

AN EIGENSYSTEM REALIZATION ALGORITHM (ERA) FOR MODAL PARAMETER IDENTIFICATION AND MODEL REDUCTION

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ABSTRACT

A method, called the Eigensystem Realization Algorithm (ERA) is developed for modal parameter identification and model reduction of dynamic systems from test data. A new approach is introduced in conjunction with the singular value decomposition technique to derive the basic formulation of minimum order realization which is an extended version of the Ho-Kalman algorithm. The basic formulation is then transformed into modal space for modal parameter identification. Two accuracy indicators are developed to quantitatively identify the system modes and noise modes. For illustration of the algorithm, examples are shown using simulation data and experimental data for a rectangular grid structure.

I. INTRODUCTION

The state space model has received considerable attention for system analyses and design in recent control and systems research programs. One of these areas, in particular, is control of large space structures. In order to design controls for a dynamic system it is necessary to have a mathematical model which will adequately describe the system's motion. The process of constructing a state space representation from experimental data is called system realization.

During the past two decades, numerous algorithms for the construction of state space representations of linear systems have appeared in the control literature. Among the first were the works of Gilbert [1] and Kalman [2], introducing the important principles of realization theory in terms of the concepts of controllability and observability. Both techniques use the transfer function matrix to solve the realization problem. Ho and Kalman [3] approached this problem from a new viewpoint. They showed that the minimum realization problem is equivalent to a representation problem involving a sequence of real matrices known as Markov parameters (pulse response functions). By minimum realization is meant the smallest state-space dimension among systems realized that has the same input-output relations within a specified degree of accuracy. Questions regarding the minimum realization from various types of input-output data and the generation of minimum partial realization are studied by Tether [4], Silverman [5], and Rossen and Lapidus [6] using Markov parameters. Rossen and Lapidus [7] successfully applied Ho-Kalman [3] and Tether [4] methods to chemical engineering systems. A common weakness of the above schemes is that effects of noise on the data analysis were not evaluated. Zeiger and McEwen

[8] proposed a combination of the Ho-Kalman algorithm [3] with the singular value decomposition technique for the treatment of noisy data. However, no theoretical or numerical studies were reported in Reference [8]. Among follow-up developments along similar lines, Kung [9] presented an algorithm in conjunction with the singular value decomposition technique to incorporate the presence of the noise. Note that the singular value decomposition technique [10-11] has been widely recognized as being very effective and numerically stable. Although several techniques of minimum realization are available in the literature, formal direct application to the modal parameter identification for flexible structures was not yet addressed.

In the structures field, the finite-element technique is used almost exclusively for constructing analytical models. This approach is well established and normally provides a model accurate enough for structural design purposes. Once the structure is built, static and dynamic tests are performed. These test results are used to refine the finite-element model, which is then used for final analyses. This traditional approach to analytical model development may not be accurate enough for use in designing a vibration control system for flexible structures. Another approach is to realize a model directly from the experimental results. This requires the construction of a minimum-order model from the test data that characterizes the dynamics of the system at the selected control and measurement positions. The present state-of-the-art in structural modal testing and data analysis is one of controversy about the best technique to use. Classical test techniques, which may provide only good frequency and moderate mode shape accuracy, are often considered adequate for finite-element model verification purposes. On the other hand, advanced data analysis techniques which offer significant reductions in test time and improved accuracy, have been available [12-16] but are not yet fully used. For example, Ibrahim [13] presented a method based on state space for the direct identification of modal parameters from free responses. Recently, Void and Russell [16] presented a method using frequency response functions and time domain analysis which can also identify repeated eigenvalues. A comparison of contemporary methods using data from the Galileo spacecraft test is provided by Chen [17].

Although structural dynamics techniques are generally successful for ground data, further incorporation with work from the controls discipline is needed to solve modal parameter identification/control problem. For example, it is known from control theory [18] that a system with repeated eigenvalues and independent mode shapes is not identifiable by single input and single output. Methods which allows only one initial condition (input) at a time [13], will miss repeated eigenvalues. Also, if the realized system is not of a minimum order and matrix inversion is used for constructing an oversized state matrix, numerical errors may become dominant.

Under the interaction of structure and control disciplines, the objective of this paper is to introduce an Eigensystem Realization Algorithm (ERA) for modal parameter identification and model reduction for dynamical systems from test data. The algorithm consists of two major parts, namely, basic formulation of the minimum order realization and modal parameter identification. In the section of basic formulation, the Hankel matrix which represents the data structure for Ho-Kalman algorithm is generalized to allow random distribution

of Markov parameters generated by free decay responses. A unique approach based on this generalized Hankel matrix is developed to extend the Ho-Kalman algorithm in combination with the singular value decomposition technique [10-11]. Through the use of the generalized Hankel matrix, a linear model is realized for dynamical system matching the input and output relationship. The realized system model is then transformed into modal space for modal parameter identifications. As part of ERA, two accuracy indicators, namely, the modal amplitude coherence and the modal phase collinearity, are developed to quantify the system modes and noise modes. The degree of modal excitation and observation are evaluated. The ERA method thus forms the basis for a rational choice of model size determined by the singular values and accuracy indicators.

