THE DEFINITION AND COMPUTATION OF A METRIC ON PLANE CURVES

THE MEANING OF A "FACE" ON A GEOMETRIC MODEL

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I shall talk about two separate topics as indicated in the title. The first topic concerns the comparison of plane curves. This work arose from the following problem. A surface on a part is generated by a plane curve, which is an interpolating spline. The part is manufactured by an outside vendor using the given interpolation points, but an unknown interpolation technique. The points on the machined curve must be checked to verify that they lie sufficiently close to the original definition. In this case the vendor supplies what is known as the APT-CL-file, which is a set of points defining a piecewise linear curve that defines the actual movement of the machine tool. APT (Automatic Programmed Tool) is a computer program for controlling numerical machine tools. The problem was to calculate a "distance" between these two curves.

There are many other cases where a curve is approximated by some process or algorithm. For example, plotting a curve is usually an approximation technique, and the generation of an offset curve may involve approximation. Other applications include the comparison of surfaces and the inspection of parts. Surfaces may be compared by examining nets of plane curves. Part inspection may involve the determination of a curve or surface from a finite set of measurements.

When the curves are parametric we can use function approximation theory to compare their coordinate functions. However, this would make the comparisons dependent on the particular parameterizations. We could use a canonical parameterization, such as parameterization by arc length, but this is hard to compute and the two curves that we wish to compare may not have the same total arc length.

We shall define a curve to be a locus of points without any underlying parameterization. We shall define a metric on a class of plane curves, give a finite computation of this metric for the case of piecewise linear curves, and show how to approximate curves that have bounded curvature by piecewise linear curves. In this way we can compute a bound on the "distance" between two curves. These techniques have been implemented in Fortran.

We shall prove some standard preliminary propositions. Let (M,d) be a metric space and A a subset of M. Define

$$d(x,A) = \inf\{d(x,y) : y \text{ in } A\}$$

DEFINITION. Let A and B be compact sets. The distance between A and B is defined to be

$$d(A,B) = \text{Maximum}\{ \sup_{x \in B} (d(x,A)), \sup_{x \in A} (d(x,B))\}$$

This is the radius of the largest circle that has center on one of the curves, and just touches the other. Note: we have given different meanings to "d." The particular meaning intended is determined by context.

PROPOSITION. Either there exists an x in A so that $d(A,B) = d(x,B)$, or there is an x in B so that $d(A,B) = d(x,A)$.

PROPOSITION. Given a family F of compact sets, (F,d) is a metric space.

We will prove the triangle inequality. Without loss of generality assume $d(A,C) = d(x,C)$ for some x in A. Since $f(w)=d(x,w)$ is continuous on the compact set B, there exists a y in B so that $d(x,B) = d(x,y)$. By definition

$$d(A,B) \geq d(x,B) = d(x,y)$$

Also there exists a z in C so that $d(y,C) = d(y,z)$

We now have

$$d(x,C) \leq d(x,z) \leq d(x,y) + d(y,z)$$

Thus

$$d(A,C) = d(x,C) \leq d(x,B) + d(y,C) \leq d(A,B) + d(B,C)$$

DISTANCE FROM A POINT TO A LINE SEGMENT

This computation is fairly obvious. The distance is either the distance from the point to the line containing the segment, or the distance from the point to one of the end points of the segment.

FINITE COMPUTATION FOR PIECEWISE LINEAR CURVES

The piecewise linear curve is a union of line segments. We call the end points of the segments knots. Given two curves A and B we say that they satisfy the knot condition if the metric is realized at either a knot of A or a knot of B. That is, either there is a knot a in A so that

