WHEN IS A BERNSTEIN-BÉZIER CURVE THE GRAPH OF A FUNCTION?

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In this paper we consider the question of determining when a Bernstein-Bézier cubic curve in the plane can be represented as the graph of function in some fixed orthogonal coordinate system.

One lets θ be an arbitrary point in the plane and D and N be nonzero orthogonal vectors in the plane. Then every point P in the plane can be written in the form $\theta + uD + vN$ for a unique choice of scalars u and v. The triple (θ,D,N) is an orthogonal coordinate system and the coordinates of P in this coordinate system are u and v. If for $t_1 \leqslant t \leqslant t_2$, $t_1 < t_2$,

$$B_3(t) = \sum_{k=0}^{3} P_k \binom{3}{k} t^k(1-t)^{3-k}$$

where the P_k's are given points in the plane, then one can also write

$$B_3(t) = \theta + u(t)D + v(t)N, \quad t_1 \leqslant t \leqslant t_2$$

for some functions u(t) and v(t). The curve $B_3(t)$ is said to be the graph of a function in the coordinate system (θ,D,N) if v(t) can be written

$$v(t) = f(u(t)), \quad t_1 \leqslant t \leqslant t_2$$

for some real valued function f defined on the range of u. The function v(t) has such a representation if and only if: whenever u(t) takes on the same value for two distinct values of t, then so does v(t).

To aid in the analysis one can introduce the notion of a curve being monotone in a given direction. The curve, $B_3(t)$, $t_1 \leqslant t \leqslant t_2$, is monotone in the direction D if whenever $t_1 \leqslant \alpha < \beta \leqslant t_2$, one has $(B_3(\beta)-B_3(\alpha))\cdot D > 0$. If $B_3(t)$ is monotone in the direction, D, then $B_3(t)$ is the graph of a function in the coordinate system (θ,D,N). However, the converse is not true in general.

We classify those curves of the form, $B_3(t)$, which can be represented as the graph of a function in the coordinate system (θ,D,N), by studying separately those which are monotone in the direction $\pm D$ and those which are not.

Those curves, $B_3(t)$, which are not monotone in the directions $\pm D$ and which are graphs of functions in (θ,D,N), are necessarily straight lines.

For the other case one has, as a preliminary theorem: the curve, $B_3(t)$, $t_1 \leq t \leq t_2$, is monotone in the direction D if and only if for each $t_1 < t_0 < t_2$, either

(1) $B_3'(t_0) \cdot D > 0$

or

(2) $B_3'(t_0) = 0$ and $B_3''(t_0) \neq 0$ and $B_3''(t_0) \cdot D = 0$

or

(3) $B_3'(t_0) . D = 0$ and $B_3'(t_0) \neq 0$

and

$$B_3''(t_0) = \lambda B_3'(t_0) \text{ for some scalar } \lambda$$

and

$$B_3'''(t_0) \neq \mu B_3'(t_0) \text{ for all scalars } \mu$$

The goal of the study has been to translate these analytic statements into geometric constraints on the control points, P_k's, of the curve $B_3(t)$. Necessary and sufficient conditions are generated by first studying necessary and sufficient conditions on the control points for cusps and then for inflection points, since each of these phenomena has to be dealt with to obtain the final theorem. The analysis is quite lengthy and will be reported on (in print) elsewhere.

This research has disclosed several unsolved elementary interpolation problems, each of which appears to be important.

Finally, it is noted that the notion of monotone in a direction D may be useful to those who study shapes of curves and surfaces since it is a way to describe the empirical notion that a given curve appears to be "headed in a given direction."