A UNIFIED REPRESENTATION SCHEME FOR SOLID GEOMETRIC OBJECTS USING B-SPLINES[†]

Extended Abstract

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1. Introduction. A critical task facing those who would construct integrated design systems dealing with solid objects is how best to define and structure the geometric information to achieve maximum flexibility, efficiency, and functionality of the system. In particular, the need arises for an efficient geometric representation scheme capable of representing a broad class of objects in a unified way. In this study we define a geometric representation scheme we call the **B-spline cylinder** [1,2] , which consists of interpolation between pairs of uniform periodic cubic B-spline curves. This approach carries a number of interesting implications. For one, a single relatively simple database schema can be used to represent a reasonably large class of objects, since the spline representation we will describe is flexible enough to allow a large domain of representable objects at very little cost in data complexity. The model is thus very storage-efficient. A second feature of such a system is that it reduces to one the number of routines which the system must support to perform a given operation on objects. Third, the scheme enables easy conversion to and from other representations.

The formal definition of the cylinder entity is given in section 2. In section 3, we explore the geometric properties of the entity, and define several operations on such objects. In section 4 we introduce some general purpose criteria for evaluating *any* geometric representation scheme, and evaluate the B-spline cylinder scheme according to these criteria.

2. The B-Spline Cylinder. A B-spline cylinder is a boundary-determined three-dimensional object defined by $\mathbf{C}(u,v) = (1-v)\,\mathbf{R}_0(u) + v\,\mathbf{R}_1(u)$ where $\mathbf{R}_0(u)$ and $\mathbf{R}_1(u)$ are uniform periodic cubic B-spline curves. It will prove useful for our purposes to think of these curves as consisting of segments. For a curve R, the equation for each segment r_i can be written as

$$\mathbf{r}_i(u) = [u^3\ u^2\ u\ 1]\,\frac{1}{6}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{p}_{i-1}\\ \mathbf{p}_i\\ \mathbf{p}_{i+1}\\ \mathbf{p}_{i+2}\end{bmatrix} \tag{1}$$

where the **p**'s are neighboring vertices of the associated control polygon. Call the segment endpoints, where $u = 0$ above, **nodes**. The linear interpolation to create the cylinder is taken to mean an order-preserving labeling of the nodes as $\alpha_0\beta_n, \alpha_1\beta_{n-1}, \ldots, \alpha_{n-1}\beta_1$ which minimizes $\sum_{i=0}^{n-1}(\alpha_i - \beta_{n-i})^2$. (This criterion corresponds roughly to minimizing the surface area of the resulting object.) An analogous interpolation between corresponding vertices of

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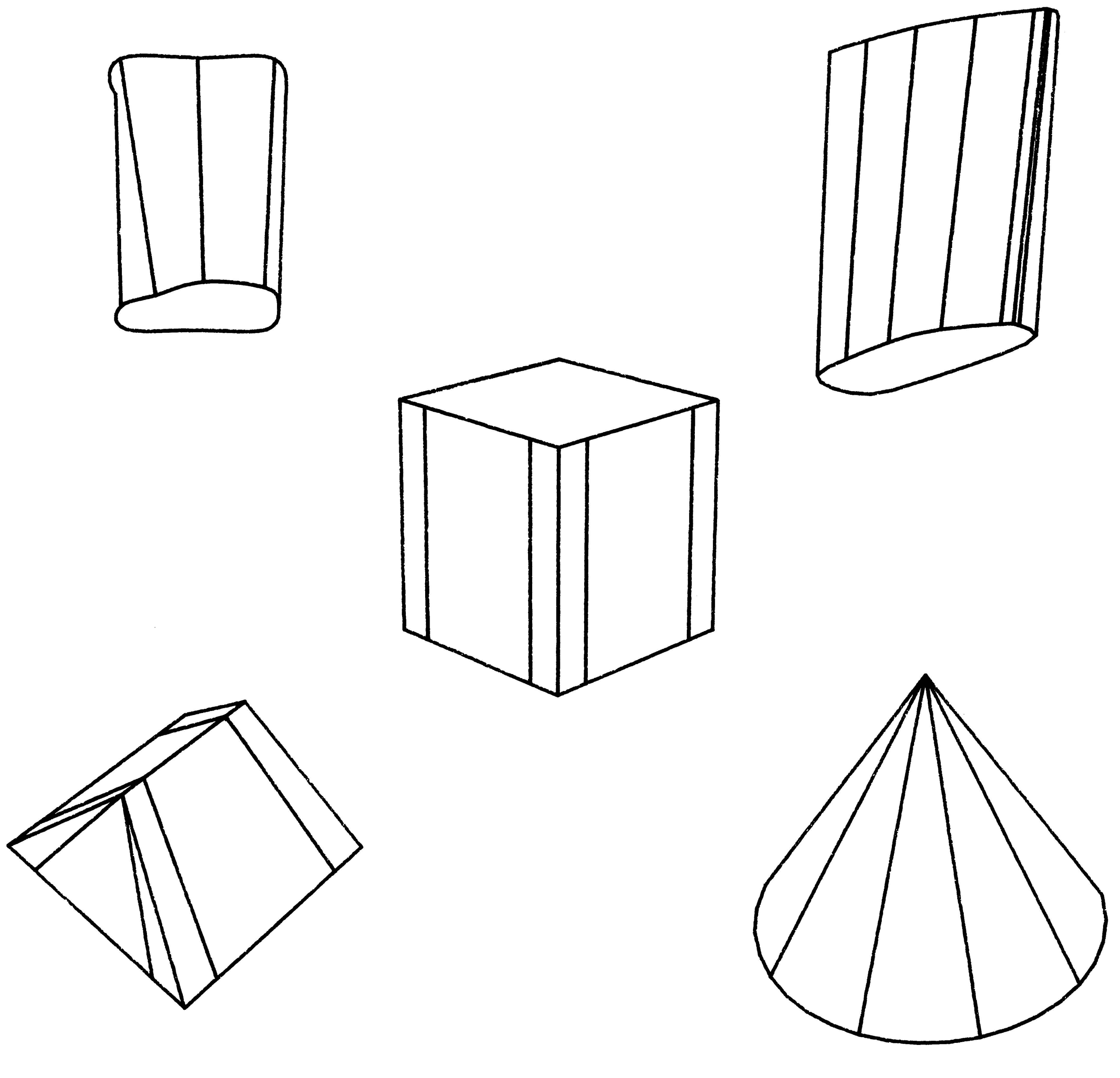

