A UNIFORM SUBDIVISION METHOD FOR TRIANGULAR BEZIER PATCHES

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1. Introduction

Bézier curves and tensor-product Bézier surface patches have been used extensively in the computer graphics/computer aided design community for the modeling of physical objects for automated design. In 1980, Lane and Riesenfeld [Lane-Riesenfeld, 1980] presented algorithms for the evaluation of Bézier curves and surfaces, and showed how these algorithms could be used in solving many of the geometric problems in CAD, and how they could be used in the computer graphic rendering problems for such surfaces. These algorithms were based upon an elegant subdivision technique, which was developed directly from the mathematical properties of the Bézier curve. This technique allows the manipulation of a curve/surface without the need to evaluate the blending functions and their derivatives.

Triangular Bézier patches have also been studied for the same reasons. Farin [Farin, 1982] has used them successfully in the interpolation of scattered 3-d data and Petersen [Petersen, 1984] has recently exhibited their use in a contouring algorithm. Subdivision of these patches has centered on a method [Farin, 1982] which subdivides the patch into three subpatches using intermediate control points calculated by the de Casteljau algorithm [de Casteljau, 1959]. However, due to the fact that one side of each of the three subtriangles is one of the sides of the parent triangle, the sides of the resulting subtriangles do not reduce uniformly in length. This implies that the polygonal rendering algorithms [Lane-Carpenter, 1979; Blinn, Carpenter, Lane, Whitted, 1980] (which require surfaces to be subdivided until they are flat) cannot be used with this subdivision method. (It is noted that Petersen used this method in his contouring algorithm, however, an extra subdivision step was added to insure that the lengths of the sides reduced uniformly.)

This paper presents a subdivision method for Bézier triangles that parallels the development of Lane-Riesenfeld for Bézier Curves. The method is developed from the mathematical description of the patch, and the resulting algorithm does not require evaluation of the blending functions. This method subdivides the triangle uniformly so that the lengths of the sides of the subtriangles are uniformly reduced. This method is a member of the class of methods first presented by Goldman [Goldman, 1983], which contain subdivision methods that arise from degenerate subdivision of Bézier tetrahedra. In this paper, we show that a certain method is a natural extension of the Lane-Riesenfeld results for Bézier curves.

2. Bézier Triangles

The Bézier triangle of degree n, defined by the set of control points $\overline{P} = \{P_{i,j,k} : i+j+k=n\}$ over the triangle T is given by

$$B_n(\bar{P}\,;T) = \sum_{i+j+k=n} P_{i,j,k}\, B_{i,j,k}(\mu_1,\mu_2,\mu_3)$$

where

$$B_{i,j,k}(\mu_1,\mu_2,\mu_3) = \frac{n!}{i!\,j!\,k!}\,\mu_1^i\,\mu_2^j\,\mu_3^k$$

are the bivariate Bernstein polynomials of degree n, using the barycentric coordinate system defined on the triangle T.

3. Subdivision of Bézier Triangles

Consider the triangle T with vertices P_1, P_2, P_3, where $P_i \in \mathbf{R}^2$. Let T be parameterized by the barycentric coordinates (μ_1,μ_2,μ_3). Consider the subdivision of T given by the following sets

$$T_1 = \left\{ P \in T : P = \mu_1 P_1 + \mu_2 P_2 + \mu_3 P_3, \text{ where } \mu_1 \geq \frac{1}{2}, \mu_2 \geq 0, \mu_3 \geq 0 \right\}$$

$$T_2 = \left\{ P \in T : P = \mu_1 P_1 + \mu_2 P_2 + \mu_3 P_3, \text{ where } \mu_2 \geq \frac{1}{2}, \mu_1 \geq 0, \mu_3 \geq 0 \right\}$$

$$T_3 = \left\{ P \in T : P = \mu_1 P_1 + \mu_2 P_2 + \mu_3 P_3, \text{ where } \mu_3 \geq \frac{1}{2}, \mu_1 \geq 0, \mu_2 \geq 0 \right\}$$

$$T_4 = \left\{ P \in T : P = \mu_1 P_1 + \mu_2 P_2 + \mu_3 P_3, \text{ where } \mu_1 \leq \frac{1}{2}, \mu_2 \leq \frac{1}{2}, \mu_3 \leq \frac{1}{2} \right\}$$

The subdivision algorithm then proceeds as follows.

Subdivision Algorithm

Let $\bar{P} = \{P_{i,j,k} : i+j+k=n\}$ be a polygonal mesh defined over the triangle T and let

$$P_{i,j,k}^{[n_1,n_2,n_3]} = \begin{cases} \frac{1}{2}\left(P_{i,j,k}^{[n_1-1,n_2,n_3]} + P_{i-1,j+1,k}^{[n_1-1,n_2,n_3]}\right) & \text{if } n_1 > 0 \\[2ex] \frac{1}{2}\left(P_{i,j,k}^{[n_1,n_2-1,n_3]} + P_{i,j-1,k+1}^{[n_1,n_2-1,n_3]}\right) & \text{if } n_2 > 0 \\[2ex] \frac{1}{2}\left(P_{i,j,k}^{[n_1,n_2,n_3-1]} + P_{i+1,j,k-1}^{[n_1,n_2,n_3-1]}\right) & \text{if } n_3 > 0 \\[2ex] P_{i,j,k} & \text{if } n_1 = n_2 = n_3 = 0 \end{cases}$$

Then

$$B_n(\overline{P}\ ;\ T) = B_n(\overline{Q}_i\ ;\ T_i),\ \text{for } i = 1, 2, 3, 4$$

where

$$\overline{Q}_1 = \left\{ P_{i,0,k}^{[m,0,k]} : i+k=n\ ,\ m = 0, \ldots, i \right\}$$

$$\overline{Q}_2 = \left\{ P_{i,j,0}^{[i,m,0]} : i+j=n\ ,\ m = 0, \ldots, j \right\}$$

$$\overline{Q}_3 = \left\{ P_{0,j,k}^{[0,j,m]} : j+k=n\ ,\ m = 0, \ldots, k \right\}$$

$$\overline{Q}_4 = \left\{ P_{i,j,k}^{[i,j,k]} : i+j+k=n \right\}$$

Lemma

Let $\overline{Q} = \left\{ Q_{i,j,k} : i+j+k = n \right\}$ be the control points for the subdivided Bézier triangle over T_4; then we have

$$\begin{aligned} |\ Q_{i+1,j,k} - Q_{i,j,k}\ | &\leq M \\ |\ Q_{i,j+1,k} - Q_{i,j,k}\ | &\leq M \\ |\ Q_{i,j,k+1} - Q_{i,j,k}\ | &\leq M \end{aligned}$$

where

$$M = \frac{1}{2}\max\left\{ \begin{array}{l} |\ P_{i+1,j,k} - P_{i,j,k}\ | \\ |\ P_{i,j+1,k} - P_{i,j,k}\ | \\ |\ P_{i,j,k+1} - P_{i,j,k}\ | \end{array} : i+j+k = n\ , \text{and}\ i,j,k < n \right\}$$

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