Two examples are given to illustrate the ERA method. The first example uses simulated data from an assumed structure. The effect of repeated eigenvalues on the parameter identification is shown. The second example uses experimental data from a simple grid structure. Comparison of the ERA results with a finite element model of the grid is performed. Experimental results for a more complex structure--the Galileo spacecraft--are shown in Ref. [19].

II. BASIC FORMULATIONS

A finite-dimensional, discrete time, linear, time-invariant dynamical system has the state-variable equations

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$

$$y(k) = Cx(k) \quad (2)$$

where x is an n dimensional state vector, u is an m dimensional control input, and y is an p dimensional output or measurement vector. The integer k is the sample indicator. The transition matrix A characterizes the dynamics of the system. For flexible structures, it is a representation of mass, stiffness and damping properties. The problem of system realization is then the following. Given the measurement functions $y(k)$, construct constant matrices $[A, B, C]$ such that the functions y are reproduced by the state-variable equations. With different sets of inputs and outputs, several cases can be obtained. The simplest case, namely, single input and single output, is treated first to allow the reader familiar with notations for the treatment of multi-input and multi-output cases.

Single input and single output

For the system (1) with free pulse-response (or initial-state-response), the time-domain description is given by the function known as Markov parameter

$$y(k) = CA^{k-1}B \quad [\text{or } y(k) = CA^kx(0)] \quad (3)$$

where $x(0)$ is the system initial conditions and k is an integer. Note that the matrix B is a column vector (single input) whereas the matrix C is a row vector

(single output). For free initial-state-response, the matrix B only represents the information of initial conditions rather than the control influence matrix as shown in Eq.(1). The problem of system realization is to construct matrices [A, B, C] in terms of the measurement function y(k) such that the identities of Eq.(3) hold. Now observe that

$$\bar{y}(k) = VA^{k-1}B \quad [\text{or } \bar{y}(k) = VA^kx(0)] \quad (4)$$

where

$$\bar{y}(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix}; \quad \text{and} \quad V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (5)$$

Assume that this nth order system has no repeated eigenvalues. There exists a row vector C from observability theory (Ref. 18) such that V has rank n. Consequently, rearranging Eq. (4) becomes

$$\bar{y}(k+1) = VA^k B = VAV^{-1}\bar{y}(k) \quad (6)$$

Given the sequence of measurement vectors $\bar{y}(k+1)$, the generalized Hankel matrix H(k) is defined as

$$H(k-1) = [\bar{y}(k), \bar{y}(k+1), \dots, \bar{y}(k+n)] = \begin{bmatrix} y(k) & y(k+1) & \dots & y(k+n-1) \\ y(k+1) & y(k+2) & \dots & y(k+n) \\ \vdots & \vdots & \ddots & \vdots \\ y(k+n-1) & y(k+n) & \dots & y(k+2n-2) \end{bmatrix} \quad (7)$$

It immediately follows from Eq.(6) that

$$H(k) = VAV^{-1}H(k-1) = VA^kV^{-1}H(0) \quad (8)$$

or from Eq. (4) that

$$H(k) = VA^k[B, AB, \dots, A^{k-1}B] = VA^k W \quad (9)$$

where W is a controllability matrix (Ref. 18). Again if the system with order n has no repeated eigenvalues, there exists a column vector B such that W has rank n. This means that H(k) is invertible if the system is controllable and observable. Letting k = 1, Eq. (8) will then determine the state matrix A in the following way

$$VAV^{-1} = H(1)H^{-1}(0) \quad (10)$$

To rigorously prove this result, define E as the column vector $E^T = [1, 0, \dots, 0]$. The measurement function y(k+1) can then be written by

$$y(k+1) = E^T H(k) E = E^T H(k) H^{-1}(0) H(0) E = E^T [H(1)H^{-1}(0)]^k H(0) E \quad (11)$$

with the aid of Eqs. (8) and (10). Hence by Eq. (3), the triple $[H(1)H^{-1}(0), H(0)E, E^T]$ is a realization in the sense that if the triple [A, B, C] in the

system equations (1) and (2) is replaced by the $[H(1)H^{-1}(0), H(0)E, E^T]$, the measurement functions $y(k)$ are reproduced as proved in Eq. (11). In other words, state variable equations (1) and (2) are transformed to the following equations

$$\bar{x}(k+1) = H(1)H^{-1}(0)\bar{x}(k) + H(0)Eu(k) \quad (12)$$

$$y(k) = E^T\bar{x}(k) \quad (13)$$

where $\bar{x}(k) = V^{-1}x(k)$. (14)

Let us summarize the case as follows.