Figure 1. A variety of B–spline cylinders.

the control polygons produces C's **associated prism.** Figure 1 shows the variety of geometric shapes that can be expressed as B-spline cylinders. As can be seen, B-spline cylinders can be used to represent such objects as cubes, regular cylinders, cones, prisms, airfoils, etc. using exactly the same database schema.

3. Operations on Cylinders. It is possible to define a number of unary operators which transform objects into other objects or which yield attributes of objects. Normal vectors, surface area and volumetric properties can be calculated by unary operators. The set membership classification problem for B-spline cylinders (given a point $\mathbf{p}$ and an object C, is $\mathbf{p}$ inside, outside, or on the surface of C?) can be solved using a three-dimensional extension of the angle-sum algorithm for point-in-polygon location.

One way to create new objects from existing ones is to "cut" an object into contiguous smaller objects, which can then be manipulated independently. Refer to Figure 2. The intersection of a B-spline cylinder C with an arbitrary plane P, called a **cut plane**, can be calculated as follows. Define P by the general linear equation $ax + by + cz + d = 0$. Denote the two control polygons defining the faces of C by Q and R. Denote an arbitrary vertex in Q by $\mathbf{q}_i$ and its corresponding vertex in R by $\mathbf{r}_i$. The line segment joining these vertices, which is an edge of C's associated prism, can be defined parametrically by $\mathbf{s}_i = (1 - t_i)\,\mathbf{q}_i + t_i\,\mathbf{r}_i$

These values can be substituted to yield

$$t_i = \frac{aq_{i_x} + bq_{i_y} + cq_{i_z} + d}{a(q_{i_x} - r_{i_x}) + b(q_{i_y} - r_{i_y}) + c(q_{i_z} - r_{i_z})}$$

which can then be used to find the point of intersection $\mathbf{s}_i$ of the cut plane with the line $\mathbf{r}_i - \mathbf{q}_i$. Repeating for all pairs of corresponding vertices in the two faces, a set of intersection points is derived, forming a polygon S in the cut plane. Even in the case of a so-called regular intersection, however, the intersection of the cut plane with C is not always identical to the B-spline curve generated using S as a control polygon. Thus, when a cylinder is segmented with a plane in a regular intersection, it is sometimes not possible to generate the curve of intersection simply by using the polygon resulting from intersecting the associated prism. Fortunately, a polygon can be derived such that it is associated with a curve passing through any set of points, by considering those points as nodes of the curve and solving a simple linear system. The same technique can be used also to derive a control polygon representation for *any* set of points, however arrived at. For example, a curve defined in some other spline formulation, or a set of points obtained from a digitizing device, can be converted into B-spline form. The given points need simply be considered as nodes of a B-spline curve. In this way the B-spline cylinder resembles the generalized cylinder representation used in computer vision. [3]

