# MULTIVARIATE SPLINE ALGORITHMS FOR CAGD

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## ABSTRACT

The theory of multivariate splines was developed in the early 1980s, and since then has generated increasing interest. Researchers in computer-aided geometric design (CAGD) hoped to get a new, useful tool for the representation and handling of surfaces. This interest in multivariate splines is based on three fruitful ideas:

Schoenberg's geometric definition of splines (Schoenberg, 1966)

Geometric recursion and subdivision

The Bernstein-Bézier representation

Multivariate splines are easily developed from the geometric definition of splines, given by Schoenberg in 1966: A polyhedron P that spans $\mathcal{R}^m$ is affinely mapped into $\mathcal{R}^s$, $s < m$. All points $p \in P$ that are mapped onto a single point $x \in \mathcal{R}^s$ from an m-s dimensional polyhedron Q(x). The volume V(x) of Q is called a spline function. It is a piecewise polynomial of order m-s and "smooth". Note that one can easily find different polyhedra defining the same spline.

The image of a vertex of P will be called a knot, while the image of an edge of P will be called a knot line, connecting two knots. These knot lines correspond to the segment boundaries.

Two special polyhedra present themselves for the definition of B-splines: a simplex S and a box or parallelepiped B, where the edges of S project into an irregular grid, while the edges of B project into the edges of a regular grid, as shown in figure 1. More general splines may be found by forming linear combinations of these B-splines, where the three-dimensional coefficients are called the spline "control points".

Note that univariate splines are simplex splines, where s = 1, whereas splines over a regular triangular grid are box splines, where s = 2.

Two simple facts render the development of the construction of B-splines:

Any "face" of a simplex or a box is again a simplex or box but of lower dimension.

Any simplex or box can be easily subdivided into smaller simplices or boxes.

The first fact gives a geometric approach to Mansfield-like recursion formulas that express a B-spline in B-splines of lower order, where the coefficients depend on x. By repeated recursion, the B-spline will be expressed as B-splines of order 1; i.e. piecewise constants. Considering the corresponding "nets" of control points, one gets de Boor - like algorithms for the calculation of a given linear combination of B-splines at x.

In the case of a simplex spline, the second fact gives a so-called "insertion algorithm" that constructs the new control points if an additional knot is inserted. In the case of a box spline, this fact gives the so called "subdivision" algorithm, which constructs a "refinement" of the control net (see figure 2).

If more than one vertex of a simplex is mapped into one knot, the corresponding spline function will be degenerate. In particular, if the knots form a simplex of $R^s$, the spline will be a "truncated Bernstein polynomial". In this case the Mansfield-like recursion formula for simplex splines degenerates to the well-known recursion formula for Bernstein polynomials, while the de Boor - like algorithm degenerates to the algorithm of de Casteljau, where the control points are called Bézier points. Clearly, the Bernstein expansion of a B-spline corresponds to a suitable simplicial decomposition of the simplex or box, such that the Bernstein representation of a B-spline can easily be constructed (see figure 3). Furthermore, especially if the B-spline is a truncated Bernstein polynomial, one gets the subdivision algorithm for Bernstein polynomials as well as a refinement of the Bézier net.

Fortunately, two of the above algorithms seem to fall in the class of non-tensor-product box splines:

The approximation of the spline by a repeated refinement for global representation

The construction of the Bézier net for application of the full Bernstein-Bézier method for local representation

It should be mentioned that both algorithms start from the control net and use - in the known cases - repeated "filling and/or averaging" procedures, as shown in figure 4 for the case of a triangular grid:

A rhombic scheme is filled with data from the previous step,

A new scheme is formed by line averaging these data.