A finite-dimensional, discrete time, linear time invariant dynamical system with a single input and a single output is realizable if the state matrix A has no repeated eigenvalues, and the system is controllable and observable.

Multi-input and Multi-output

The time-domain description for this case is given by the pulse-response (or initial-state-reponse) function known as Markov parameter

$$Y(k) = CA^{k-1}B \quad (\text{or } Y(k) = CA^k[x_1(0), x_2(0), \dots, x_m(0)]) \quad (15)$$

where $x_i(0)$ represents the i th set of initial condition and k is an integer. Note that B is a $n \times m$ matrix and C is a $p \times n$ matrix. The problem of system realization is that, given the functions $Y(k)$, construct constant matrices $[A, B, C]$ in terms of $Y(k)$ such that the identities of Eq. (15) hold. The algorithm begins by forming the $r \times s$ block matrix (generalized Hankel matrix)

$$H_{rs}(k-1) = \begin{bmatrix} Y(k) & , Y(k+t_1) & \dots, Y(k+t_{s-1}) \\ Y(j_1+k) & , Y(j_1+k+t_1) & \dots, Y(j_1+k+t_{s-1}) \\ \vdots & \vdots & \ddots \vdots \\ Y(j_{r-1}+k) & , Y(j_{r-1}+k+t_1) & \dots, Y(j_{r-1}+k+t_{s-1}) \end{bmatrix} \quad (16)$$

where $j_i (i=1, \dots, r-1)$ and $t_i (i=1, \dots, s-1)$ are arbitrary integers. For the system with initial-state-response measurements, simply replace $H_{rs}(k-1)$ by $H_{rs}(k)$. It is easy to prove from Eq. (15) that Eq. (9) also holds for this multi-input and multi-output case,

$$H_{rs}(k) = V_r A^k W_s \quad ; \quad V_r = \begin{bmatrix} C \\ CA^{j_1} \\ \vdots \\ CA^{j_{r-1}} \end{bmatrix} \quad \text{and } W_s = [B, A^{t_1}B, \dots, A^{t_{s-1}}B] \quad (17)$$

where V_r and W_s are respectively the observability and controllability matrices in a general sense. Note that V_r and W_s are rectangular matrices with dimensions $r \times n$ and $n \times s$ respectively. Assume that there exist a matrix $H^\#$ satisfying the relation

$$W_S H^\# V_R = I_n \quad (18)$$

where I_n is an identity matrix of order n . Define O_p as a null matrix with order p , $E_p^T = [I_p, O_p, \dots, O_p]$ and $E_m^T = [I_m, O_m, \dots, O_m]$. In view of Eqs. (16) and (18), the measurement function $Y(k+1)$ can be obtained through either of two algorithms A1 and A2. The algorithm A1 is

$$\begin{aligned} Y(k+1) &= E_p^T H_{RS}(k) E_m = E_p^T V_R A^k W_S H^\# V_R W_S E_m \\ &= E_p^T [V_R A W_S H^\#]^k V_R W_S E_m \\ &= E_p^T [H_{RS}(1) H^\#]^k H_{RS}(0) E_m \end{aligned} \quad (19)$$

and the algorithm A2 is

$$\begin{aligned} Y(k+1) &= E_p^T H_{RS}(k) E_m = E_p^T V_R W_S H^\# V_R A^k W_S E_m \\ &= E_p^T V_R W_S [H^\# V_R A W_S]^k E_m \\ &= E_p^T H_{RS}(0) [H^\# H_{RS}(1)]^k E_m \end{aligned} \quad (20)$$

Hence, by Eq.(15), $[H_{RS}(1) H^\#, H_{RS}(0) E_m, E_p^T]$ or $[H^\# H_{RS}(1), E_m, E_p^T H_{RS}(0)]$ is a realization. There is no doubt that the matrix $H^\#$ plays a major role in solutions (19) and (20). What is $H^\#$? Observe that, from Eqs. (17) and (18),

$$H_{RS}(0) H^\# H_{RS}(0) = V_R W_S H^\# V_R W_S = V_R W_S = H_{RS}(0) \quad (21)$$

$H^\#$ is a pseudo-inverse of $H_{RS}(0)$ in a general sense. When the rank of $H_{RS}(0)$ equals to the column number of $H_{RS}(0)$, then $H^\# = [[H_{RS}(0)]^T H_{RS}(0)]^{-1} [H_{RS}(0)]^T$.

If the rank equals to the row number, then $H^\# = [H_{RS}(0)]^T [H_{RS}(0) [H_{RS}(0)]^T]^{-1}$. The matrix $H_{RS}(1) H^\#$ has been used in structural dynamics area to identify system modes and frequencies.¹³ Both are special cases representing either single input or single output which can not realize a system that has repeated eigenvalues, or a noise-free system unless the system order is a priori known. A general solution for $H^\#$ is given below.

