N85-31207

ROBUST ADAPTIVE CONTROL

K. S. Narendra and A. M. Annaswamy
Yale University
New Haven, CT 06520

ABSTRACT

The paper discusses several concepts and results in robust adaptive control and is organized in three parts. The first part surveys existing algorithms. Different formulations of the problem and theoretical solutions that have been suggested are reviewed here. The second part contains new results related to the role of persistent excitation in robust adaptive systems and the use of hybrid control to improve robustness. In the third part promising new areas for future research are suggested which combine different approaches currently known.

1. INTRODUCTION

The stable adaptive control of linear time invariant plants, in what is now termed "the ideal case", was resolved in 1980 [1-4]. The assumptions made in [1-4] regarding the plant to prove global stability are quite stringent. Specifically, knowledge of the sign of the high frequency gain K_p, the relative degree n* and an upper bound n on the order of the plant transfer function are assumed to be known. Further it is assumed that the zeros of the plant transfer function lie in the left half plane, the plant parameters are constant (though unknown) and the system is disturbance free. However, in practice, these assumptions are rarely met. No actual plant is truly linear, finite dimensional or noise free. Further, in practical situations, the rationale for using adaptive control is to compensate for large variations in plant parameters. In the presence of such deviations from ideal conditions, the algorithms suggested in [1-4] no longer assure the boundedness of the signals in the adaptive loop. This accounts for the wide interest in the past few years in what is termed robust adaptive control to achieve satisfactory performance in the presence of both modeling and operating uncertainties. This paper attempts to survey some of the modest gains that have been made in this direction, presents some new results for improving robustness and discusses promising directions for future research.

Adaptive systems are special classes of nonlinear systems and many questions which arise in such systems can be stated as problems in the stability theory of differential equations. In particular, questions of robustness can be addressed using amply discussed results on practical stability and total stability. Since such results are bound to find increased application in adaptive systems, some of the more frequently used concepts, definitions and theorems are collected in section 2.

Recent years have witnessed many contributions to the robustness problem. Among these some assume additional prior information regarding the uncertainties to suitably modify the adaptive algorithms [5- 9] while others assume that the reference inputs possess properties which make the ideal system exponentially stable. In all cases it is shown that boundedness of solutions is assured when the true situation deviates in specific ways from the ideal. Some of these analytical results which are currently known are presented in section 3.

Sections 4 and 5 contain some new results on persistent excitation and hybrid adaptive control which are relevant to the problem of robustness. In section 4 a nonlinear error equation of second order is discussed in detail. While the ideal system is uniformly asymptotically stable it is shown that unbounded solutions can result if the disturbance is sufficiently large. It is also shown that by increasing the degree of persistent excitation of the reference input the overall system can be made practically stable. Section 5 discusses hybrid control algorithms recently introduced by the authors [10]. The same algorithms can also be modified to adaptively control discrete plants by updating control parameters infrequently. Some plausible arguments are given towards the end of the section as to why such algorithms may be more robust than continuous algorithms when external bounded disturbances are present.

Finally, in section 6, possible ways of combining known methods are discussed in the hope that it will stimulate research in these new directions. While no hard results exist in these areas the suggestions are based on extensive simulation studies.

2. MATHEMATICAL PRELIMINARIES AND STABILITY RESULTS

Some well known concepts and results of stability theory which find frequent application in the analysis of adaptive systems are included in this section. While they can be readily found in any good text [11-13] we present them here for easy reference as well as to place some of the problems discussed in the following sections in proper perspective. We start with the definitions of uniform asymptotic and exponential stability of the solution $x = 0$ of an equation $\dot{x} = f(x,t)$, $f(0,t) = 0$. We assume that f is continuous and satisfies conditions which guarantee the existence and uniqueness of solutions and continuity of their dependence on the initial conditions. The general solution of the differential equation is denoted as $p(t,x_0,t_0)$ with $p(t_0,x_0,t_0) = x_0$.

(i) Definition (Uniform Asymptotic Stability): The equilibrium $x = 0$ of the differential equation $\dot{x} = f(x,t)$ is uniformly asymptotically stable if it is uniformly stable and for some $\varepsilon_1 > 0$ and all $\varepsilon_2 > 0$ there is a $T(\varepsilon_1,\varepsilon_2) > 0$ such that $\|x_0\| < \varepsilon_1$ implies $\|p(t,x_0,t_0)\| < \varepsilon_2$ for all $t \geq t_0 + T$.

(ii) Definition (Exponential Stability): The equilibrium state of the equation $\dot{x} = f(x,t)$ is exponentially stable if two positive constants α and β which are independent of the initial values exist such that for sufficiently small initial values, $\|p(t,x_0,t_0)\| < \beta\|x_0\|e^{-\alpha(t-t_0)}$.

A linear time-invariant system with $f(x,t) = Ax$ where A is a constant matrix is asymptotically stable if the eigenvalues of A are in the open left half of the complex plane. Asymptotic, uniform asymptotic and exponential stability are equivalent in this case. For linear time-varying systems, asymptotic stability does not imply uniform asymptotic stability whereas the latter is equivalent to exponential stability. For linear systems, all stability properties hold in the large. In general, for nonlinear systems exponential stability implies uniform asymptotic stability but not vice versa. If $f(x,t)$ is autonomous or periodic in t, all stability properties are uniform.

In robust adaptive control we are interested in deducing the properties of the solutions of a perturbed system (S_p) from the behavior of the solutions of an unperturbed system (S). These are described by the differential equations

$$\dot{x} = f(x,t) \qquad (S) \; ; \qquad \dot{x} = f(x,t) + g(x,t) \qquad (S_p) \tag{1}$$

Let the equilibrium state of (S) be exponentially stable. If $\|g(x,t)\| < L\|x\|$ for sufficiently small b and δ, and $\|x\| < \delta$, then the equilibrium state of (S_p) is also exponentially stable [11]. In physical situations the condition $g(0,t) = 0$ required above is not generally met and this gives rise to the concept of total stability.

(iii) Definition (Total Stability) [11]: The equilibrium state $x = 0$ of (S) is totally stable if for every $\varepsilon > 0$ two positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ exist such that every solution $p(t,x_0,t_0)$ of (S_p) satisfies $\|p(t,x_0,t_0)\| < \varepsilon$, $t \geq t_0$ provided $\|x_0\| < \delta_1$ and $\|g(x,t)\| < \delta_2$.

In the Russian literature this is also referred to as stability under persistent disturbances. The uniform asymptotic stability of the unperturbed system implies total stability [11] and is frequently used to prove robustness of adaptive systems in the presence of sufficiently small perturbations. Recently the magnitude constraint on $\|x_0\|$ in the definition of total stability has been relaxed by Anderson and Johnstone [8] at the expense of stronger conditions on $f(x,t)$.

In practical systems we are interested in the uniform boundedness of the solutions in the presence of perturbations as well as in the magnitudes of this bound. This leads to the concept of practical stability defined below.

(iv) Definition (Practical Stability) [12]: Let $Q_0 = \{x \,|\, \|x\| < \delta_1\}$ be open set in $\mathbb{R}^n$ and $\delta_2 > 0$ a constant such that $\|g(x,t)\| < \delta_2$ for all x and $t \geq t_0$. If the solutions of (S_p) lie within a closed bounded set $Q \supset Q_0$ for $x_0 \in Q_0$ then the system (S) is said to be practically stable.

Total stability assures the existence of Q_0 and δ_2 relative to which the system (S) is practically stable but provides no way of estimating the size of Q_0 or the magnitude of δ_2. In adaptive control applications this is not adequate. One is more interested in determining an estimate of Q from a knowledge of δ_2.

An alternative method for treating the effect of perturbations is by considering them as bounded independent functions of time. This leads to the well known concept of bounded input - bounded output (BIBO) stability.