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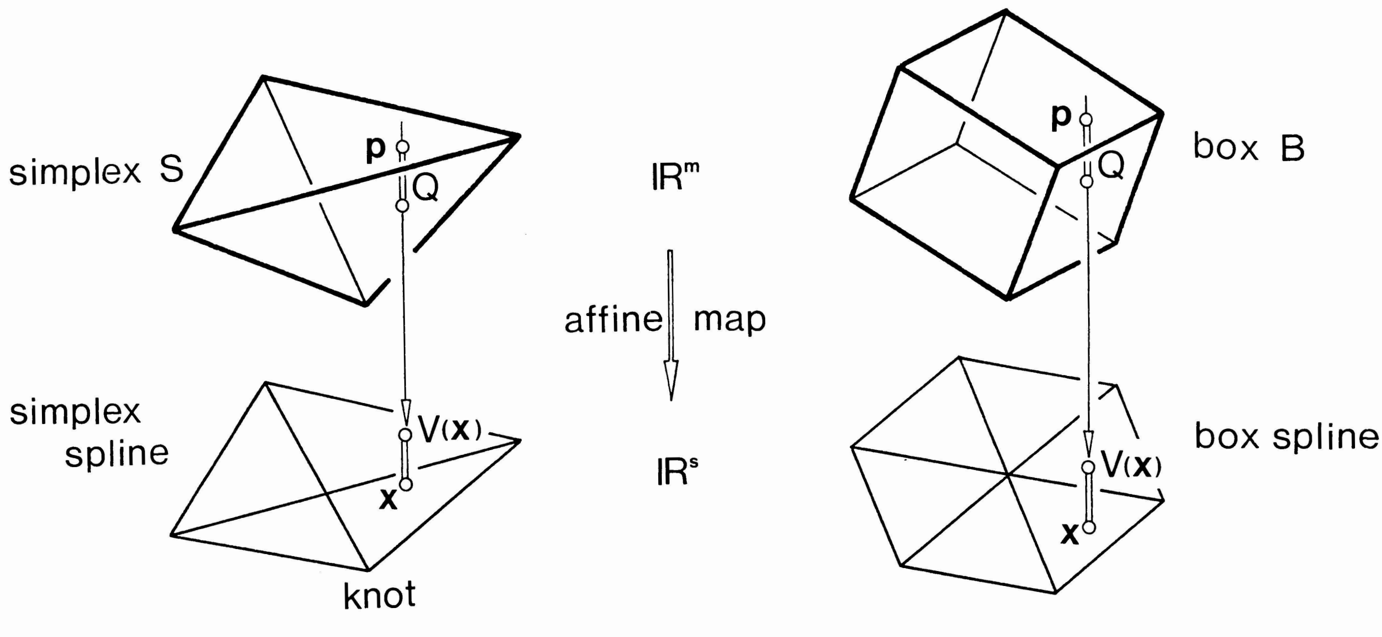