For an n th order system, find the nonsingular matrices P and Q such that^{10,11}

$$H_{RS}(0) = P D Q^T \quad (22)$$

where the $r \times n$ matrix P and the $n \times m$ matrix Q^T are isometric matrices (all the columns are orthonormal), leaving the singular values of $H_{RS}(0)$ in the diagonal matrix D with positive elements $[d_1, d_2, \dots, d_n]$. The rank n of $H_{RS}(0)$ is determined by testing the singular values for zero (relative to desired accuracy)¹² which will be described in the next section. Define

$$H_{RS}(0) = P D Q^T = [P D] [Q^T] = P_d Q^T \quad (23)$$

Each of the four matrices $[P_d^T, Q^T, W_s, V_r^T]$ has rank and row number n . By Eq.(17) with $k=0$,

$$V_r W_s = H_{rs}(0) = P_d Q^T \quad (24)$$

Multiplying on the left by P_d^T and solving for Q^T yields

$$T W_s = (P_d^T P_d)^{-1} P_d^T V_r W_s = Q^T \quad (25)$$

T is nonsingular because if $U = W_s Q (Q^T Q)^{-1} = W_s Q$, then $TU = I$ by Eq. (25). Since $TU = I = UT$ for nonsingular T and U then

$$W_s [Q (P_d^T P_d)^{-1} P_d^T] V_r = I_n \quad (26)$$

Hence, by Eq. (18)

$$H^{\#} = [Q] [(P_d^T P_d)^{-1} P_d^T] = [Q] [D^{-1} P^T] = Q P_d^{\#} \quad (27)$$

The dimension of matrices Q and $P_d^{\#}$ with rank n are respectively $m \times n$ and $n \times p$. To this end, summarize the case as follows.

A finite-dimensional, discrete time, linear time-invariant dynamical system with multi-input and multi-output is realizable in terms of the measurement function if the system is controllable and observable.

Note that no restrictions on system eigenvalues are given for this case. In other words, this technique can realize a system with repeated eigenvalues. The system (1) with this realization will be transformed into the following equation

$$\bar{x}(k+1) = H_{rs}(1) H^{\#} \bar{x}(k) + H_{rs}(0) E_m u \quad (28)$$

$$y(k) = E_p^T \bar{x}(k) \quad (29)$$

where $x(k) = V_s H^{\#} \bar{x}(k)$. Or

$$\bar{x}(k+1) = H^{\#} H_{rs}(1) \bar{x}(k) + E_m u \quad (31)$$

$$y(k) = E_p^T H_{rs}(0) \bar{x}(k) \quad (32)$$

where $\bar{x}(k) = W_s x(k)$ (33)

The realizations (28)-(33) are not of minimum order, since the dimension of \bar{x} is the number of either columns or rows of the matrix $H_{rs}(0)$ which is larger than the order n of the state matrix A for multi-input and multi-output cases.

With the aid of Eqs.(17), (18) and (27), a minimum order of realization can be obtained from

$$\begin{aligned}
Y(k+1) &= E_p^T H_{rs}(k) E_m = E_p^T V_r A^k W_s E_m \\
&= E_p^T V_r W_s H^{\#} V_r A^k W_s H^{\#} V_r W_s E_m \\
&= E_p^T H_{rs}(0) Q P_d^{\#} V_r A^k W_s Q P_d^{\#} H_{rs}(0) E_m \\
&= E_p^T H_{rs}(0) Q [P_d^{\#} H_{rs}(1) Q]^k P_d^{\#} H_{rs}(0) E_m \\
&= E_p^T P_d [P_d^{\#} H_{rs}(1) Q]^k Q^T E_m
\end{aligned} \tag{34}$$

where Eq.(23) has been used to obtain the last equality. This is the basic formulation of realization for ERA.

The triple $[P_d^{\#} H_{rs}(1) Q, Q^T E_m, E_p^T P_d]$ is a minimum realization, since the order n of $P_d^{\#} H_{rs}(1) Q$ equals to the dimension of the state vector x . The same solution in a different form for the case where $j_i = t_i = i$ ($i=1, \dots, r-1$) can be obtained by completely different approach as shown in Refs. [3 & 20]. The system (1) with this realization is written as

$$\bar{x}(k+1) = P_d^{\#} H_{rs}(1) Q \bar{x}(k) + Q^T E_m u \tag{35}$$

$$y(k) = E_p^T P_d \bar{x}(k) \tag{36}$$

where

$$x(k) = W_s Q \bar{x}(k) \tag{37}$$

A simple exercise such as replacing $Y(k+1)$ by $Y(k)$ in Eqs.(19), (20) and (34) shows that all the algorithms developed above are also true for the realization of a system with initial-state-response.

