# Triangular Bernstein-Bézier Patches, A Survey and New Results

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Triangular Bernstein-Bézier patches are an alternative to the standard rectangular ones. They are maps of triangular domains into $R^3$, where the patches are described by control nets similar to the rectangular case. One difference is the definition of surface normals at the patch corners: rectangular patches have inherent problems with so-called degenerate corners, i.e. edges collapsing into one point. Triangular patches do not have these problems. Since triangular shapes arise rather frequently in the design of complicated surfaces (e.g. interior car body panels), it seems that for such surfaces triangular patches are a promising addition to the usually employed rectangular patches.

Another major application is the formulation of interpolants to solve the scattered data problem: here Bernstein-Bézier patches provide a very powerful tool; they can be used to discover new schemes as well as to discover new properties of existing methods.

A survey of known results is given, including

*Subdivision:* The recursive evaluation of triangular Bernstein-Bézier patches generates control nets for the three subtriangles of the original patch that are generated by the point of evaluation. This algorithm is important for applications such as contouring and rendering: since every patch is contained in the convex hull of its control net, a simple rejection procedure exists. If a patch cannot be rejected, it will be subdivided and the rejection test will be performed for all subtriangles.

*Continuity conditions:* For complicated surfaces, continuity of adjacent patches must be ensured. Conditions for continuity of arbitrary order are derived from the recursive evaluation procedure. In the design of surfaces, simple continuity is sometimes too restrictive and the concept of "visual continuity" must be invoked. This means that along common patch boundaries, tangent planes from both patches must coincide, but partials or directional derivatives need not be coplanar. Smoothness conditions of this form are easily formulated in terms of control nets.

*Connection to rectangular patches:* For hybrid surfaces, composed of both rectangular and triangular patches, continuity conditions must be established. In most cases this again means "visual continuity". The resulting conditions are similar to the ones for triangle/triangle conditions.

*Interpolants in Bernstein-Bézier form:* Triangular interpolants, such as the Clough-Tocher element[1], are presented in Bernstein-Bézier form. These interpolants aim at the solution of the scattered data problem and define a surface that is piecewise defined over a triangulation of the given data points. The surface will be a Bernstein-Bézier patch over each triangle and will satisfy certain imposed continuity conditions. The Bernstein-Bézier method turns out to be a very useful tool in analyzing such interpolants as well as facilitating their development.

Several new results are presented, including

*Interpolants in three or more dimensions:* New generalizations of the above interpolants are presented. These have applications in scientific computing such as interpolation to measurements collected from 3D data sites. These data sites will be tessellated into tetrahedra and a piecewise polynomial surface will be defined over each of them, again with certain imposed continuity conditions. The Bernstein-Bézier form facilitates the construction and understanding of these interpolants considerably. Higher dimensional problems can be handled by an inductive generalization of the two- and three-dimensional interpolants.

*Reflection lines for Bernstein-Bézier triangles:* Reflection line simulation is a particularly effective surface interrogation tool. Its origins are in the car industry, where the aesthetics of a car body are judged by the reflection pattern generated by a set of parallel fluorescent light bulbs. Bernstein-Bézier patches lend themselves easily to the computation of reflection lines.

*The solvability of interpolation problems:* Interpolants over arbitrary triangulations are of interest in the field of scattered data interpolation. The dimensions of the underlying linear spaces are unknown. So of course B-spline-like basis functions are not known either. Some recent progress is described.[2]

## References


1. G. Strang and F. Fix (1973): An Analysis of the Finite Element Method. Prentice Hall.

2. W. Böhm, G. Farin, J. Kahmann (1984): A Survey of Curve and Surface Methods in CAGD. Computer Aided Geometric Design, vol. 1, pp. 1-60.