(v) Definition (BIBO Stability): A system $\dot{x} = f(x,u,t)$ with $f(\cdot,0,t) = 0$ is BIBO stable if for every $\alpha \geq 0$ and every $a \geq 0$ there is a $\beta = \beta(\alpha,a)$ such that $\|p_u(\tau,x_0,t_0)\| \leq \beta$ for all $\tau \geq t_0$ for every initial condition (x_0,t_0) with $\|x_0\| \leq \alpha$ and $\sup_t \|u(t)\| \leq a$, where $p_u(\cdot,x_0,t_0)$ is the solution of the system with input $u(\cdot)$.

A linear system $\dot{x} = A(t)x + b(t)u$ is BIBO stable if the homogeneous part is uniformly asymptotically stable. This is a property which is frequently used in robust adaptive control using the concept of persistent excitation. In contrast to the above, uniform asymptotic stability of a nonlinear system does not imply BIBO stability. An example of this was given by Desoer et al [14]. A similar situation arises in the discussion of robustness of a second order nonlinear system in section 4.

Stability Problems in Adaptive Systems: The study of the stability of adaptive systems (as shown in the following sections) can be conveniently carried out using a set of nonlinear time-varying error differential equations. Even in the "ideal" or disturbance free case the time-variations arise due to the presence of the reference input r(•). The following are some noteworthy features of many of the stability questions which arise in adaptive systems.

(i) In the ideal case, a Lyapunov function $V > 0$ with $\dot{V} \leq 0$ can be found. The negative semi-definiteness of $\dot{V}$ cannot be avoided and is a result of the adaptive law used.

(ii) As a result of (i) even the unforced (autonomous) system is uniformly stable. Even when the reference input is persistently exciting, $\dot{V} \leq 0$ but the system can be shown to be uniformly asymptotically stable [15]. We note that LaSalle's theorem cannot be directly applied to prove this since the system is nonautonomous.

(iii) Since the system is exponentially stable with a persistently exciting reference input, Malkin's theorem can be used to conclude that the solutions will be bounded for some initial set Q_0 and perturbation of magnitude $\leq \delta_2$. However, very little can be said directly about either Q_0 or δ_2.

(iv) Another consequence of the semidefiniteness of $\dot{V}$ is that assuring even the boundedness of solutions using Lyapunov's Direct method for given bounds on perturbations is no longer trivial. Some of these cases are considered in section 3. In section 4 it is shown that even when the reference input is persistently exciting, if the disturbance is large the solutions can be unbounded. Alternately, for a given bound on the disturbance the persistent excitation can be made sufficiently large to assure the boundedness of the solutions.

3. RECENT RESULTS IN ADAPTIVE CONTROL

In this section we attempt to survey briefly some of the theoretical results currently known in the area of robust adaptive control. The aim of the section is to provide an understanding of the qualitative ideas that led to these results as well as the analytical tools used in deriving them. Since the ideal system forms the starting point of all perturbation analyses, we shall briefly outline the statement of the problem and the proof of stability in this case. Further, while several stable adaptive algorithms have been suggested in the literature, we shall discuss the proof of stability using only the algorithm proposed in [16]. The proofs using all the other algorithms follow along similar lines.

a) Ideal System: The plant to be controlled is described by the state equations

$$\dot{x}_p = A_p x_p + b_p u \quad ; \quad y_p = c_p^T x_p \tag{2}$$

and a reference model is described by

$$\dot{x}_m = A_m x_m + b_m r \quad ; \quad y_m = c_m^T x_m \tag{3}$$

where x_p, u and y_p are respectively the state input and output of the plant and x_m and y_m are the state and output of the model. The transfer functions of the plant and model are

$$W_p(s) = c_p^T(sI-A_p)^{-1}b_p = \frac{K_p Z_p(s)}{R_p(s)} \quad ; \quad W_m(s) = c_m^T(sI-A_m)^{-1}b_m = \frac{K_m}{R_m(s)}$$

The following assumptions are made regarding $W_p(s)$ and $W_m(s)$

(i) $Z_p(s)$, $R_p(s)$ and $R_m(s)$ are monic polynomials of degrees m, n and $n^*=n-m$

(ii) $Z_p(s)$ and $R_m(s)$ are strictly stable polynomials

and (iii) r is a piecewise continuous uniformly bounded reference input.

The objective is to control the plant in such a fashion that the output error between plant and model $e_1 \overset{\Delta}{=} y_p - y_m$ tends to zero asymptotically, while the signals and parameters of the system remain uniformly bounded. It is now well known that knowledge of the exact relative degree n* of the plant, an upper bound n on its order, the sign of the gain K_p and the condition that $Z_p(s)$ be Hurwitz as given in (ii) are needed to solve the problem. n* enables the model to be constructed while the value of n determines the order of the controller to be used. The sign of K_p and the constraint on $Z_p(s)$ are needed to prove the stability of the overall system.

Structure of Controller: In the following we shall assume that K_p is known and $K_p = K_m = 1$. To meet the control objective a controller described by the following equations is used:

$$\dot{\omega}^{(1)} = F\omega^{(1)} + gu \quad ; \quad \dot{\omega}^{(2)} = F\omega^{(2)} + gy_p \quad ; \quad u = \theta^T\omega + r \tag{4}$$

where F is an asymptotically stable nxn matrix, (F,g) is controllable, $\omega^T = [\omega^{(1)^T}, \omega^{(2)^T}]$ and θ(t) is a 2n dimensional parameter vector which is to be adjusted adaptively. It is well known [17] that a unique constant vector θ* exists such that the transfer function of the plant together with the controller matches that of the model exactly, when $\theta(t) \equiv \theta^*$. The aim of the adaptive law is to adjust θ(t) in such a manner that the overall system is globally stable and $\lim_{t\to\infty} e_1(t) = 0$.

While several special cases of the adaptive control problem have been considered, we discuss below the general case when $W_p(s)$ has a relative degree $n^* \geq 2$. If $\theta(t) - \theta^* \overset{\Delta}{=} \phi(t)$, then ϕ is the parameter error vector and the output of the plant can be expressed as

$$y_p(t) = W_m(s)[r(t) + \phi^T(t)\omega(t)] \tag{5}$$

The Adaptive Scheme: To generate the adaptive law an auxiliary error signal $y_a(t)$ is added to $e_1(t)$ to generate an augmented error $\varepsilon_1(t)$. If

$$y_a(t) \triangleq [\theta^T(t)W_m(s)I - W_m(s)\theta^T(t)]\omega(t) \tag{6}$$

then

$$\phi^T(t)\zeta(t) = e_1(t) + y_a(t) = \varepsilon_1(t) \tag{7}$$

where $W_m(s)I\omega = \zeta$. The adaptive law for updating $\theta(t)$ then depends on the augmented error $\varepsilon_1(t)$ and the signal $\zeta(t)$ and is given by

$$\dot{\phi}(t) = \dot{\theta}(t) = \frac{-\varepsilon_1(t)\zeta(t)}{1 + \zeta^T(t)\zeta(t)} \tag{8}$$

This has been shown to result in global stability of the adaptive loop [16].

Proof of Global Stability: If $V(\phi) = 1/2\ \phi^T\phi$, the adaptive law (8) yields

$$\dot{V}(\phi) = \frac{-\varepsilon_1^2(\phi)}{1 + \zeta^T(t)\zeta(t)}$$

from which it follows that

(i) ϕ and $\dot{\phi}$ are uniformly bounded

(ii) $\dot{\phi} \in L^2$ (9)

and (iii) $\varepsilon_1(t) = \nu(t)[1 + \zeta^T(t)\zeta(t)]^{1/2}$, $\nu \in L^2$

Since the complete proof is too long and involved to be included here in its entirety we merely outline the principal steps involved.