FIGURE 1

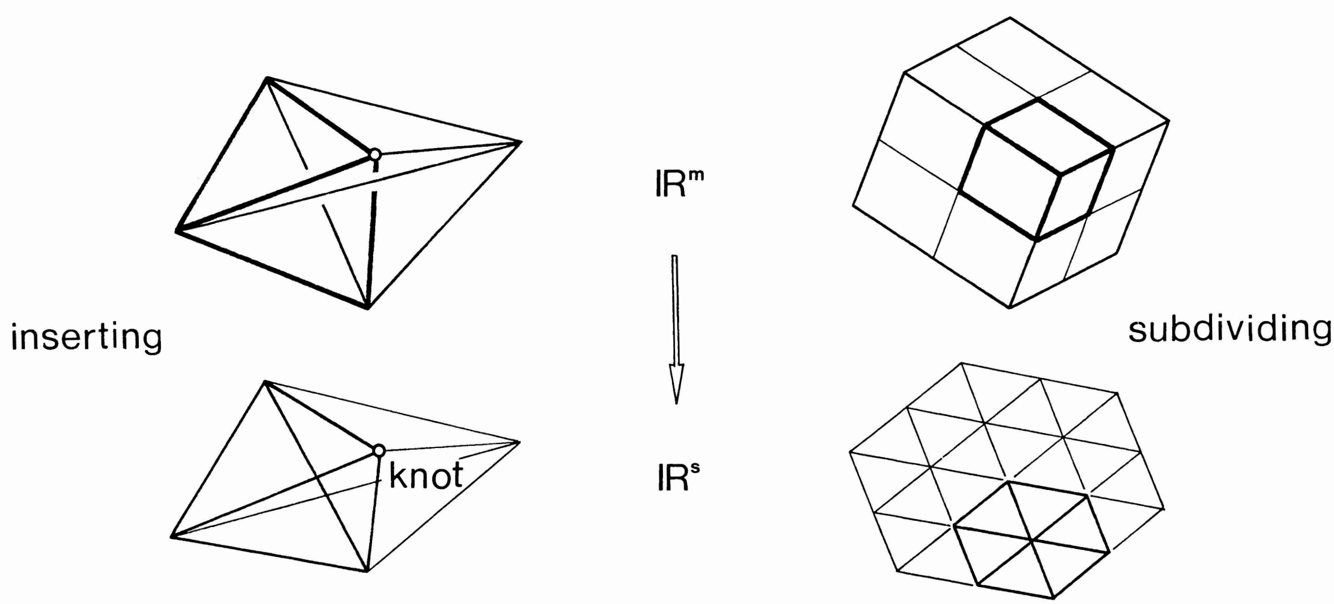


FIGURE 2

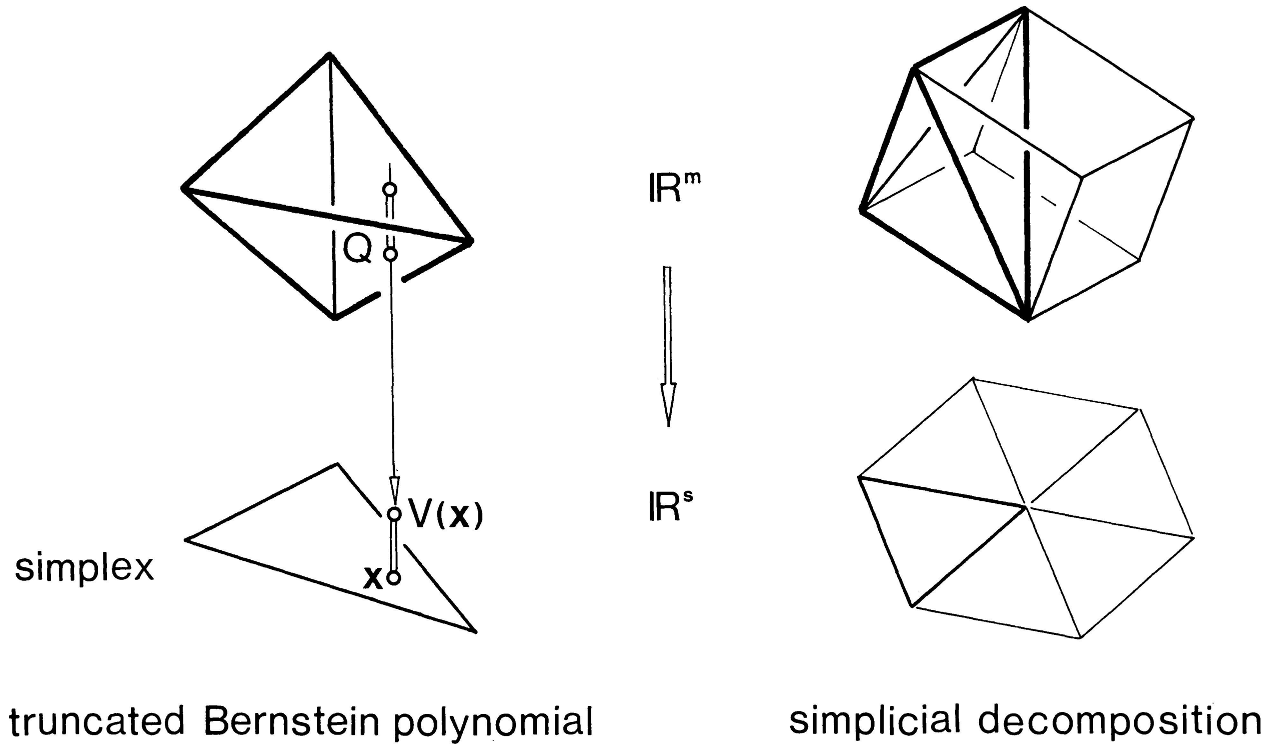


FIGURE 3

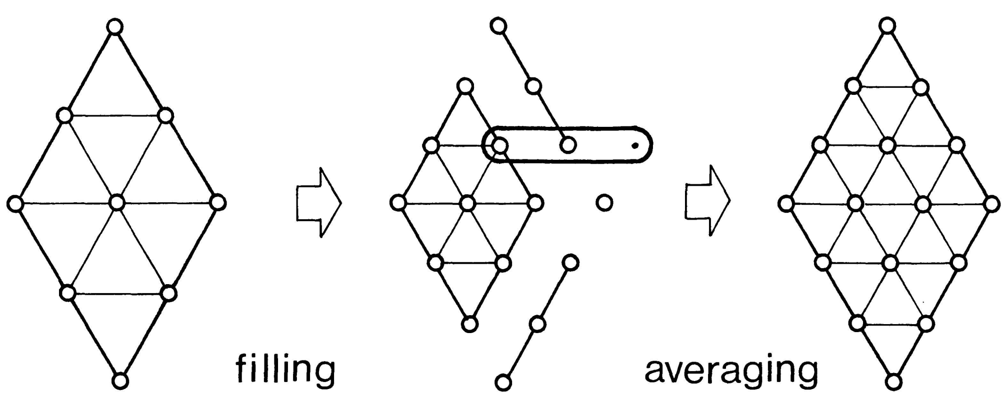


FIGURE 4