$$d(A,B) = d(a,B)$$

or there is a knot b in B so that

$$d(A,B) = d(b,A)$$

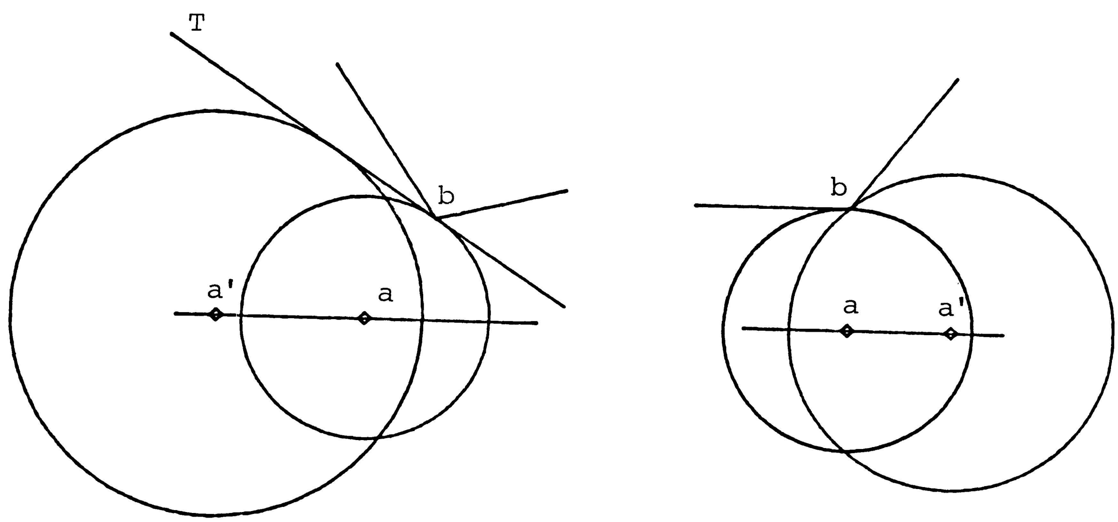
PROPOSITION. Suppose the pair (A,B) does not satisfy the knot condition. Suppose $d(A,B) = \sup\{d(x,B)\}$. Then there exists a point a in A so that $d(A,B) = d(a,B)$ and the circle of radius $r = d(A,B)$ and center a meets B in two or more points.

PROOF. Define $L(x)$ to be a line segment that contains point x (there will be two such segments if x is a knot). Assume that for every a such that $d(a,B) = d(A,B)$ the circle of radius $r = d(A,B)$ and center a meets B in only one point. Let a be a point so that

$$d(a,B) = d(A,B)$$

Then the circle of radius $r = d(A,B)$ and center a meets B in only one point b and

$$d(A,B) = d(a,b)$$



Let T be the line tangent to the circle at b. There are three cases.

(1) $L(a)$ and T are not parallel.

(2) $L(a)$ is parallel to T and some $L(b)$ is not parallel to T.

(3) $L(a)$ and every $L(b)$ are parallel to T.

Case 1. The line T separates the plane into two half spaces. The circle lies in one and every $L(b)$ lies in the other. This follows from the definition of $d(A,B)$. Because the circle touches B only at b and T is not parallel to $L(a)$, we may move the center a to a new center a' so that the radius of the circle increases, and does not contain a point of B in its interior (see the left figure). But this contradicts the definition of $d(A,B)$.

Case 2. T is parallel to $L(a)$ and some $L(b)$ does not lie in T. Again we may slightly alter a to a' to increase the radius of the circle (see the right figure). This is a contradiction.

Case 3. L(a) is parallel to T and every L(b) lies in T. Then we may translate the circle along L(a) until one of the following occurs.

(i) The center becomes a knot of A.

(ii) The circle contacts a second point of B.

(iii) Some L(b) does not lie in T.

Event (i) cannot happen because our curves do not satisfy the knot condition, and (ii) cannot happen because our assumption was that a could not be chosen so that the circle contacts B in two or more points. Therefore (iii) must occur. However, we have shown that cases (1) and (2) above are contradictions. Thus (iii) is a contradiction. Therefore our original assumption is false and the proposition is proved.

With the aid of this proposition we can construct an algorithm to compute d(A,B). Obviously the circle can not contact a single segment in two points. Thus we need only compute all circles that have centers on A and just touch B at two points. The points where the circle contacts B must be either tangent points, or knots. That is if the circle C(a,r), which has center a and radius r, contacts B at b where b is an element of segment I, and b is not an end point, then C(a,r) is tangent to I. There are only three cases.

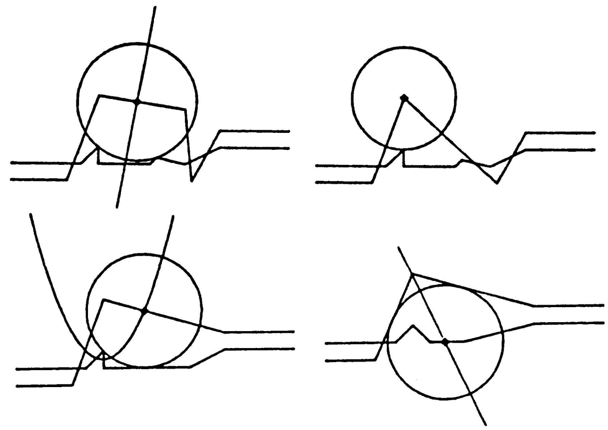
(1) The center a lies on the locus of points equidistant from two knots of B.