Examination of Eqs. (19), (20) and (34) reveals that algorithms (A1) and (A2) are special cases of ERA. A1 is formulated by inserting the identity matrix (18) into the right hand side of the state matrix A as shown in Eq. (19). On the other hand, A2 is obtained by inserting the identity matrix (18) into the left hand side of the state matrix A as shown in Eq. (20). However the algorithm ERA is formed by inserting the identity matrix (18) into both sides of the state matrix A as shown in Eq. (34). Because of the different insertion, A1 and A2 do not minimize the order of the state transition matrix. Mathematically, if the singular value decomposition technique is not included in the computational procedures, A1 and A2 can not be numerically implemented, unless a certain degree of artificial noise and/or system noise are present. Noises tend to make up the rank deficiency of the generalized Hankel matrix $H_{rs}(0)$ for algorithms A1 and A2. Since the degree of noise presence is generally unknown, algorithms A1 and A2 are not recommended.

III. MODAL PARAMETER IDENTIFICATION AND MODEL REDUCTION

The presence of almost unavoidable noise and structural nonlinearity introduces uncertainty about the rank of the generalized Hankel matrix and,

hence, about the dimension of resulting realization. By employing the singular value decomposition (SVD) technique, the rank structure of the Hankel matrix can be quantitatively displayed. The set of singular values can be used to judge the distance of the matrix with determined order to a lower-order one. Therefore, the structure of the generalized Hankel matrix can be properly exploited to efficiently solve the realization problem. These include an excellent numerical performance, stability of the realization and flexibility in determining order-error tradeoff.

Assume that, by Eq. (22)

$$D = \text{Diag. } [d_1, d_2, \dots, d_n, d_{n+1}, \dots, d_N] \quad (38)$$

with

$$d_1 \geq d_2 \geq \dots \geq d_n \geq d_{n+1} \geq \dots \geq d_N \quad (39)$$

If the matrix $H_{rs}(0)$ has a rank n then all the singular values $d_j (j=n+1, \dots, N)$ should be zero. When singular values $d_j (j=n+1, \dots, N)$ are not exactly zero but very small, then one can easily recognize that the matrix $H_{rs}(0)$ is not far away from a n -rank matrix. However, there can be real difficulties in determining a gap between the computed last nonzero singular value and what should be effectively considered zero, when measurement noise is present. Possible sources of the noise can be attributed to the measurement signal, computer round-off and instrument imperfections.

Look at the singular value d_n of the matrix $H_{rs}(0)$. Choose a number δ based on measurement errors incurred in estimating the elements of $H_{rs}(0)$ and/or round-off errors incurred in a previous computation to get them. If δ is chosen as "zero threshold" such that $\delta < d_n$, then the matrix $H_{rs}(0)$ is considered to have rank n . Unless information about the certainty of the measurement data are given, the number δ is defined as a function of the precision limit in the computer machine. For example, $\delta = d_n/d_1$ cannot exceed the precision limit. Further details are found in Ref. [11].

After the test of singular values, assume that the matrix $[P_d^{\#} H_{rs}(k) Q]$ has rank n . Find the eigenvalues z and eigenvectors ψ such that

$$\psi^{-1} [P_d^{\#} H_{rs}(k) Q] \psi = z \quad (40)$$

The modal damping rates and damped natural frequencies are simply the real and imaginary parts of s , after transformation from the z - to the s - plane using the relationship

$$s = [(1/nz) \pm 2j\pi] / (k\Delta\tau) \quad (41)$$

where $\Delta\tau$ is the data sampling interval and j is an integer. Note that k is generally chosen as 1 for simplicity. Although z and ψ are in complex domain, computation of Eq.(40) can be performed in the real domain (Ref. 21) since the state matrix realized for most flexible structures has independent eigenvectors.

The triple $[z, \psi^{-1} Q^T E_m, E_p^T P_d \psi]$ is obviously a minimum order of realization simply by observing Eq. (34). $E_p^T P_d \psi$ is called sensor modal displacements and $\psi^{-1} Q^T E_m$ initial modal amplitudes. To quantify the system modes and noise modes, two indicators are developed as follows.

Modal Amplitude Coherence γ

If the information about the uncertainties of the measurement is minimum, the rank thus determined by the SVD becomes larger than the number of excited and observed system modes to represent the presence of noises in modal space. In modal parameter identification, the indicator referred to as modal amplitude coherence is developed to quantitatively distinguish the system and noise modes. Based on the accuracy parameter, the degree of the modal excitation (controllability) is estimated.