4. Evaluation of the Cylinder Representation. Defining all objects in the way described carries a number of important advantages, some of which accrue from the properties of B-spline curves themselves, and some from this particular use of them. B-spline curves, and by extension B-spline cylinders, exhibit a number of well-known attractive properties from the standpoint of design. [4] In particular, corners, cusps, and other discontinuities may be introduced easily by using multiple polygon vertices; that is, placing several vertices at the same location. A multiplicity of 3 at two successive vertices will yield an embedded linear segment which is coincident with the corresponding span of the control polygon. It is by this means that objects such as cubes are represented as B-spline cylinders. Moreover, B-spline cylinders are a concise analytic representation for a large class of physical objects, since such objects are determined unambiguously by the control points of the two curves which compose their faces. These control points can be retrieved from a relational database at a cost of $2n + 1$ tuple fetches, where n is the number of vertices in (each of) the two polygons. This can be compared to older spline techniques, for example Hermite

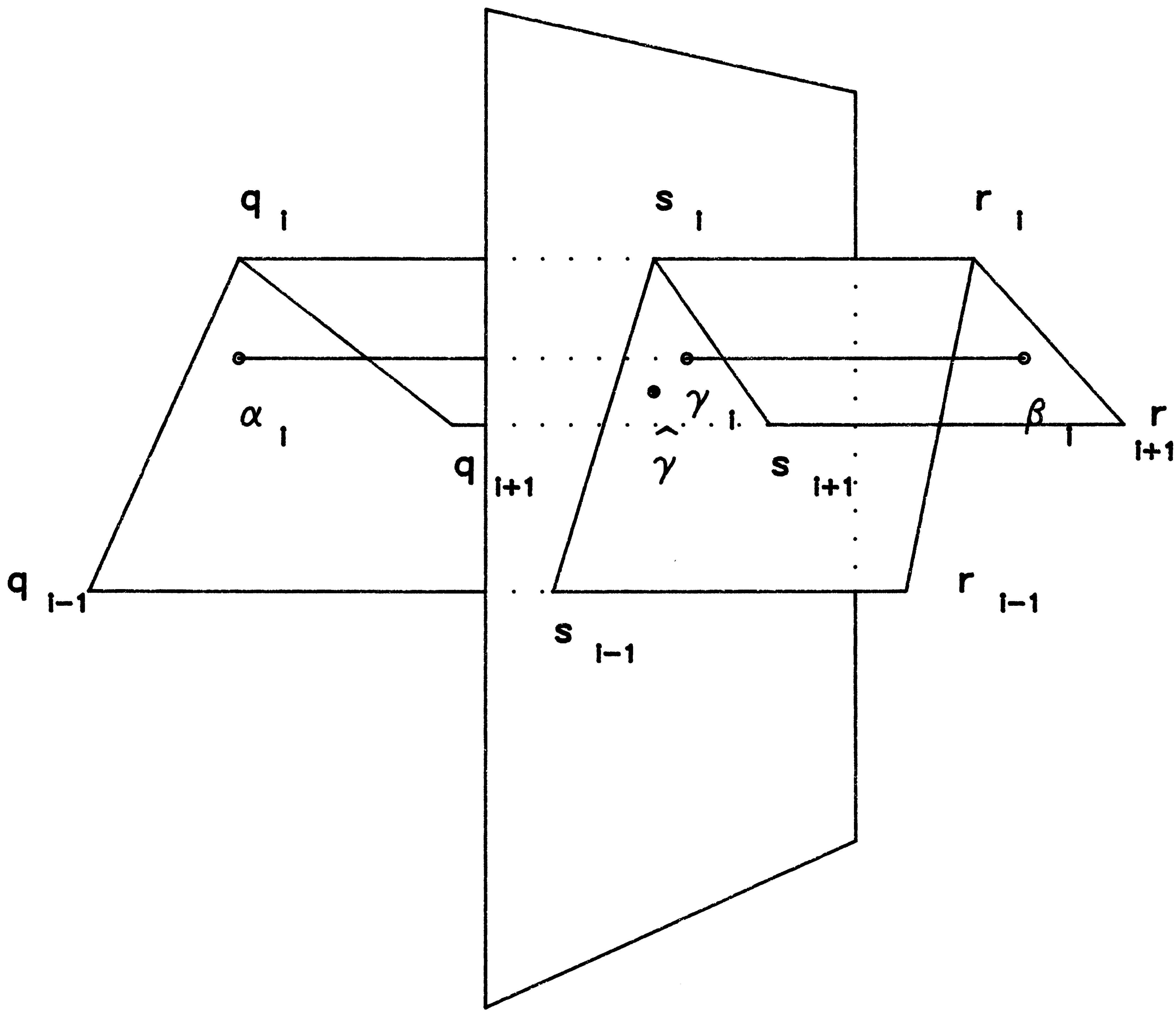

Figure 2. The Plane Cut Problem.

interpolation, which are typically far more verbose.