(2) The center lies on the locus of points equidistant from two lines. The lines contain segments of B.

(3) The center lies on a locus of points equidistant from a knot of B and a line, which contains a segment of B.

Thus we need only compute perpendicular bisectors of segments, angle bisectors of angles, and parabolas. The parabolas are defined by the knot-line pairs of B. The knot is the focus and the line is the directrix.

To find all the candidates for centers, we intersect the various loci with the curve A. We must also of course reverse the roles of A and B in case d(A,B) = d(b,A) for some b in B. There are then four cases in total, including the case of a center occurring at a knot. These cases are illustrated below.



The various loci are computed using projective representations and being careful to avoid excessive round-off error and making sure that special cases are handled correctly.

We can avoid doing the computations for all point-point, point-line, and line-line pairs by computing certain distances between segments and establishing certain inequalities.

THE PIECEWISE LINEAR APPROXIMATION

We will show that if we know a bound on the curvature then we can construct a piecewise linear curve that approximates a given curve arbitrarily closely.

Let g_r be a function whose graph is a circular arc of radius r. The center of the arc is below the x-axis and it passes through the points (a,0) and (b,0) where $a < b$. Let $\kappa(f)$ be the magnitude of the curvature of f.

PROPOSITION. Let f be a twice differentiable function such that $f(a) = f(b) = 0$. If f is greater than g_r anywhere on the interval (a,b), then there is a point in (a,b) where

$$\kappa(f) > 1/r.$$

PROOF. Let $h = f - g_r$. Suppose that f is greater than g_r at some point of (a,b). Let x be a point of (a,b) where the maximum value of h is attained. Therefore

$$h'(x) = 0 \quad \text{and} \quad h''(x) \le 0$$

Now $g_r'' < 0$, so

$$h''(x) = f''(x) + \mathrm{Abs}(g_r''(x)) \le 0$$

Therefore

$$\mathrm{Abs}(f''(x)) \ge \mathrm{Abs}(g_r''(x))$$

$f'(x) = g_r'(x)$, because $h'(x) = 0$. It follows from the expression for the curvature that

$\kappa(f) \geq \kappa(g_r) = 1/r.$

We can repeat the argument with a slightly smaller r. Thus

$\kappa(f) > 1/r.$

COROLLARY. If $\kappa(f) \leq 1/r$ on (a,b) then $f \leq g_r$ on (a,b).

PROPOSITION. Let C be a curve connecting two points. Suppose $\kappa(C) \leq 1/r$. Let L be the line segment joining the two points. Suppose any line passing through the segment and perpendicular to it meets C in exactly one point. Let the length of the segment be ℓ. If

$\ell/2 < r$

then

$d(C,L) \leq r - \mathrm{Sqrt}(r^2 - \ell^2/4)$

This proposition provides an obvious construction of a piecewise linear curve that approximates a given curve to arbitrary accuracy provided we know a bound on the curvature.

Perhaps these techniques can be generalized to surfaces using approximating polygons. However, for curves the piecewise linear approximation in some sense approximates both curvature and arc length. For surfaces the analog is not true. It is well known that triangulations can be found for surfaces in such a manner that the triangles approximate the surface arbitrarily closely, yet have arbitrarily large surface area (see [1]).

FACES

The second topic I wish to talk about concerns faces on geometric models. Systems for representing the surface of a solid model have been constructed (see [4] for example). These systems give both a topological, that is an abstract, and a geometric representation. The concepts involved are such things as vertices, edges, loops, faces, and surfaces. The topological representation leads to an obvious triangulation and thus to a simplicial complex. Therefore the model may be treated conventionally. This simplicial complex has a geometric realization. There is a dual representation involving faces on specific surfaces. These faces are bounded by "loops." The questions to be discussed are, "Under what circumstances do these geometrical faces make sense?" and "How can they be explicitly defined?" Also, "When are these 'geometrical' faces homeomorphic to the realization of the abstract (topological) face?"

Faces here are essentially discs with holes. Algebraic topology in effect treats faces as discs with no holes. Storage and computational constraints force us to deal with this new type of face.

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