The modal amplitude coherence is done by calculating the coherence between each modal amplitude history and an idea one formed by extrapolating the initial value of the history to latter points using the identified eigenvalue. Let the control input matrix (initial condition) be expressed

$$\psi^{-1}Q^T E_m = [b_1, b_2, \dots, b_n]^* \quad (42)$$

where * means transpose and complex conjugate, and the $1 \times m$ column vector b_j corresponds to the system eigenvalue $s_j (j=1, \dots, n)$. Consider the sequence

$$\bar{q}_j^* = [b_j^*, \exp(t_1 \Delta \tau s_j) b_j^*, \dots, \exp(t_{s-1} \Delta \tau s_j) b_j^*] \quad (43)$$

which represents the ideal modal amplitude in complex domain containing informations of the magnitude and phase angle with time step $\Delta \tau$. Now, define vectors q_j such that

$$\psi^{-1}Q^T = [q_1, q_2, \dots, q_n]^* \quad (44)$$

The complex vector q_j represents the modal amplitude time history from the real measurement data obtained by the decomposition of the Hankel matrix. Let γ_j be defined as the coherence parameter for the j th mode, satisfying the relation

$$\gamma_j = |\bar{q}_j^* q_j| / (|\bar{q}_j^* q_j| |\bar{q}_j^* q_j|)^{1/2} \quad (45)$$

where $|\cdot|$ represents the absolute value. The parameter γ_j takes only the values between 0 and 1. $\gamma_j \rightarrow 1$ as $\bar{q}_j \rightarrow q_j$ indicates that the realized system eigenvalue s_j and the initial modal amplitude b_j are very close to the true values for the j th mode of the system. On the other hand, if γ_j is far away from the value 1, the j th mode is a noise mode. However, to make a clear cut between the system modes and noises requires further studies. Obviously, the parameter γ_j quantifies the degree to which the modes were excited by a specific input, i.e. the degree of controllability.

Modal Phase Collinearity μ

For lightly damped structure, normal mode behavior should be observed. An indicator referred to as the modal phase collinearity is developed to measure the strength of linear functional relationship between the real part and the imaginary part of the sensor modal displacement (mode shape) for each mode. Based on the accuracy indicator, the degree of the modal observation is estimated. Define

$$E_p^T P_d \psi = [c_1, c_2, \dots, c_n] \quad (46)$$

where $c_j (j=1,2,\dots,n)$ is the sensor modal displacement corresponding to the j th realized mode. Let the column vector $\underline{1}$ of order p be

$$\underline{1}^T = [1, 1, \dots, 1] \quad (47)$$

in which p is the number of sensors. Now compute the following quantities for the j th mode shape.

$$\bar{c}_j = c_j^T \underline{1} / p \quad (48)$$

$$c_{rr} = [\text{Real}(c_j - \bar{c}_j \underline{1})]^T [\text{Real}(c_j - \bar{c}_j \underline{1})] \quad (49)$$

$$c_{ri} = [\text{Real}(c_j - \bar{c}_j \underline{1})]^T [\text{Imag}(c_j - \bar{c}_j \underline{1})] \quad (50)$$

$$c_{ii} = [\text{Imag}(c_j - \bar{c}_j \underline{1})]^T [\text{Imag}(c_j - \bar{c}_j \underline{1})] \quad (51)$$

$$e = (c_{ii} - c_{rr}) / 2c_{ri} \quad (52)$$

$$\theta = \arctan[e + \text{sgn}(e)(1 + e^2)^{1/2}] \quad (53)$$

where $\text{Real}(\)$ and $\text{Imag}(\)$ respectively are the real part and imaginary part of the complex vector $(\)$, and $\text{sgn}(\)$ is the sign of the scalar $(\)$. The modal phase collinearity μ_j for the j th mode is then defined as (Ref.22)

$$\mu_j = \{ c_{rr} + c_{ri} [2(e^2 + 1) \sin^2(\theta) - 1] / e \} / (c_{rr} + c_{ii}) ; j=1,2,\dots,n \quad (54)$$

This indicator checks the deviation from $0^\circ - 180^\circ$ behavior for components of j th identified sensor modal displacement. The parameter μ_j takes only the values between 0 and 1. $\mu_j \rightarrow 1$ indicates that the accuracy of the modal displacement is high. On the other hand, if μ_j is away from 1, the j th mode is either a noise mode or high damping is present.

Model Reduction

The dynamical system is composed of an interconnection of all the ERA identified modes. The accuracy indicators allow one to determine the degree of individual mode participation. Model reduction can then be made by truncating all the modes with low accuracy indicators. The accuracy of the complete modal decomposition process can be examined by comparing a reconstruction of $Y(k)$ formed by Eq.(35) with the original free decay responses, using the reduced model.

IV. SUMMARY OF ERA

A flowchart of the procedures to be followed to use ERA in system model identification is presented in figure 1. The computational steps are summarized

as follows:

1. Construct a block-Hankel matrix $H_{rs}(0)$ by arranging the measurement data into its rows with given r, s, t_i ($i = 1, 2, \dots, s-1$) and j_i ($i = 1, 2, \dots, r-1$), (Eq. 16).
2. Decompose $H_{rs}(0)$ using singular value decomposition (Eq. 23).
3. Determine the order of the system by examining the singular values of the Hankel matrix $H_{rs}(0)$ (Eq. 38).
4. Construct a minimum-order realization (A, B, C) using a shifted block-Hankel matrix (Eq. 34).
5. Find eigensolutions of the realized state matrix (Eq. 40) and compute the modal damping rates and frequencies. (Eq. 41).
6. Calculate the coherence parameter (Eq. 45) and the collinearity parameter (Eq. 54) to quantify system modes and noise modes.
7. Determine the reduced system model based on accuracy indicators, reconstruct function $Y(k)$ (Eq. 35) and compare with measurement data.

