PARAMETERIZATION IN GRID GENERATION*

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The mathematical description of a physical object is an absolute necessity in solving nearly any problem in computational fluid dynamics or related fields where one must compute the numerical solution of partial differential equations. The selection of points on which to compute a numerical simulation differs from the geometry definition procedures in computer-aided design. Whereas in the latter case decisions are often based on aesthetics, the distribution of grid points for calculating the solution of partial differential equations must be chosen so as to include consideration of truncation error, stability, and the resolution of the solution near boundary layers and shocks (ref. 1). It is therefore important to be able to specify the distribution of points along a grid line.

The problem of distributing points along a curve will now be considered. It will be assumed that the curve is defined parametrically. The objective is to select a set of parameter values so that the corresponding points on the curves are properly distributed. The distribution may be based on some intrinsic property of the curve such as arc length or curvature.

Suppose a curve is given parametrically by the equation

$$r = r(\eta), \ 0 \le \eta \le 1$$

where $r = (x,y,z)$. The desired set of values for η will be defined by introducing a reparameterization of the curve

$$r = r(\eta(t)), \ 0 \le t \le 1$$

For each value of t, the arc length derivative $d(t)$ will be defined so that

$$[d(t)]^2 = r_t \cdot r_t$$

The function $d(t)$ cannot be completely arbitrary since it must satisfy

$$\int_0^1 d(t)dt = L \qquad (1)$$

where L is the length of the curve. Since r is a composite function of t, we also have

$$r_t \cdot r_t = (r_\eta \cdot r_\eta)\eta_t^2$$

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*Research supported by NASA Langley Research Center under Grant No. NSG 1577.

Therefore, when d(t) is given, the function $\eta(t)$ is the solution of the initial value problem

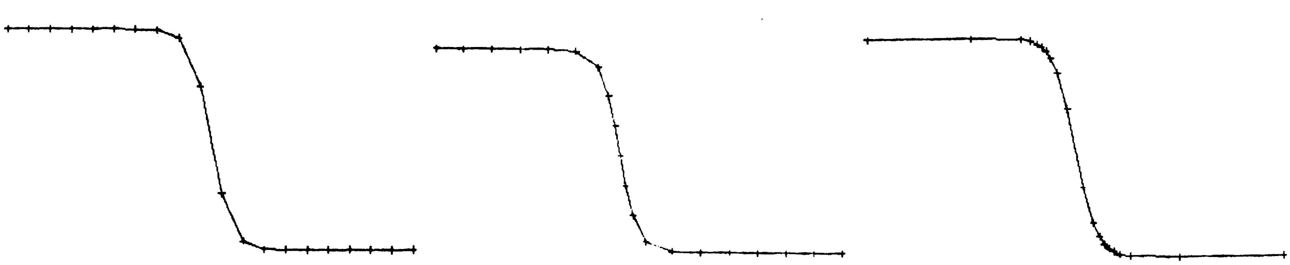
$$\eta_t = \frac{d(t)}{(r_\eta \cdot r_\eta)^{1/2}} = F(t,\eta),\ \eta(0) = 0 \qquad (2)$$

This problem can be solved accurately and efficiently by various numerical algorithms. It can be further noted that stability will be enhanced if $r_\eta \cdot r_\eta$ is an increasing function of η, that is, grid spacing increases with uniform increments of η. The numerical solution of this initial value problem may not exactly satisfy the condition $\eta(1) = 1$. This may be due to error in the numerical solution or the fact that d(t) is not exactly normalized by the above integral condition (1). In either case, the solution is computed until the value $\eta = 1$ is reached, and then the independent variable t is scaled so that, as a function of the new variable, $\eta(1) = 1$. This scaling process will alter the grid spacing, but the ratio of the grid spacing at any two points will be unchanged. Thus, if only the relative spacings at points along the curve are to be prescribed, no normalization of d(t) is necessary. An example would be the case when equal spacing of grid points along the curve is desired. Any constant value for d(t) would suffice.

The parameterization algorithm has been used to distribute points along various plane curves. For the first example, consider the curve defined by

$$y = \tanh(5x) \quad , -2 \leq x \leq 2$$

The parameter η is introduced, with $x = 4\eta - 2$, and the initial value problem (2) is solved using a fourth-order Runge-Kutta scheme with variable step length. The following figures illustrate the effect of reparameterization. Figure 1(a) is the point distribution resulting from equal spacing of the parameter η. Figure 1(b) has points uniformly spaced relative to arc length while 1(c) concentrates points where the curvature is large.



(a) (b) (c)

Figure 1. Grid point distributions for (a) uniform η, (b) d(t) = c, c = constant, and (c) $d(t) = c/(1 + 5|\kappa|)$, κ = curvature.

The above example was selected because the graph is typical of solutions for problems with shocks. In such cases a grid as in 1(c) will minimize smearing or oscillations in the solution.

The second example applies to the construction of a two-dimensional grid between an ellipse and a circle. On each grid line connecting the boundary components, a function of the form

$$d(t) = ae^{t} + be^{-t}$$

is used with the constants a and b selected so that (1) holds and also the grid spacing at the elliptical $t = 0$ boundary component is some specified value. An example of such a grid, with a small uniform spacing at the ellipse, is indicated in Figure 2. In this example a curvature-based reparameterization was also used to redistribute grid points along the ellipse. Finally, Figure 3 contains the plot of the same region with a grid having uniform spacing along every grid line. This was accomplished in an iterative process with the reparameterization performed in alternating coordinate directions.

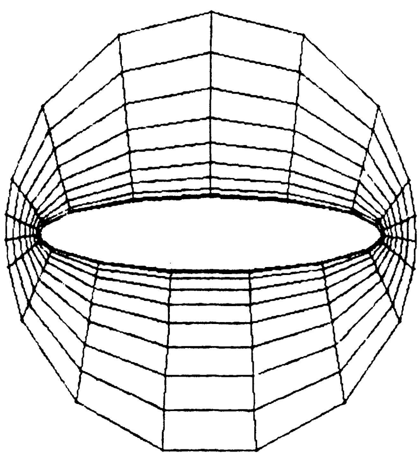


Figure 2. Grid concentration near inner boundary.

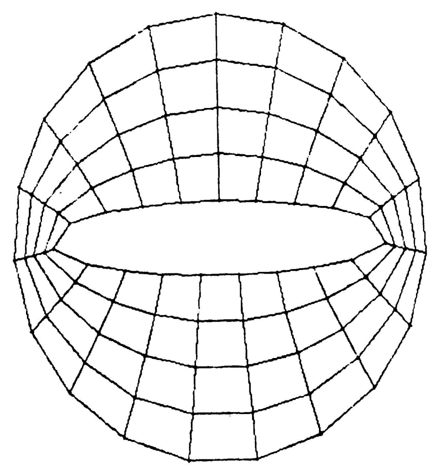


Figure 3. Uniform spacing in both directions.

Many of the potential applications of reparameterization in grid generation have not been addressed. One particularly attractive possibility would be to derive a method for generating adaptive grids by selecting d(t) to be a solution dependent function. Of course, with this or any grid redistribution scheme the skewness of the grid lines and the overall smoothness of the grid should be examined. A detailed study of the effect of grid properties on the numerical solution of partial differential equations may be found in the following reference.

REFERENCE

1. Thompson, J. F.; Warsi, Z.U.A.; and Mastin, C. W.: Numerical Grid Generation, Elsevier/North-Holland, New York, 1985.