(a) Since the parameter vector is bounded by (i) it is first shown that

$$\sup_{\tau\leq t}|y_p(\tau)| \sim \sup_{\tau\leq t}\|\omega^{(2)}(\tau)\| \sim \sup_{\tau\leq t}\|\omega(\tau)\| \sim \sup_{\tau\leq t}\|\zeta(\tau)\| \tag{10}$$

Here $\sim$ is an equivalence relation and implies that the corresponding signals in (10) grow at the same rate [18].

(b) Since $\dot{\phi} \in L^2$ it can be shown that $y_a(t)$ grows at a slower rate than $\sup_{\tau\leq t}\|\omega(\tau)\|$ denoted by $y_a(t) = o[\sup_{\tau\leq t}\|\omega(\tau)\|]$. (11)

(c) From (5), (9-iii) and (11) it follows that

$$e_1 = W_m\phi^T\omega = \nu[1 + \zeta^T\zeta]^{1/2} + o[\sup_{\tau\leq t}\|\omega(\tau)\|] \tag{12}$$

(d) Since $\nu \in L^2$ using equation (4) we conclude that $\sup_{\tau\leq t}\|\omega^{(2)}(\tau)\| = o \sup_{\tau\leq t}\|\omega(\tau)\|$ which contradicts (10).

Hence all the signals in the system are uniformly bounded and $\lim_{t\to\infty} e_1(t) = 0$.

The importance of demonstrating the boundedness of $\phi(t)$ and $\dot{\phi} \in L^2$ in the proof of stability is worth noting. [In some cases it may be possible to show that $\lim_{t\to\infty} \dot{\phi}(t) = 0$, which serves the same purpose.] The former assures that the relevant signals in (10) grow at the same rate while the latter is used to prove that $|y_p(t)|$ and $\|\omega_2(t)\|$ should grow at different rates if the adaptive control is used, leading to a contradiction.

Asymptotic Stability of the Ideal System: Once the boundedness of all the signals in the adaptive system has been established, interest shifts to the convergence of the parameter vector $\theta(t)$ to its desired value θ^* or equivalently of $\phi(t)$ to the null vector. Since the adaptive law (8) can be represented as

$$\dot{\phi}(t) = \frac{-\zeta(t)\zeta^T(t)}{1+\zeta^T(t)\zeta(t)}\,\phi(t) \tag{13}$$

the conditions that have to be imposed on $\zeta(t)$ to accomplish this is of interest. Following the results of Morgan and Narendra [19] if $\dfrac{\zeta(t)}{\sqrt{1+\zeta^T(t)\zeta(t)}}$ is persistently exciting $\lim_{t\to\infty} \phi(t) = 0$ and the convergence is exponential. Since $W_m(s)I\omega = \zeta$, a sufficient condition for $\zeta(\cdot)$ to be persistently exciting is that $\omega(\cdot)$ is persistently exciting [15]. Hence conditions under which $\omega(\cdot)$ will be persistently exciting have been investigated by several authors [15,20-22].

Persistent Excitation (PE) of $\omega(t)$ and $\omega^*(t)$: Early results on the convergence of the parameter vector to the null vector were stated in terms of the PE of $\omega(t)$. However since $\omega(t)$ is a dependent variable within the adaptive loop, very little can be said directly about its persistent excitation. Hence attempts were made to express this condition in terms of the PE of signals in the model which are at the discretion of the designer. Since the adaptive system and model transfer functions are identical when $\theta(t) \equiv \theta^*$, the model can be parametrized in such a fashion that a signal ω^* in it would correspond to the signal $\omega(t)$ in the adaptive loop. Further since the model is time invariant, conditions on $r(t)$ which would assure the PE of $\omega^*(t)$ can be derived. If $\tilde{\omega}(t) \triangleq \omega(t) - \omega^*(t)$, the adaptive law assures that $\lim_{t\to\infty} \tilde{\omega}(t) = 0$. Hence, in the ideal case the PE of $\omega^*(t)$ ensures the PE of $\omega(t)$ and hence the convergence of the parameter vector $\theta(t)$ to its true value.

Comments:

(i) The above arguments have focussed attention on several interesting questions related to persistent excitation and transformations under which the property is preserved [15].

(ii) The convergence of $\tilde{\omega}(t)$ to 0 is used above to show the PE of $\omega(t)$ and hence the convergence of $\phi(t)$ to 0. This is no longer possible when an external disturbance is present since even the boundedness of the signals is not assured in such a case.

(iii) From the results of several authors it is now known that an almost periodic reference input with n-distinct frequencies results in the PE of $\omega(t)$.

b) Adaptation Under Perturbations: The adaptive control system described in section (3a) assumed ideal conditions. The plant was linear and time-invariant and no external disturbances were present. In addition, considerable prior knowledge of plant transfer characteristics was assumed to help in setting up a reference model and deriving stable adaptive laws. As mentioned earlier, plants are rarely strictly linear or finite dimensional and in many practical situations the need for adaptive control arises due to large parameter variations. Also, external input and output disturbances are invariably present in real systems. Hence there is a definite need to extend the theory developed for the ideal case to situations with modeling errors and external disturbances. Some of the schemes that have been proposed in recent years to achieve robustness in the presence of such perturbations are briefly reviewed in this section and some new results are reported in sections 4 and 5.

The basic adaptive system in the ideal case is only uniformly stable. This implies that bounded perturbations can theoretically produce unbounded outputs. When the reference input is persistently exciting, the nonlinear system is uniformly asymptotically stable in the large and exponentially stable when the initial state x_0 lies in a finite ball around the origin. The latter fact allows BIBO results to be derived using theorems of the type described by Malkin, provided the perturbations are sufficiently small. However, as pointed out in section 2, very little can be said using such an approach about the effect of bounded perturbations of a specified maximum amplitude on the global behavior of the solutions of the adaptive system. In addition to such perturbation methods a few global methods have also been used to derive results in robust adaptive systems. The principal concepts involved in deriving some of these are discussed below.

(i) Use of Dead-Zone [5]: The problem statement is similar to that given for the ideal system with the exception that $y_p = c_p^T x_p + v_1$ where v_1 is a bounded disturbance. Using the same adaptive law (8) as in the ideal case, the error equations can be expressed as

$$\phi^T(t)\zeta(t) + v(t) = \epsilon(t) \qquad (14)$$

and

$$\dot{\phi}(t) = \frac{-\Gamma\epsilon(t)\zeta(t)}{1 + \zeta^T(t)\Gamma\zeta(t)} \qquad \text{(adaptive law)}$$

where v is an equivalent output disturbance due to v_1. The difficulty arises due to the presence of v(t) in (14). When $\mathrm{sgn}[\phi^T\zeta + v] = \mathrm{sgn}[\phi^T\zeta]$ the adaptation is in the right direction. Otherwise the parameter vector may be adjusted away from its desired value. This implies that problems of convergence may arise when $\epsilon(t)$ is of the order of the vound v_0 of v(t). The modification in the algorithm suggested in [5] is to use a dead-zone so that the adaptive parameters are not adjusted when $\epsilon(t)$ lies inside it. Hence the overall system operates in two modes--a linear time-invariant mode when $|\epsilon(t)| \leq v_0 + \delta$ (for some constant $\delta > 0$) and an adaptive mode otherwise. In [5] it is shown that such an algorithm results in a system with bounded signals. Further, adaptation takes place for only a finite time. This implies that in practice the system will converge to a linear time-invariant system in a finite time after which the output error will lie entirely in the dead-zone and hence adaptation ceases entirely.

(ii) Bound on $\|\theta^*\|$: An alternate approach to the bounded disturbance problem was taken by Kreisselmeier and Narendra [6]. While the statement of the problem as well as the structure of the controller are identical to that in (i), it is assumed that no knowledge of a bound on the disturbance is available. Instead, it is assumed that the desired vector θ^* has a norm less than a specified value $\|\theta^*\|_{max}$. Hence the search procedure can be confined essentially to the set $S:\{\theta | \|\theta\| \leq \|\theta^*\|_{max}\}$. The adaptive law used to update $\theta(t)$ is identical to that in the ideal case when θ lies in the interior of S and is modified when it reaches the boundary of S, or lies outside it. In [6] it is shown that such a scheme results in the boundedness of all signals in the system.

