} \bar{u} &= \bar{L} \bar{q} \\ \bar{T} &= \bar{R} \bar{\beta} \end{aligned} \right\} \text{on } \partial V_n$	$\left. \begin{aligned} \bar{u} &= \bar{U} \bar{\beta} \\ \bar{T} &= \bar{R} \bar{\beta} \\ \bar{u} &= \bar{L} \bar{q} \end{aligned} \right\} \text{on } \partial V_n$
$\pi = \pm \int_V \left( \frac{1}{2} \bar{\beta}^T \bar{H} \bar{\beta} - \bar{\beta}^T \bar{G} \bar{q} + \bar{q}^T \bar{Q}_n \right) dV$		
$\bar{H} = \int_{V_n} \bar{P}^T \bar{S} \bar{P} dV$		$\bar{H} = \frac{1}{2} \int_{\partial V_n} (\bar{R}^T \bar{U} + \bar{U}^T \bar{R}) dS$
$\bar{G} = \int_{V_n} \bar{P}^T \bar{\theta} dV$	$\bar{G} = \int_{\partial V_n} \bar{R}^T \bar{L} dS$	
$\partial \pi / \partial \bar{\beta} = 0 \rightarrow \bar{\beta} = \bar{H}^{-1} \bar{G} \bar{q}$		
$\pi = \pm \int_V \left( \frac{1}{2} \bar{q}^T \bar{K} \bar{q} - \bar{q}^T \bar{Q}_n \right) dV = \pm \left( \frac{1}{2} \bar{q}^T \bar{K} \bar{q} - \bar{q}^T \bar{Q} \right)$		
$\delta \pi = 0 \rightarrow \bar{K} \bar{q} = \bar{Q}$		



deficient even if the above inequality is satisfied.

A kinematic mode corresponds to nodal and boundary displacements for which no work is done by the assumed stress distribution, hence is also called zero energy mode. This means that

$$\underline{\beta} = \underline{H}^{-1} \underline{G} \underline{q} = 0 \quad \text{or} \quad \underline{G} \underline{q} = 0 \quad (9)$$

Since the assumed stresses are equilibrating, all rigid body modes will involve no external work. A kinematic deformation mode refers to element boundary displacements which indicate element deformation but involve no work from the assumed stresses.

One of the cases that kinematic deformation modes appear when the condition is satisfied is a twelve degree of freedom rectangular plate element under Kirchhoff assumption<sup>[13]</sup>. It is derived by assuming linear distributions in stress couples ( $M_x$ ,  $M_y$ ,  $M_{xy}$ ) within the element, and cubic distribution in lateral displacement ( $w$ ) and linear distribution in normal slopes ( $w_{,n}$ ) along each edge. The condition of Eq. (8) is satisfied. The element has three rigid body degrees of freedom yet the stiffness matrix has five zero eigenvalues. One can verify that the boundary displacements depicted in Fig. 1 and represented by the following equations will yield no work due to linear distributions of moments  $M_n$  and  $M_{ns}$  and hence uniform distribution in Kirchhoff shear  $V_n$  along the boundary:

$$\begin{aligned} \text{along } y = b, w = -x(a^2 - x^2); \quad w_{,y} = 0 \\ y = -b, w = x(a^2 - x^2); \quad w_{,y} = 0 \\ \text{along } x = +a, w = 0; \quad w_{,x} = 2a^2y/b \\ x = -a, w = 0; \quad w_{,x} = 2a^2y/b \end{aligned} \quad (10)$$

Another kinematic mode involves similar displacements with nonzero values of  $w$  and  $w_{,y}$  along  $x = \pm a$ , and  $y = \pm b$  respectively. To suppress these kinematic modes it is only necessary to add stress terms such as  $M_x = \beta_{10}xy$  and  $M_y = \beta_{11}xy$  or  $M_{xy} = \beta_{12}x^2 + \beta_{13}y^2$  which will lead to linear distribution of