Note that the determination of r, s, t_i and j_i in Step 1 above requires further development. This determination is related to the choice of the measurement data to minimize the size of the Hankel matrix $H_{rs}(0)$ with the rank unchanged.

V. EXAMPLES: SIMULATION AND EXPERIMENT RESULTS

To illustrate the ERA method, two examples are given. First, a numerical problem will be posed and solved for an assumed structure with distinct and repeated frequencies. Second, experimental data for a simple, two-dimensional, grid structure as shown in Fig. 2 is used and realized in terms of a linear system. Experimental results are compared with those predicted by a finite element model.

Numerical Simulation

Figure 2 shows a representation of a typical flexible structure. The dynamical equation for this typical structure with initial-state-response in terms of system modes in modal space can be written as:

$$dg/dt = \Lambda g \quad (55)$$

$$y = \bar{C} g \quad (56)$$

where Λ is a canonical matrix with the diagonal blocks $\{\Lambda_1, \dots, \Lambda_k\}$,

g is the generalized modal amplitude and C is the generalized sensor influence matrix. The quasi-diagonal matrix Λ_j ($j=1, \dots, k$) has the matrix form

$$\Lambda_j = \begin{bmatrix} \delta_j & \omega_j \\ -\omega_j & \delta_j \end{bmatrix} \quad (57)$$

The complex values $\delta_j \pm i\omega_j$ are the eigenvalue of the frame structure.

Given a model described as in Eq. (55), results of some numerical simulation using the ERA scheme can be summarized in the sequel. Two cases will be given including systems with and without repeated eigenvalues. The numerical test is performed by taking as "data" y the output values of the solution of a model with the form (55) whose parameters Λ , C and initial condition $g(t_0)$ are known. In the analysis of physical systems, experimental methods generate the measurement data y . It is then desired to realize a system by using the data y and compare with the known model.

Case I: A model with distinct eigenvalues

Assume that parameters such as bending rigidity, mass density and damping coefficient of the assumed structure are adjusted to give

$$\Lambda_j = \begin{bmatrix} -0.01xj & j \\ -j & -0.01xj \end{bmatrix}; \quad j = 1, 2, 3, 4, 5 \quad (58)$$

To illustrate applications of ERA in a single input and single output case, A sensor is chosen and located to give

$$\bar{C} = [1, 0, 1, 0, 1, 0, 1, 0, 1, 0] \quad (59)$$

Let the initial condition for free decay responses be

$$g^T(t_0) = [0, 1, 0, 1, 0, 1, 0, 1, 0, 1] \quad (60)$$

Then the functions y with a sample time interval 0.05 second generated from the model (55) with known parameters (58), (59) and (60) are used as measurement data for the ERA procedure.

Using $j_i = t_i = i$ and $r=s=90$ in Eq. 16, the ERA realization of a dynamical system is

$$C = [0.709, 2.529, -0.347, -1.706, 0.814, -1.183, -1.382, -0.276, 1.129, 1.257] \quad (61)$$

$$g^T(t_0) = [0.103, 0.367, -0.114, -0.563, 0.395, -0.574, -0.696, -0.139, 0.396, 0.440] \quad (62)$$

and Λ is identical to that shown in Eq. (58) with the accuracy close to the precision limit of the computer. In the process of realization, the number $\delta = d_n/d_1$ as defined in Eq. (38) is set to be 10^{-12} . The singular values of the generalized Hankel matrix $H_{rs}(0)$ are

$$D = [49.86, 44.84, 33.69, 27.64, 23.69, 21.04, 13.57, 10.95, 6.374, 5.508] \quad (63)$$

All the values d_j ($j = 1, \dots, 90$) which has the number d_j/d_1 less than 10^{-12} are considered to be zero. The rank of the Hankel matrix $H_{rs}(0)$ is obviously ten which is identical to the order a priori given in Eq.(58). The realized state matrix is a minimum order of 10 and the eigensolutions are obtained from this 10×10 matrix. All the parameters for modal amplitude coherence (Eq.45) and modal phase collinearity (Eq. 54) are 100%. Although Eqs. (61) and (62) are a different realization from the system (59) and (60), they are equivalent in the sense that a unitary transformation and normalization will make them equal.

By forming the matrices V in Eq. (5) and W in Eq. (9) with the aids of Eqs.(58)-(60), the reader can see that this realization is controllable and observable.