Apart from the obvious differences between the schemes suggested in [5] and [6], there are mathematical differences in the proof that are worth stressing. As in [1-4], the proofs of stability in [5] use limiting arguments as $t \to \infty$ to show that $\dot{\phi} \in L^2$. Such a procedure cannot be used in [6], since $\phi(t)$ does not tend to any limit as $t \to \infty$. Hence all arguments are based on the analysis of the behavior of the system over a finite interval. As shown in section 6 the approaches in [5] and [6] complement each other and can be combined to have wider application in adaptive systems in the future.

(iii) The σ-modification Scheme: In approaches (i) and (ii) certain prior information is assumed to implement the adaptive laws. In contrast to this, a scheme suggested by Ioannou and Kokotovic [7] assures boundedness of all signals in the system, without any assumptions regarding the bounds on either the disturbance or the control parameters. However, to the authors' knowledge, the method has been shown to result in global boundedness only for the special case when the reference model is strictly positive real.

The method is based on the following simple ideas. If $V(e,\phi)$ is a quadratic Lyapunov function candidate, the time derivative $\dot{V}(e,\phi)$, along a trajectory, is generally a quadratic function of e and hence is negative semidefinite. When a disturbance is present, $\dot{V}(e,\phi)$ has the general form $-e^TQe + e^T\alpha v$, where $Q = Q^T > 0$, α is a constant vector and v is a bounded disturbance. Very little can be concluded regarding stability from this and accounts for the modifications suggested in [5] and [6]. In [7], an additional term $-\sigma\theta$ is used in the adaptive law, as a result of which $\dot{V}(e,\phi)$ becomes negative definite outside a bounded region in the (e,ϕ) space. From this it is concluded that all signals in the system are bounded.

(iv) Adaptive Systems and Time-Varying Plants: The methods outlined in sections 3b(i-iii) deal with the global behavior of the adaptive systems when bounded perturbations are present. In contrast to this Anderson and Johnstone [8] examine adaptive control problems where the assumptions made regarding the system deviate slightly from the ideal. While [8] addresses primarily the problem of time-varying plant parameters the authors claim that the same methods with remarkably little change allow examination of the effect of measurement noise, plant nonlinearity and undermodelling of the plant order.

As in our discussions in section (3a), the authors first consider the ideal system and demonstrate uniform or exponential stability in the presence of persistent excitation. For the various types of perturbations considered, their aim is then to show that the resulting equations can be cast in such a form that the total stability of the overall system can be demonstrated using modifications of Malkin's theorems. However, as mentioned earlier, the theorems are useful primarily for establishing the existence of robustness in the presence of sufficiently small perturbations rather than for providing guidance in the choice of the control input to assure boundedness of solutions when the class of perturbations is specified.

4. PERSISTENT EXCITATION AND ROBUSTNESS

In the last section, we discussed two approaches of studying the robustness problem in adaptive systems. The approach in 3-d assumed that the perturbations were sufficiently small and derived BIBO results local in nature, using Malkin's theorem, whereas in 3a-3c the approach was global in nature and used additional information regarding plant dynamics and the external perturbations. Also, the first approach made use of the PE of the reference input which was not needed in the second.

In this section, we present some new results which demonstrate global boundedness of all signals in the adaptive system in the presence of bounded disturbances when the reference input is sufficiently persistently exciting. We show that by analyzing a set of nonlinear error differential equations, we can establish the global robustness behavior of the adaptive systems. In particular, it is shown that if the persistent excitation of the model output is larger than the disturbance, the solutions will be globally bounded and that if the maximum amplitude of the disturbance is greater than that of the model output, the system can have unbounded solutions. The basic idea is stated here by considering the adaptive control of a first order plant and studying the corresponding second order nonlinear differential equations in detail. The same methodology is applicable to the general adaptive control problem.

Nonlinear Error Equations: The plant to be adaptively controlled, the corresponding reference model and the resulting error equations are as follows:

$$\begin{array}{ll}\text{Plant:} & \dot{y}_p = a_p y_p + u + v; \quad u = \theta y_p + r \\ \text{Model:} & \dot{y}_m = -y_m + r \\ \text{Error Equations:} & \dot{e}_1 = -e_1 + \phi y_p + v \\ \text{Adaptive Law:} & \dot{\phi} = -e_1 y_p \end{array} \tag{15}$$

where r is the reference input, v is a bounded input disturbance, e_1 is the output error defined as $e_1 \triangleq y_p - y_m$ and ϕ is the parameter error. In the ideal case, when $v(t) \equiv 0$, by considering

$$\begin{aligned}\dot{e}_1(t) &= -e_1(t) + \phi(t)y_p(t) \\ \dot{\phi}(t) &= -e_1(t)y_p(t)\end{aligned} \tag{16}$$

it immediately follows that the system is uniformly stable and if $y_p(t)$ is persistently exciting, the system is exponentially stable. When a disturbance v(t) is present, it is tempting to proceed as in the ideal case and require $y_p(t)$ in (15) to be persistently exciting so that the unperturbed system is exponentially stable resulting in a bounded error vector for bounded perturbations. Since stability of the overall system has not been established, $y_p(t)$ cannot be assumed to be bounded and proving that it is PE becomes specious. Hence we have to express the right hand side of (15) in terms of the model output $y_m(t)$ which is an independent variable rather than the dependent variable $y_p(t)$. This results in the following nonlinear error differential equations:

$$\begin{aligned} \dot{e}_1(t) &= -e_1(t) + \phi(t)y_m(t) + \phi(t)e_1(t) + v \\ \dot{\phi}(t) &= -e_1(t)y_m(t) - e_1^{\,2}(t) \end{aligned} \qquad (17)$$

By analyzing the above nonlinear differential equation, we demonstrate the global behavior of the adaptive system in the presence of v(t).

The Ideal System: In the absence of external perturbations, the nonlinear system

$$\begin{aligned} \dot{e}_1(t) &= -e_1(t) + \phi(t)y_m(t) + \phi(t)e_1(t) \\ \dot{\phi}(t) &= -e_1(t)y_m(t) - e_1^{\,2}(t) \end{aligned} \qquad (18)$$

can be shown to be uniformly asymptotically stable in the large as follows: If $W(e_1,\phi) = \frac{1}{2}[e_1^{\,2} + \phi^2]$, the time-derivative $W[e_1,\phi] = -e_1^{\,2} \leq 0$. Hence the system $e_1(t)$ and $\phi(t)$ are uniformly bounded for all $t \geq t_0$, if $W[e_1(t_0),\phi(t_0)] < \infty$. Since $e_1 \in L^2$ and $\dot{e}_1$ is bounded, $\lim_{t\to\infty} e_1(t) = 0$. The nonlinear vector $[\phi e_1, -e_1^{\,2}]^T$ can be considered to be the input to the linear part which is exponentially stable if $y_m(t)$ is PE. As $e_1 \to 0$ as $t \to \infty$, this input tends to zero and hence $x(t) \to 0$ as $t \to \infty$ where $x \triangleq [e_1,\phi]^T$. Since all the arguments are independent of the initial time t_0 and the magnitude of the initial conditions, the system is u.a.s.l. It is also worth noting that when $y_m(t)$ is PE, the linear part of (18) is exponentially stable but the nonlinear system is exponentially stable only when the initial state $x(t_0)$ lies in a finite ball around the origin and not globally exponentially stable.

Perturbed System: To provide some insight into the behavior of the nonlinear system, we shall discuss three cases where the perturbed nonlinear system (17) is autonomous.

Case (i) $y_m(t) \equiv 0$: When $v(t) \equiv 0$, the system is uniformly stable. If $v(t) \equiv v_{max}$, $\lim_{t\to\infty} \phi(t) = -\infty$ and $\lim_{t\to\infty} e_1(t) = 0$.

