A Survey of Patch Methods

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Introduction

There are two broad classes of interpolation methods for surfaces: patch methods (Barnhill, 1985) and distance-weighted methods. We discuss patch methods in this paper. (Readers interested in distance-weighted methods should consult Barnhill and Stead (1984) and Franke (1982) and the references therein.) Patch methods are somehow a response to the fact that surface geometry is local, that is, only small parts of a surface are created at a time. We present the two categories of patches, transfinite patches and finite dimensional patches, followed by a discussion of trivariate patches. We do not discuss how to create the domains for patches nor data that are needed for the various schemes. Information on creating triangles and tetrahedra is given in Barnhill and Little (1984) and the references therein and information on creating gradient data in Stead (1984) and the references therein.

Transfinite Patches

In the car industry one has whole curves of information to interpolate. S. A. Coons (1964) fixed his attention on one rectangular set of four curves and constructed interpolants to position and cross-boundary derivative information all along the curves. W. J. Gordon (1969) called such interpolation "transfinite" because entire curves of data are interpolated. R. E. Barnhill, J. A. Gregory, and L. E. Mansfield noticed that the C^1 Coons patch did not interpolate to the cross-boundary derivative on two of the four curves. (Gregory, 1974; Barnhill, 1977).

The difficulty was traced to the lack of commutativity in the (1,1) derivatives (the "twists") in the Coons patch. Shortly thereafter Barnhill and Gregory (1975) created "compatibility correction" terms which introduced a variable twist, that is, both of the two implied twists are included.

The idea of transfinite triangular patches was initiated by Barnhill, Birkhoff, and Gordon (1973). They created a C^1 triangular Coons patch on the standard triangle with vertices (1,0) (0,1) and (0,0). For several years attempts were made to generalize these patches to an arbitrary triangle; one successful effort is in Klucewicz' (1977) thesis.

Barycentric coordinates are natural for triangles, and transforming between cartesion xy-coordinates and the barycentric coordinates b_i is defined by the equation

$$(1) \qquad (x,y) = \sum_{i=1}^{3} b_i V_i \qquad\qquad 1 = \sum_{i=1}^{3} b_i$$

where the V_i are the vertices of the triangle.

This equation has a solution if and only if the triangle has positive area, i.e., the V_i are not collinear. Barycentric coordinates are very useful in working with triangles which have been used by the finite element engineers for a long time. (Finite elements enter the story below.) Finally, F. F. Little proposed a barycentric calculus which enables us to take a derivative with respect to a given direction. More precisely, if $F = F(x,y)$, then F can be rewritten in terms of barycentric coordinates by substituting equations (1) for x and y: call $F(x,y) = F(b_1V_1 + b_2V_2 + b_3V_3) = G(b_1,b_2,b_3)$. Then we can take derivatives of F with respect to a direction e by use of the chain rule:

$$\frac{\partial F}{\partial e} = \sum_{i=1}^{3} \frac{\partial G}{\partial b_i} \frac{\partial b_i}{\partial e}$$

The Barnhill, Birkhoff, and Gordon interpolants for an arbitrary triangle have been used at Utah for some time, but have only recently been published: A C^1 BBG scheme (and its "discretization", defined below) is in Barnhill (1983). A C^2 BBG scheme is in Alfeld and Barnhill (1984). A bivariate BBG scheme, a trivariate BBG scheme, and a bivariate "radial Nielson" interpolant together with their discretizations are in Barnhill and Little (1984). (Trivariate patches are discussed below.) Other transfinite triangular interpolants are Gregory's (1983) symmetric schemes.

Finite Dimensional Patches

For some time engineers in finite element analysis have used piecewise polynomials defined over rectangles or triangles. (Strang and Fix, 1973; Zienkiewicz, 1977) These polynomials are the basis functions for interpolation schemes. However, the finite element method is used to calculate what we would call the positions and so the cardinal form of the interpolant, that is, the form in which the data functionals occur explicitly, is not needed. Thus, for example, the well-known 18 degree of freedom C^1 quintic's cardinal form was discovered only recently by Barnhill and Farin (1981) although the scheme has been used in finite element calculations for years.

Kolar, Kratochvil, Zenisek, and Zlamal (1971) discuss a "hierarchy" of polynomial interpolants defined over triangles. They give precise statements on the degree of the polynomial needed to obtain a certain global smoothness when the polynomial scheme is applied over each triangle in a network, for example, linear C^0, quintic C^1, nonic C^2, etc. Recently at Utah Whelan (1985) and others have developed cardinal forms for the C^2 nonic.

The finite element engineers have developed a second type of piecewise polynomial where a given triangle is subdivided and a piecewise scheme is defined over the subdivisions. In order to distinguish these two types of triangular schemes, we call the first type "macro" triangles and the second type (subdivision) "micro" triangles. The best known microtriangle scheme is the cubic C^1 Clough-Tocher element for which the macrotriangle is subdivided at its centroid into three microtriangles. Barnhill, Farin, and Little worked on creating a C^2 quintic Clough-Tocher interpolant with the macrotriangle subdivided into seven microtriangles, but have not yet succeeeded. Alfeld

(1985) has created such a scheme over nine microtriangles.

An important development for the discovery and description of piecewise polynomial schemes over triangles is Farin's (1980) generalization of the "Bernstein-Bezier" methods to arbitrary triangles. There are two levels of generalization here, since Bezier's original development was for rectangles. Sabin (1976) described triangular patches over equilateral triangles. Bernstein-Bezier methods are applicable to any piecewise polynomial scheme because they are one representation of the polynomial. The fact that geometric information can be fairly easily determined from this representation gives the methods its power.

Transfinite patches can be specialized to finite dimensional patches by discretizing the transfinite data, for example, by replacing a curve by its (univariate) cubic Hermite interpolant. The "serendipity elements" of the engineers can be viewed as examples of the discretization of transfinite patches.

A very recent idea is that of "visual continuity". C^1 visual continuity (VC^1) means tangent plane continuity. The concept becomes important when fitting together patches whose domains do not "match" such as a triangle and a rectangle. By not matching we mean that, for example, the isoparametric lines from a rectangle do not correspond to any standard lines in a triangle. Gregory and Charrot (1980) fit a triangular patch into a network of rectangular patches in a VC^1 way. Visual continuity has been discussed for triangular patches by Herron (1985) and for triangular and rectangular Bezier patches by Farin (1982).

Trivariate Patches

There are many applications which involve the creation of a surface in four-space, that is, a trivariate function. We mentioned above that trivariate BBG interpolants over tetrahedra are given by Barnhill and Little (1984). Another transfinite interpolant is formed by means of a convex combination of BBG projectors by Alfeld (1984a). A C^1 Clough-Tocher-like tetrahedral interpolant, which is quintic over four subtetrahedra and requires C^2 data, is presented by Alfeld (1984b). A second C^1 tetrahedral interpolant, which is cubic over twelve subtetrahedra and requires C^1 data, has just been announced by Worsey and Farin (manuscript in preparation). In fact this is a special case of their n-dimensional simplicial interpolants. Gregory (1985) has generalized his symmetric triangular interpolant (a convex combination of projectors) to n-dimensional simplices.

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