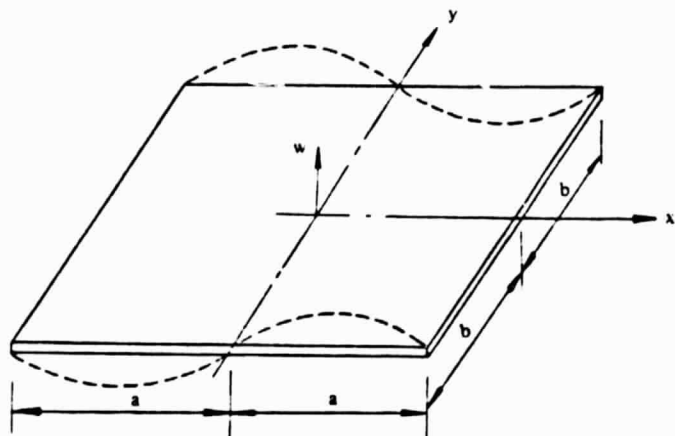


Fig. 1 Kinematic deformation mode of a rectangular plate element under linear moment distribution

shear force along the edges. For an irregular shaped quadrilateral plate element, the stiffness matrix obtained by linear distribution of  $M_x$ ,  $M_y$  and  $M_{xy}$ , would, in general, not be rank deficient.

Another 12 DOF plate elements derived by assuming linear distributions in stress-couple is a triangular element with  $w$ ,  $w_{,x}$  and  $w_{,y}$  at each corner and  $w_{,n}$  at

the midpoint of each side as nodal displacements [14,15] as shown in Fig. 2. For the twelve nodal displacements shown in the figure, the distribution of normal rotation  $w_{,n}$  along all edges are in the form of

$$w_{,n} = f(s) = C[1 - 6(s/l) + 6(s/l)^2] \quad (11)$$

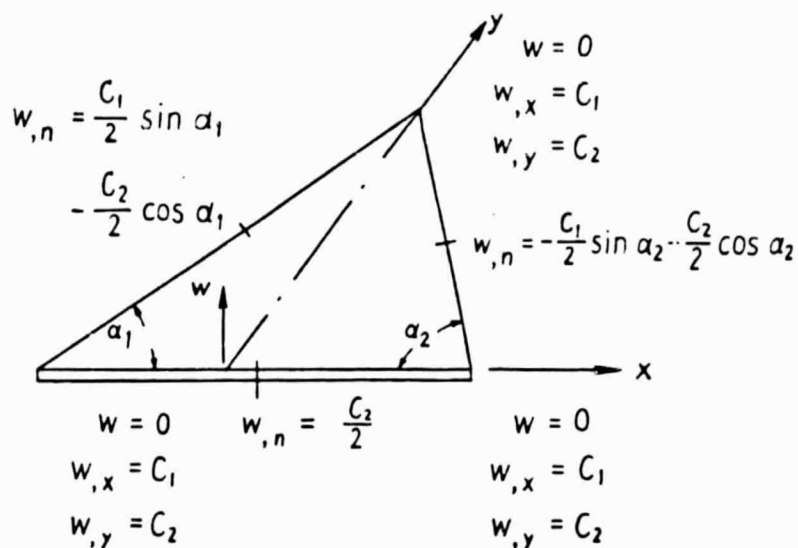


Fig. 2 kinematic deformation modes of a 12-DOF plate element under linear moment distribution

where  $s$  is the coordinate along the edge and  $l$  is the length of the edge. Here  $f(s)$  is symmetric about the midside point and its integral over  $l$  vanishes, thus the integral over the product of  $f(s)$  and any linear function in  $s$  will be zero. Also, along all edges the  $w$  distribution is cubic and is antisymmetric about their

midpoints. Now under linear distributions in stress couples, the bending moment  $M_n$  is linear along  $s$  and the Kirchhoff shear  $V_n$  is constant. They should do no work under the boundary of displacements described above. Since  $w$  is zero at all corners, the corner forces also do no work. The deformation pattern shown in Fig. 2 represents the combination of two independent kinematic modes and the rank of the stiffness matrix is seven.