Case (ii) $y_m(t) \equiv y_{max}$: The unforced system in this case is autonomous and, by LaSalle's theorem, is u.a.s.l. since the largest invariant set in $E = \{x | e_1^{\,2} = 0\}$ is the origin. However, since the system is nonlinear, it no longer follows that a bounded input will result in a bounded output. If, for example, $v(t) \equiv -v_{max}$, where $v_{max} > y_{max}$, we can show that $\lim_{t\to\infty} e_1(t) = -y_{max}$ and $\lim_{t\to\infty} \phi(t) = -\infty$.

Case (iii) $y_m(t) \equiv y_{max}, y_{max} > v_{max}$: The system is Lagrange stable. When $v(t) \equiv -v_{max}$, the system has an equilibrium state at $(0, \frac{v_{max}}{y_{max}})$ which is u.a.s.l. Similarly when $v(t) \equiv v_{max}$, the system has an equilibrium state at $(0, -\frac{v_{max}}{y_{max}})$.

The above special cases reveal that the behavior of the nonlinear system is very much dependent on y_{max} and v_{max}. In particular, when $y_m(t) \equiv y_{max}$ and $v(t) \equiv -v_{max}$, the system has unbounded solutions when $v_{max} > y_{max}$ and all solutions are bounded when $y_{max} > v_{max}$. The results also carry over to the general case when both $v(t)$ and $y_m(t)$ are time-varying and are stated in the following main theorem of the paper. (Fig. 1)

Theorem 1:Let $|y_m(t)| \le y_{max}$, $|v(t)| \le v_{max}$ and $y_m(t)$ be a smooth persistently exciting signal in the sense described in [23]. This implies that positive numbers T_0, ε_0 and δ_0 exist such that given any $t_1 \ge 0$, there exists a $t_2 \in [t_1, t_1 + T_0]$, with $[t_2, t_2 + \delta_0] \subset [t_1, t_1 + T_0]$ and $\frac{1}{T_0} \left| \int_{t_2}^{t_2+\delta_0} y_m(\tau)d\tau \right| \ge \varepsilon_0$. Then

(a) If $y_{max} < v_{max}$, by choosing an input $v(t)$ as

$$\begin{aligned} v(t) &= -\mathrm{sgn}(y_m(t))v_{max} && \text{when } |e_1(t)| \ge y_{max} \\ &= \mathrm{sgn}(e_1(t))v_{max} && |e_1(t)| < y_{max} \end{aligned}$$

where $\mathrm{sgn}(a(t)) \triangleq \frac{a(t)}{|a(t)|}$ when $a(t) \neq 0$ and is equal to unity when $a(t) = 0$, there exist initial conditions for which $\lim_{t\to\infty} \phi(t) = -\infty$ and $e_1(t)$ approaches asymptotically the region $|e_1| \le y_{max} + \varepsilon$, where ε is an arbitrary positive constant.

(b) If $\varepsilon_0 > v_{max} + \delta$ where δ is any arbitrary positive constant, then all the solutions of the differential equation (17) are bounded.

Proof:

a) Let D_1 be the open domain enclosed by the line $e_1 = -v_{max}$ and the curve $\phi = \frac{e_1 + v_{max}}{e_1 + y_{max}}$ with $\phi \le 0$. When $y_m(t) \equiv y_{max}$ and $v(t) \equiv -v_{max}$ all solutions that start on the boundary $\partial(D_1)$ enter D_1. Since the system is autonomous and contains no singularities in D_1, all solutions originating D_1 are unbounded and $\lim_{t\to\infty} \phi(t) = -\infty$, $\lim_{t\to\infty} e_1(t) = -y_{max}$.

For a time-varying signal $y_m(t)$ the proof of unboundedness is related to the above autonomous case. Consider the solution of the differential equation with initial condition $(0,\phi_0)$ with $\phi_0 < \frac{-v_{max}}{y_{max}}$, with $y_m(t) \equiv y_{max}$ and $v(t) \equiv -v_{max}$. Let Γ_+ denote the open curve along which the trajectory lies for all $t \geqslant 0$. Similarly let Γ_- denote the curve along which the solution lies for all $t \geqslant 0$ when $v(t) = v_{max}$ and $y(t) = -y_{max}$. Let $\Gamma(\phi_0) = \Gamma_+ \cup \Gamma_-$. $\Gamma(\phi_0)$ divides the plane into two open regions D_2 and D_2^c where $(0,\phi) \in D_2$ if $\phi < \phi_0$. Then all solutions of the differential equation with $|y_m(t)| \leqslant y_{max}$ and $|v(t)| \leqslant v_{max}$ with initial conditions on $\Gamma(\phi_0)$ lie either in $\Gamma(\phi_0)$ or enter D_2. Since this is true for every ϕ_0, the solutions are unbounded and $\lim_{t\to\infty} \phi(t) = -\infty$.

b) Let $x \triangleq [e_1,\phi]^T$. Let D denote the region in $\mathbb{R}^2$ $D \triangleq \{x \mid |e_1| \leq v_{max}\}$ and let D^c denote the complement of D. If $W(x) = 1/2\, x^T x$, the time derivative of W along a trajectory is $\dot{W}(x) = -e_1^2 + e_1 v < 0$ for $x \in D^c$. Hence $\|x\|$ decreases in D^c and can increase only in D. We wish to show that a constant c exists so that if $\|x(t_0)\| = c_1 \geq c$ over an interval $[t_0, t_0 + T_0]$, then $\|x(t_0 + T_0)\| < c_1$.

If $\|x(t_0)\| = c_0$, integrating the equation for $\dot{e}_1$ in (17) it can be shown that if $x(t_0) \in D$, then $x(t_1) \in D^c$ for some $t_1 \in [t_0, t_0 + T_0]$ if $c_0 > \frac{v_{max}}{\sin\theta}$, where $\cot\theta = \frac{2[T+1]}{(\varepsilon_0 - v_{max})T}$. Hence under the conditions specified in the theorem, the trajectory invariably enters D^c during every period T_0. By increasing $\|x(t_0)\|$ monotonically, the trajectory can be made to lie in a subdomain of D^c for a finite time Δ with $0 < \Delta < \delta_0$ over every period. Since $\|x(t)\|$ decays exponentially in this subdomain, a constant $c > c_0$ exists satisfying the conditions of the theorem.

Comments: 1. The positive limit set of any solution x(t) lies in D.

2. ε_0 will be referred to as the degree of persistent excitation. By the theorem, the solutions are bounded if $\varepsilon_0 > v_{max}$ but the nature of the limit set depends on T_0, ε_0 and δ_0.

3. From the theorem it follows that for a given bound v_{max} on the perturbations, the system can be made robust by increasing the degree of persistent excitation. Note that this is an example of practical stability.

4. The conditions for boundedness and unboundedness of solutions are given in this case in terms of $y_m(t)$. For design purposes it is more appropriate to express them in terms of the reference input r(t).

5. HYBRID ADAPTIVE CONTROL

In continuous adaptive systems of the type described in the previous sections, the plant operates in continuous time and the controller parameters are adjusted continuously. Recent advances in microprocessor and related digital computer technology favor the use of discrete systems in which signals are defined at discrete instants. Practical systems on the other hand may contain both discrete and continuous elements. Such systems may be described as hybrid systems. In a recent report [10] the authors have developed analytical models of hybrid systems in which control parameters are adjusted in discrete time even as the continuous plant signals are processed in real time. The same algorithms can also be extended to control discrete time plants so that the overall discrete system operates on two time scales - a fast time scale in which the system operates and a slow time scale in which the control parameters are updated. We shall refer to such a system as a discrete hybrid system.