The B-spline cylinder scheme can also be evaluated with respect to a more formal model of geometric representation, although the definitions and criteria given here are intended to be applicable to any representation scheme. A **representation scheme** S is a mapping $X : M \longrightarrow R$ where M is the set of all suitable subsets of E^3, and R is the set of all syntactically correct terminal strings of some grammar. If $r \in R$ and there exists an $m \in M$ such that $X(m) = r$ then r **represents** or **generates** m through X. The **domain** D of a representation scheme X is the set

$$D = \{m \in M \mid \exists r \in R \text{ and } X(m) = r\}$$

A set $V \subseteq R$ of valid objects is defined by

$$V = \{r \in R \mid \exists d \in D \text{ and } X(d) = r\}$$

where $D \subseteq M$ is the domain of X. Any $r \in V$ is called a **valid** representation. The **range** of a representation scheme is the set V of representations which are valid.

A number of criteria can now be identified by closer examination of the representation scheme mapping X. First, if a representation can correspond to more than one physical object, ambiguity results when the modeling system is called upon to reproduce or reconstruct a previously defined object. A representation $X(m_i)$ of an object $m_i \in D$ is said to be an **unambiguous representation** if for all $m_j \in D$,

$$m_i \neq m_j \longrightarrow X(m_i) \neq X(m_j)$$

that is, X is injective. If for all $v \in V, v$ is an unambiguous representation, then X is an **unambiguous representation scheme**. The B-spline cylinder representation scheme is unambiguous; that is, such objects are unambiguously determined by the polygons of their face curves. Second, in a modeling system the intent is to deal as much as possible with representations of objects rather than to have to reconstruct objects explicitly through the mapping X^{-1}. Clearly, it is desirable that an object be representable by one and only one representation; that is, for the representation scheme to be unique. Moreover, uniqueness is necessary in order to draw conclusions about *sets* of objects solely from examination of their representations. One might, for example, wish to detect the congruence of two objects simply by examining their respective representations. If there is a unique $v \in V$ such that $X(d) = v$ then v is a **unique representation** of $d \in D$. X is a **unique representation scheme** if for all $v \in V, v$ is a unique representation. Unfortunately, this condition has proven to be extremely difficult to achieve. Virtually no common representation scheme -- wire frame, constructive solid geometry, spatial enumeration, oct-trees -- is unique. The B-spline cylinder scheme shares this shortcoming.

5. The BCYL System. Many of the ideas discussed in this report have been implemented in a simple geometric modeling and data management system called BCYL. The present version of the BCYL system is written in C and Pascal and runs on a VAX-11/780 under UNIX† 4.2BSD. BCYL employs hidden-surface color graphic display and a relational database management system as a back end. It also contains an editing capability for the interactive creation and modification of objects using a graphics terminal.

6. Conclusions. The definition of what entities and structures are necessary and sufficient for adequately representing geometric information has rarely been addressed. More usually, systems lack coherence and are subject to a combinatorial explosion if system designers attempt to provide users with any capability beyond graphic display. The simplified representation scheme that has been defined in this study and employed in the BCYL system was made for the express purpose of keeping secondary storage accesses to an absolute minimum. All objects in the system are so-called B-spline cylinders. The retrieval of an

† UNIX is a Trademark of Bell Laboratories.

object therefore consists simply of retrieving the control polygon coordinates used to generate its two face curves. This approach appears to be both computationally and storage efficient.

When one is designing a geometry based upon a single representation scheme, there is need for extreme care in selecting that scheme. With this in mind, terminology and concepts have been elaborated by which geometric representation schemes can be described and evaluated by means of a model of the geometric modeling process itself. It has been found that B-spline cylinders possess the essential properties of efficiency and unambiguity and share a common failing among representation schemes in that they lack uniqueness. Moreover, objects defined in this way inherit most of the attractive design characteristics which have made B-spline curves so popular. A B-spline curve can also be interpolated smoothly through an arbitrary collection of data points.

A number of common operations have been defined on B-spline cylinders, the most important of which is regular intersection with a plane. It has been found that the curve of intersection is not in general associated with the polygon of intersection, and an additional operation is necessary on the polygon to re-establish this association. This finding has an important bearing on a system which would permit creation, manipulation, and relinking of cylinders by means of regular intersection.

7. References.

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