Finally, a remark should be made about the rectangular membrane element derived by using 5  $\beta$  terms which was used as an example when the assumed stress hybrid element was first introduced. The element has eight degrees of freedom and three rigid body modes, hence, Eq. (8) is satisfied and for general arrangement of the reference axes the resulting element stiffness matrix will have a rank of five. However, for a square element if the diagonal lines are used as the reference axes as shown in Fig. 3, the resulting stiffness matrix will have a rank of only three. If the stress assumption is

$$\begin{aligned} \sigma_x &= \beta_1 + \beta_4 y \\ \sigma_y &= \beta_2 + \beta_5 x \\ \tau_{xy} &= \beta_3 \end{aligned} \tag{12}$$

then in the  $\xi - \eta$  coordinate system the corresponding assumed stress is

$$\begin{aligned} \sigma_\xi &= a_1 + a_4 \xi + a_3 \eta \\ \sigma_\eta &= a_2 + a_4 \xi + a_5 \eta \\ \tau_{xy} &= a_3 - a_4 \eta - a_5 \xi \end{aligned} \tag{13}$$

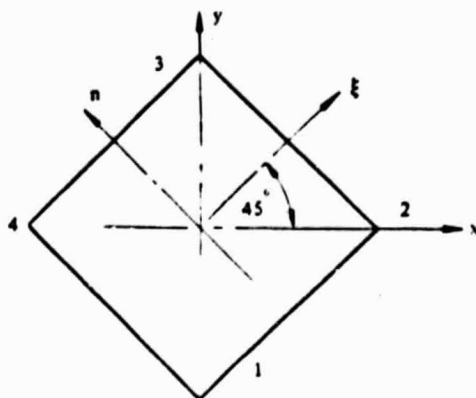


Fig. 3 Square membrane element with reference axes for stresses along the diagonals

In this case the following two nodal displacement patterns are kinematic modes:

$$(a) \quad u_{\xi_1} = -u_{\xi_2} = u_{\xi_3} = -u_{\xi_4} = 1$$

$$\text{and} \quad u_{\eta_1} = u_{\eta_2} = u_{\eta_3} = u_{\eta_4} = 0$$

$$(b) \quad u_{\xi_1} = u_{\xi_2} = u_{\xi_3} = u_{\xi_4} = 0$$

$$\text{and} \quad -u_{\eta_1} = u_{\eta_2} = -u_{\eta_3} = u_{\eta_4} = 1$$

Similarly, if the x-axis coincides with one diagonal of a general rectangular membrane element, the resulting element stiffness matrix by 5- $\beta$  assumption will have a rank of only four.

Spilker et al.<sup>(11)</sup> have studied the problem of rank deficiency of the 5- $\beta$  square element by first identifying all independent deformation modes that can uniquely define any displacements of the element and then substituting into Eq. (9) to see whether  $Q$ ,  $q$  vanishes. By changing the angle  $\theta$  between x and  $\xi$  axes

they were able to detect the two kinematic deformation at  $\theta = 45^\circ$ . This example on the membrane element also illustrates the danger of not taking complete polynomials in the stress terms in the hybrid element formulation. In that case, the resulting element stiffness matrix will be lack of invariance on and, indeed, it may be rank deficiency under certain reference coordinates.

For any newly developed hybrid stress elements it is suggested, in addition to Patch-test, an eigenvalue survey should be conducted in order to detect any kinematic deformation modes. If invariance of the stiffness matrix cannot be maintained, such a survey will have to be made for different coordinate systems used for the stresses. It should be noted that kinematic deformation modes can always be suppressed by adding appropriate stress terms.

#### 4. SEMI-LOOP ELEMENTS FOR PLATES AND SHELLS BY THE HYBRID FORMULATION

For the analyses of plates and shells, one of the difficult tasks is to match the compatibility at a node at which the reference planes of the elements are not coplanar. At such nodes all six degrees of freedom should be considered but for plate and shell elements a maximum of five degrees of freedom can be used at a node. Irons<sup>(18)</sup> has suggested the use of the so-called semi-Loop element for which the normal rotations along each edge are defined at nodes which are not located at corners of the element. By the conventional assumed displacement method it is still a difficult task to construct shape functions for a semi-Loop element. But,

it is a more or less routine procedure to formulate such an element for plates and for simple shells by the assumed stress hybrid method.