In this section we describe briefly one of the hybrid adaptive algorithms and demonstrate global stability in the ideal case of an adaptive system which uses such an algorithm. The behavior of a discrete hybrid system is then discussed when bounded external disturbances are present. Using the results of the previous section, arguments are put forward as to why hybrid schemes should result in more robust systems and simulation results are presented to show that this is indeed the case.

a) Hybrid Error Model: In this section we consider the first of several hybrid error models given in [10] and discuss its properties. Similar results can also be derived in all the other cases. The error model is described by the equation

$$\phi_k^T u(t) = e_1(t) \qquad t \in [t_k, t_{k+1}), \qquad k \in N \tag{19}$$

where $u: \mathbb{R}^+ \to \mathbb{R}^m$, $e_1: \mathbb{R}^+ \to \mathbb{R}^1$ are piecewise continuous functions which are referred to as the input and output functions of the error model. $\{t_k\}$ is a monotonically increasing unbounded sequence in $\mathbb{R}^+$ with $0 < T_{min} \leq T_k \leq T_{max} < \infty$ for $k \in N$ where $T_k = t_{k+1} - t_k$. When $T_k = T$, a constant, we shall call T the sampling period. $\phi: \mathbb{R}^+ \to \mathbb{R}^m$ is a piecewise constant function, referred to as the parameter error vector and assumes values $\phi(t) = \phi_k$, $t \in [t_k, t_{k+1})$, where ϕ_k is a constant vector.

It is assumed that ϕ_0 (and hence ϕ_k) is unknown, the values u(t) and $e_1(t)$ can be observed at every instant t and $\Delta\phi_k \triangleq \phi_{k+1} - \phi_k$ can be adjusted at $t = t_{k+1}$. The objective is to determine an adaptive law for choosing the sequence $\{\Delta\phi_k\}$ using all available input-output data so that $\lim_{t\to\infty} e_1(t) = 0$.

Theorem 2: If in the error equation (19) the vector ϕ_k is updated according to the adaptive law

$$\Delta\phi_k = \frac{-1}{T_k} \int_{t_k}^{t_{k+1}} \frac{e_1(\tau)u(\tau)}{1 + u^T(\tau)u(\tau)} \, d\tau \tag{20}$$

then

(i) if u(t) and $\dot{u}(t)$ are uniformly bounded in $\mathbb{R}^+$ $\lim_{t\to\infty} e_1(t) = 0$

(ii) if in addition to the conditions in (i) u is persistently exciting over an interval T_{min}, $\lim_{k\to\infty} \phi_k = 0$

(iii) If $u \in L_e^\infty$ then $e_1(t) = \rho(t)[1 + u^Tu]^{1/2}$, $\rho \in L^2$.

Proof: If $V(k) = \frac{1}{2} \phi_k^T \phi_k$, using the adaptive law (20) we obtain $\Delta V(k) = -\frac{1}{2} \phi_k^T [2I - R_k]R_k\phi_k \le 0$

where
$$R_k = \frac{1}{T_k} \int_{t_k}^{t_{k+1}} \frac{u(\tau)u^T(\tau)}{1 + u^T(\tau)u(\tau)} \, d\tau \, .$$

Hence $V(k)$ is a Lyapunov function and assures the boundedness of ϕ_k. Since $\sum_{k=1}^{\infty} \Delta V(k) < \infty$ it follows that $\lim_{k\to\infty} \Delta V(k) = 0$. Hence

$$\lim_{k\to\infty} \phi_k^T R_k \phi_k = \lim_{k\to\infty} \frac{1}{T_k} \int_{t_k}^{t_{k+1}} \frac{e_1^2(\tau)}{1 + u^T(\tau)u(\tau)} \, d\tau = 0.$$

(i) If u is bounded, e_1 is bounded and $e_1 \in L^2$. If $\dot{u}$ is bounded $\lim_{t\to\infty} e_1(t) = 0$.

(ii) If u is persistently exciting R_k is uniformly positive definite and hence $\phi_k \to 0$ as $k \to \infty$.

(iii) If u grows in an unbounded fashion with $u \in L_e^\infty$, $e_1 = \rho\sqrt{1 + u^Tu}$ where $\rho \in L^2$.

Comments: In the three cases given in theorem 2 the first two assume that the input u is uniformly bounded and the corresponding results are applicable to the identification problem. The third case which treats unbounded inputs is applicable to the control problem.

The fact that T_k need not be a constant is also worth noting. As shown in section 6 a time-varying period may be used to improve the transient response of the system.

b) Stable Hybrid Adaptive Control - Ideal Case: The hybrid adaptive algorithm described in the preceding section can be used to adjust the control parameters of a hybrid adaptive system. Using an approach very similar to that used in section 3 for a continuous time system the overall system can be shown to be globally stable. Using the same notation as in section 3 we have for the adaptive law

$$\Delta\phi_k = -\frac{1}{T_k} \int_{t_k}^{t_{k+1}} \frac{\varepsilon_1(\tau)\zeta(\tau)}{1 + \zeta^T(\tau)\zeta(\tau)} \, d\tau$$

From the analysis in the previous section we conclude that

(i) the parameter error vector ϕ_k is bounded

and (ii) $\varepsilon_1 = \rho\sqrt{1 + \zeta^T\zeta}$ where $\rho \in L^2$,

which conditions are the same as those obtained for the continuous case. Condition (i) assures that the signals y_p, $\omega^{(2)}$, $\|\omega(t)\|$ and $\|\zeta(t)\|$ grow at the same rate. Condition (ii) results in $|y_p(t)| = o \sup_{\tau\le t} \|\omega(\tau)\|$ which contradicts the previous assertion proving the boundedness of all the signals.

The similarity between the continuous and hybrid systems also extends to cases when external bounded disturbances are present and the methods described in sections 3 and 4 apply to the hybrid case as well. However, as shown in the following section, the use of averaged values over an interval rather than instantaneous values, results in more robust control.

c) Adaptive System with Two Time Scales: The hybrid adaptive algorithm developed in section 5a and applied to hybrid adaptive systems in section 5b can also be s[illegible] modified for discrete hybrid systems or discrete systems with two time [illegible]es, [illegible] shown below.

Let the output error $e_1(k) \varepsilon \mathbb{R}^1$ and the parameter error vector $\phi(k) \varepsilon \mathbb{R}^n$ be related by the error equation

$$\phi^T(k)w(\ell) = e_1(\ell) \qquad k,\ell \varepsilon N, \quad \ell \varepsilon[kT,(k+1)T] \tag{21}$$

where $\phi(k)$ is a constant vector over the interval $[kT,(k+1)T]$, $T \varepsilon N$ and denotes the period of the interval and $w(\ell) \varepsilon \mathbb{R}^n$ is an input vector. Using information collected over the entire interval, the parameter error vector $\phi(k)$ is updated at time $(k+1)T$ using the adaptive law

$$\begin{aligned}\phi(k+1) - \phi(k) \overset{\Delta}{=} \Delta\phi(k) &= - 1/T \sum_{i=kT}^{(k+1)T-1} \frac{e_1(i)w(i)}{1 + w(i)^T w(i)} \\ &= - R(k)\phi(k)\end{aligned} \tag{22}$$

where $R(k) \overset{\Delta}{=} 1/T \sum_{kT}^{(k+1)T-1} \frac{w(i)w(i)^T}{1 + w(i)^T w(i)}$.

In [10] it is shown that $V(k) = 1/2\ \phi^T(k)\phi(k)$ is a Lyapunov function for the system (21) from which it follows that $\phi(k)$ is bounded if $\phi(0)$ is bounded and

$$\lim_{i\to\infty} \frac{e_1(i)}{[1+w(i)^T w(i)]^{1/2}} = 0 \qquad i \varepsilon N \tag{23}$$

If the adaptive law (22) is used in a control system to update the parameters, equation (23) can be used to demonstrate global stability [10].

When an external disturbance v is present the error equation (23) have to be modified as

$$\phi^T(k)w(\ell) + v(\ell) = e_1(\ell) \qquad \ell \varepsilon[kT,(k+1)T] \tag{24}$$

Using the same adaptive law as before, the error equation has the form

$$\begin{aligned}\Delta\phi(k) &= -R(k)\phi(k) + \sum_{i=kT}^{(k+1)T-1} \frac{w(i)v(i)}{1 + w(i)^T w(i)} \\ &= -R(k)\phi(k) + s(k)\end{aligned} \tag{25}$$

where $s(k) \overset{\Delta}{=} \sum_{i=kT}^{(k+1)T-1} \frac{w(i)v(i)}{1 + w(i)^T w(i)}$.

The matrix $R(k)$ and the vector $s(k)$ in algorithm (25) are averaged values over an interval rather than instantaneous values. Hence the equivalent system may be considered to have more persistently exciting inputs in its homogeneous equation and a smaller magnitude of perturbation (if the mean value of the disturbance is small). Due to both reasons the outputs tend to be smaller. Simulation results shown in Fig. 2 indicate the dramatic improvement in performance.

6. NEW DIRECTIONS

The criteria for judging the performance of an adaptive control system are no different from those used for any conventional control system and include stability speed and accuracy of response. In the preceding sections methods using persistent excitation of reference input, and nonlinear and hybrid adaptive algorithms were described which would make the overall system stable under perturbations. A judicious combination of these different methods may improve the robustness of the system substantially and result in schemes which are practically attractive. Some of these combinations as well as extensions of known methods which appear promising are given below.

(i) Robustness of n^{th} Order System Using Persistent Excitation: A detailed analysis of a first order adaptive system containing a single control parameter was given in section 4. When a disturbance is present it was shown that a sufficiently large persistently exciting reference input would also result in bounded solutions. Further studies have revealed that similar conclusions can be drawn regarding higher order systems and research is currently being done to determine the bounds on the solutions.

(ii) Hybrid Adaptive Control: In the adaptive control system described in section 5, it was shown that the sampling interval T_k could itself be time-varying provided it lay in a bounded interval $[T_{min}, T_{max}]$ with $T_{min} > 0$. In practical systems it appears possible to adjust T_k on line to improve the transient response of the [illegible]n.

(iii) Dead-Zone, Persistent Excitation and Plant Identification: A sufficiently large dead-zone in the adaptive algorithm was shown to result in bounded solutions in section 3. The results in section 4 indicated that boundedness of solutions could also be achieved by increasing the PE of the reference input. It therefore appears likely that the same results can be achieved using a combination of a smaller dead-zone and a smaller degree of persistent excitation. Simulation studies have shown that this is indeed the case and attempts are being made to demonstrate this theoretically.

When the reference input is persistently exciting and the adaptive loop is stable, the plant parameters can be estimated on-line and used in second level adaptation to reduce the dead-zone further. Hence combining a dead-zone with PE of reference inputs appears to be of both theoretical and practical interest.

(iv) $\|\theta^*\|_{max}$ and Persistent Excitation: As in (iii) a persistently exciting input enables θ^* to be estimated and hence an attempt could be made to use the information to decrease the region of search.

(v) σ-modification and Persistent Excitation: The σ-modification scheme, in its basic form, described in section 3 is unappealing, since the parameter error can be large if $\|\theta^*\|$ is large. Using identification methods as in (iii) and (iv) and estimating θ^* on line, second level adaptive procedures may result in a smaller bias.

The second level adaptation problems stated in (ii)-(v) while practically attractive, lead to stability questions in more complex nonlinear systems. Further, it is worth pointing out that all of them consider external disturbances rather than perturbations in plant dynamics. The reduced order problem which deals with the design of a low order controller to adaptively control a higher order plant is generally agreed to be the single most important theoretical question in the field of adaptive control. While considerable research is being carried out in this area, it is acknowledged that even a proper formulation of this problem is a formidable one. It is felt that the answers to some of the questions raised in this section will contribute significantly towards this end.

Acknowledgment

The work reported here was supported by the National Science Foundation under grant ECS-8300223.

REFERENCES

1. K.S. Narendra, Y.H. Lin and L.S. Valavani, "Stable Adaptive Controller Design - Part II: Proof of Stability," IEEE Trans. on Autom. Control, AC-25, pp. 440-448, June 1980.

2. A.S. Morse, "Global Stability of Parameter Adaptive Control Systems," IEEE Trans. on Autom. Control, AC-25, pp. 433-439, June 1980.

3. K.S. Narendra and Y.H. Lin, "Stable Discrete Adaptive Control," IEEE Trans. on Autom. Control, AC-25, pp. 456-461, June 1980.

4. G.C. Goodwin, P.J. Ramadge and P.E. Caines, "Discrete Time Multivariable Adaptive Control," IEEE Trans. on Autom. Control, AC-25, pp. 449-456, June 1980.

5. B.B. Peterson and K.S. Narendra, "Bounded Error Adaptive Control," IEEE Trans. on Autom. Control, AC-27, pp. 1161-1168, Dec. 1982.

6. G. Kreisselmeier and K.S. Narendra, "Stable Model Reference Adaptive Control in the Presence of Bounded Disturbances," IEEE Trans. on Autom. Control, AC-27, pp. 1169-1175, Dec. 1982.

7. P.A. Ioannou and P V. Kokotovic, "Robust Redesign of Adaptive Control," IEEE Trans. on Autom. Control, AC-29, pp. 202-211, March 1984.

8. B.D.O. Anderson and R.M. Johnstone, "Adaptive Systems and Time Varying Plants," Int. Journal of Control, vol. 37, No. 2, pp. 367-377, 1983.

9. B.D.O. Anderson and R.M. Johnstone, "Robust Lyapunov Stability Results and Adaptive Systems," Proc. 20th IEEE CDC, San Diego, pp. 510-515, Dec. 1981.

10. K.S. Narendra, I.H. Khalifa and A.M. Annaswamy, "Error Models for Stable Hybrid Adaptive Systems - Part II," Tech. Report No. 8404, Center for Systems Science, Yale University, New Haven, Ct., Feb. 1984 (To be published in IEEE

Transactions on Automatic Control).

11. W. Hahn, *Theory and Application of Lyapunov's Direct Method with Applications*, Academic Press, New York, 1961

12. J.P. LaSalle and S. Lefschetz, *Stability by Lyapunov's Direct Method with Applications*, Academic Press, New York, 1961.

13. Krasovskii, *Stability of Motion*, Stanford University Press, Stanford, 1963.

14. C.A. Desoer, R. Liu and L.V. Auth, Jr., "Linearity vs. Nonlinearity and Asymptotic Stability in the Large," *IEEE Trans. on Circuit Theory*, pp. 117-118, March 1965.

15. K.S. Narendra and A.M. Annaswamy, "Persistent Excitation and Robust Adaptive Systems," Tech. Report No. 8302, Center for Systems Science, Yale University, New Haven, Ct., Aug. 1983.

16. K.S. Narendra and Y.H. Lin, "Design of Stable Model Reference Adaptive Controllers," *Applications of Adaptive Control*, pp. 69-130, Academic Press, New York, 1980.

17. K.S. Narendra and L.S. Valavani, "Stable Adaptive Controller Design-Direct Control," *IEEE Trans. on Autom. Control*, AC-23, pp. 570-583, Aug. 1978.

18. K.S. Narendra, A.M. Annaswamy and R.P. Singh, "A General Approach to the Stability Analysis of Adaptive Systems," Tech. Report No. 8402, Center for Systems Science, Yale University, New Haven, Ct., Jan. 1984 (To be published in *International Journal of Control*).