The equilibrium model of Fraeijs de Veubeke<sup>(19)</sup> always results to semi-Loof elements. It has been shown that this model and the assumed stress hybrid model can both be derived by the modified complementary energy principle<sup>(2)</sup>. The only difference is that for the former, tractions along each boundary are represented by generalized loading parameters and the corresponding fictitious nodal displacements are weighted integrals of the boundary displacements, while for the latter, boundary displacements are interpolated in terms of nodal displacements and the corresponding nodal forces are obtained from the variational sense. The expression for stiffness matrices are the same for both models except the G matrices in Eq. (5) are derived differently for the two models.

We can now show that the equilibrium model can also be formulated by boundary displacement interpolation in the same way as for  $\pi_{mc}$  in Table 1. Consider a problem for which certain nodal displacements are of the Semi-Loop type, hence, the corresponding boundary displacements are independent from one edge to the other. This means that for each element the boundary displacement continuity is not maintained at the corners.

For simplicity let us consider a straight boundary  $0 < s < l$ , with its traction  $T(s)$  represented by a polynomial of order  $m$ .

Such traction distribution can be represented by  $m+1$  nodal forces  $Q_1, \dots, Q_{m+1}$  at arbitrary locations  $s_1, \dots, S_{m+1}$  when the following conditions are satisfi-

ed:

$$\sum_{i=1}^{m+1} Q_i s_i^n = \int_0^l T(s) s^n ds, \text{ for } n = 0, \dots, m \quad (14)$$

We can easily show that if the boundary displacement  $u(s)$  is also a polynomial of order  $m$  then

$$\int_0^l T(s) u(s) ds = \sum_{i=1}^{m+1} Q_i u_i \quad (15)$$

where  $u_i$  is the nodal displacement at  $s_i$ . Conversely, if the boundary displacement is interpolated through the nodal displacements at the  $m+1$  nodes along the boundary, the corresponding nodal forces  $Q_i$  obtained by Eq. (15) will satisfy the conditions given by Eq. (14), hence, will be equipollent to the boundary traction distribution  $T(s)$ . When the boundaries of two neighboring elements are approximated by polynomials of the same order, a compatibility of the nodal displacements will then necessarily guarantee the reciprocity of the nodal forces and, hence, the pointwise interelement equilibrium condition is maintained. The equilibrium model by Fraeijs de Veubeke, thus, is a special semi-Loof element by the assumed stress hybrid model when the number of nodes along a boundary exceeds by one the order of the polynomial which represents the corresponding traction. An exception in this case is that for constant traction distribution there is only one node which should be located at the midpoint of the edge. For example, for the linear moment triangular plate element by Fraeijs de Veubeke<sup>[17]</sup>, the nodal displacements are lateral displacements  $w$  at the three corners and the three midside points and two normal



rotations along each edge. Here, the normal bending moment  $M_n$  is linear along each edge, hence, by using two corresponding nodal displacements  $w_n$  along the edge, the equilibrium conditions are satisfied exactly. It should be remarked that in this case the locations of the semi-Loof nodes may be arbitrary.

For the equilibrium model, kinematic deformation modes often exist in one element or in a group of elements and appropriate superelement technique is often required to suppress the mechanisms<sup>[20]</sup>. On the other hand, for the same choice of nodal displacements, the number of stress terms can be increased by the assumed stress hybrid model, hence, the possibility of any kinematic deformation modes can always be eliminated. Thus, the assumed stress hybrid method is a most promising approach for the semi-Loof element.

## 5. CONCLUDING REMARKS

For the formulation of the assumed stress hybrid finite elements there are different methods of approach to choose from and the selection of the appropriate stress fields requires sufficient physical insight. It is not likely that, in the future, hybrid stress elements can be formulated as simply as the use shape function routines and numerical integration techniques in the conventional assumed displacement method. But abundant knowledge about this method has been collected and future development of hybrid stress elements should be and can be made fool proof and should be implemented according to the most efficient method of approach.

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