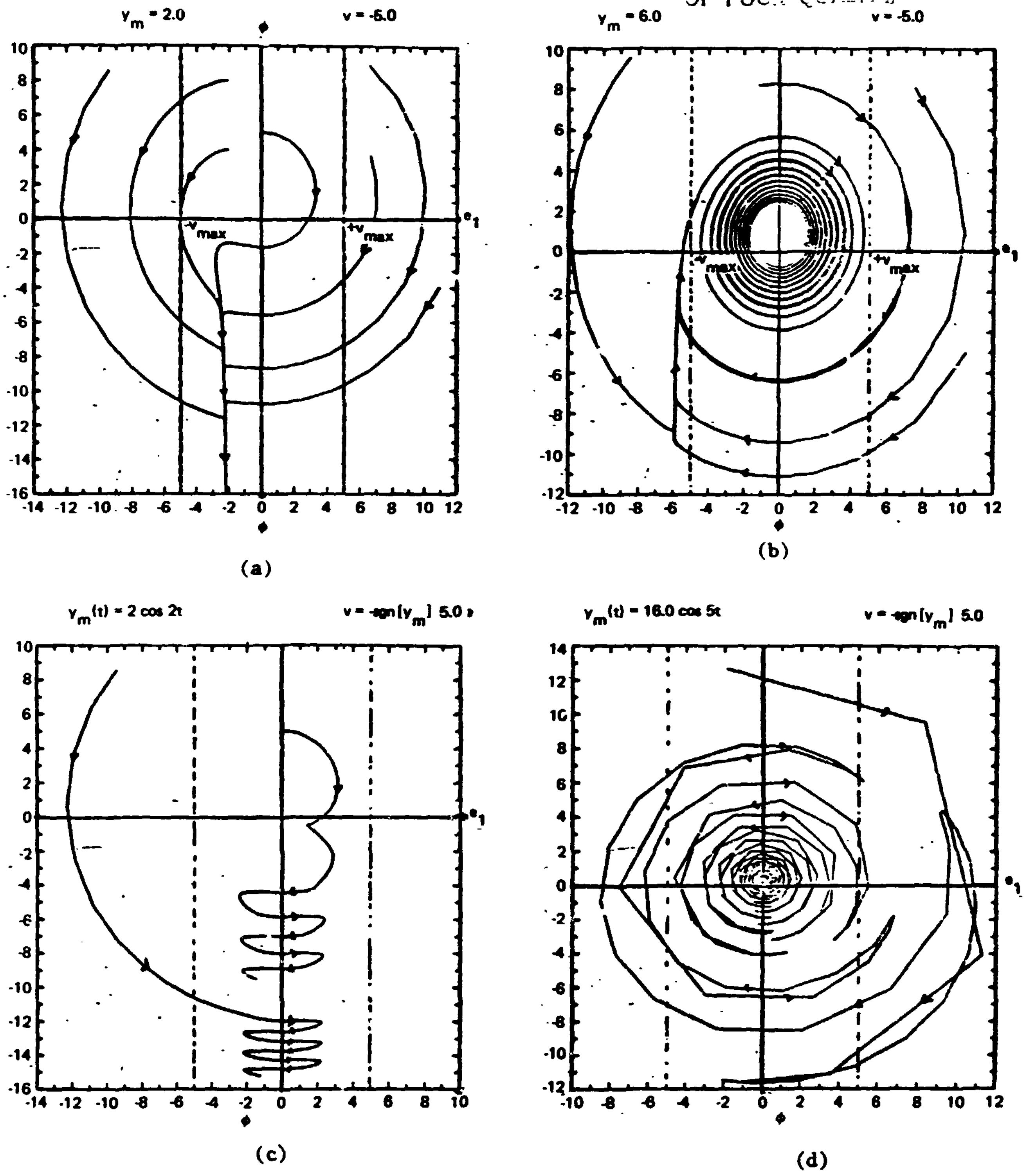
19. A.P. Morgan and K.S. Narendra, "On the Uniform Asymptotic Stability of Certain Linear Nonautonomous Differential Equations," *SIAM Journal of Control and Optimization*, vol. 15, pp. 5-24, Jan. 1977.

20. B.D.O. Anderson and C. Richard Johnson, Jr., "Exponential Convergence of Adaptive Identification and Control Algorithms," *Automatica*, vol. 18, pp. 1-13, 1982.

21. S. Dasgupta, B.D.O. Anderson and A.C. Tsoi, "Input Conditions for Continuous Time Adaptive Systems Problems," Proc. of the 22nd IEEE CDC, San Antonio, Dec. 1983.

22. S. Boyd and S.S. Sastry, "On Parameter Convergence in Adaptive Control," Tech. Report, June 1983.

23. A.P. Morgan and K.S. Narendra, "On the Stability of Nonautonomous Differential Equations $\dot{x} = [A + B(t)]x$ with Skew-Symmetric Matrix B(t)," *SIAM Journal of Control and Optimization*, vol. 15, pp. 163-176, Jan. 1977.



(a) (b) (c) (d)

$$\dot{e}_1 = -e_1 + \phi(e_1 + y_m) + v$$

$$\dot{\phi} = -e_1(e_1 + y_m)$$

Fig. 1: Persistent Excitation and Robustness.

(a) autonomous case $|y_m|<|v|$ unstable

(b) autonomous case $|y_m|>|v|$ stable

(c) non-autonomous case $y_{max}< v_{max}$ unstable

(d) non-autonomous case $\varepsilon_0 > v_{max}$ stable

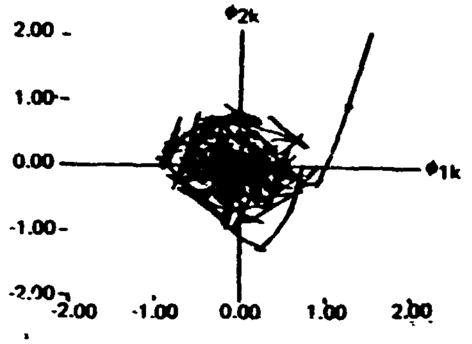


Fig. 2a: ϕ_{1k} and ϕ_{2k} with parameters adjusted at every instant

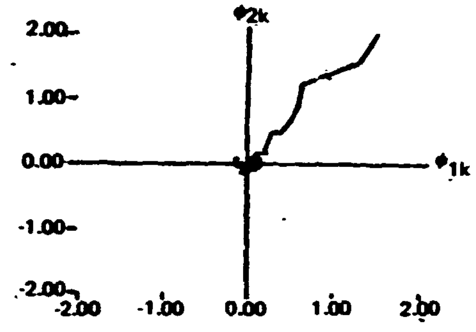


Fig. 2b: ϕ_{1k} and ϕ_{2k} with parameters held constant over [kT,(k+1)T]

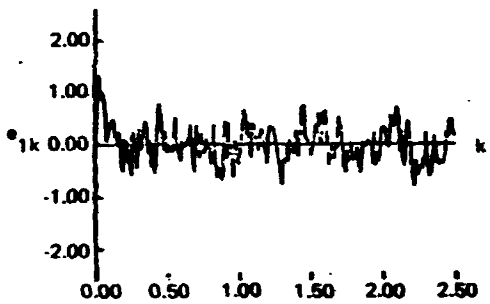


Fig. 2c: e_{1k} vs. k with parameters adjusted at every instant

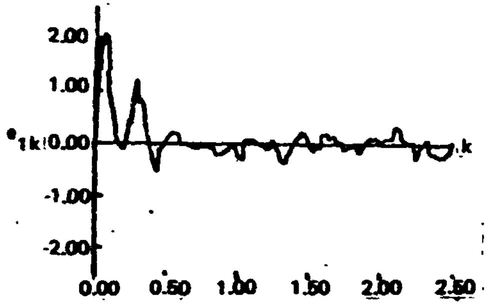


Fig. 2d: e_{1k} vs. k with parameters held constant over [kT,(k+1)T]

$\phi^T(k) = [\phi_{1k},\phi_{2k}]$ $\omega^T(t) = [\sin(.05t),\sin(.25t)]$ T = 5.0

Fig. 1

Fig. 2: Improved Robustness Using Hybrid Control