Case II: A model with repeated eigenvalues and independent eigenvectors

Assume now that the system model is represented by

$$\Lambda_1 = \Lambda_2 = \begin{bmatrix} -0.01 & 1.0 \\ -1.0 & -0.01 \end{bmatrix} \quad (64)$$

and

$$\Lambda_j = \begin{bmatrix} -0.01 \times j & j \\ -j & -0.01 \times j \end{bmatrix} \quad j = 3, 4, 5 \quad (65)$$

Using the same process as last case, the ERA realization simply miss the repeated eigenvalue Λ_1 . The result is expected since, by control theory for a linear system, single input or single output does not make a system with repeated eigenvalues and independent eigenvectors controllable or observable. It can be verified that the matrices V in Eq. (5) and W in Eq. (9) formed by Eqs. (59), (60), (64) and (65) have rank 8. Multi-input and multi-output must be used to realize such a system. Let two sensors be chosen and located such that

$$C = \begin{bmatrix} 1, 0, 1, 0, 1, 0, 1, 0, 1, 0 \\ 0, 0, 1, 0, 1, 0, 1, 0, 1, 0 \end{bmatrix} \quad (66)$$

and two initial conditions for free decay responses

$$g^T(t_0) = \begin{bmatrix} 0, 1, 0, 1, 0, 1, 0, 1, 0, 1 \\ 0, 0, 0, 1, 0, 1, 0, 1, 0, 1 \end{bmatrix} \quad (67)$$

Note that the rows in Eqs. (66) and (67) are independent. For each initial condition, a series of "measurement" function y with a sample time interval 0.05 second can be generated from the model (55) where each y in this case is a vector with two elements for two different sensors. The free decay function Y in Eq. (15) is then a 2×2 matrix. Using that $j_i = t_i = 1$ and $r = s = 45$ for Eq. (16), the ERA realization for a dynamical system is then

$$C = \begin{bmatrix} 0.135, -1.686, 0.155, -0.172, 0.111, -0.032, 0.099, 0.035, 0.195, 0.177 \\ -0.004, 0.107, 0.142, -0.136, 0.111, -0.032, 0.099, 0.035, 0.195, 0.177 \end{bmatrix} \quad (68)$$

$$g^T(t_0) = \begin{bmatrix} -0.014, -0.457, 3.840, -3.692, 8.338, -2.405, 8.956, 3.181, 2.818, 2.554 \\ -0.051, 0.092, 3.605, -3.508, 8.338, -2.406, 8.956, 3.181, 2.818, 2.554 \end{bmatrix} \quad (69)$$

where Λ is identical to that shown in Eqs.(64-65). The singular values D are

$$D = [70.16, 44.32, 37.97, 25.25, 11.18, 9.050, 7.950, 3.873, 0.127, 0.026] \quad (70)$$

The same error window $\delta = d_n/d_1$ as last is used. All the parameters for modal amplitude coherence and modal phase colinearity are 100%. Again, Eqs. (68-69) and Eqs. (66-67) are equivalent in the sense that a unitary transformation and normalization will make them equal. The reader can easily verify that this realization is controllable and observable.

Sample Experimental Results

A sample set of modal identification results that have been obtained from laboratory test data using ERA are included in this section. The test article, shown in Fig. 2, is a 7 ft by 10 ft flexible grid structure that will be used at NASA Langley for vibration control experimentation. It is constructed of overlapping aluminum bars of 1/4 in. by 2 in. cross section, riveted together at one-foot intervals. Four rivets are used at each joint to provide a tight connection. The structure is suspended from a stiff overhead beam using two short cables attached to the top horizontal member. The results to be shown are from a preliminary dynamics test of the grid. It was conducted by exciting the structure with an airjet and measuring the free vibration response using nine non-contacting proximity sensors. The response sensors were attached to a stiff frame located adjacent to the grid for the measurement of out-of-plane motions. Eight different excitation frequencies corresponding to resonant responses were used. The sampling rate was 32 samples per second.

The ERA analysis was performed using a single matrix of data from all nine response measurements and eight initial conditions. Each response function Y as shown in Eq. (16) was thus a 9×8 matrix. The Hankel matrix H_{75} of 72 rows by 400 columns was formed to perform the analysis. Table 1 provides a comparison of the ERA results with analytical prediction from a NASTRAN finite-element model. The entries in the center of table are correlation coefficients in percent between each ERA-identified mode shape and each NASTRAN mode shape. High correlation values indicate good agreement between the two shapes. The results show reasonable agreement in both frequencies and mode shapes, except for the damping result of the first mode. The main reason for the first mode discrepancy is inadequate data length. Only 50 data points were used which corresponds to less than one cycle of data for the first mode. The results can be improved by using more data points. Note that few high correlations occur for some modes with significantly different frequencies. This is because only 9 sensors were used in comparison. More detailed experimental results for a complex structure are shown in Ref.[19].

CONCLUDING REMARKS

An Eigensystem Realization Algorithm (ERA) is developed for parameter identification and model reduction for dynamical systems. Two developments are given in this paper. First, a new approach is developed to derive the basic ERA formulation of minimum realization for dynamical systems. As by-products of this approach, two alternative less powerful algorithms, identified as A1 and A2, are derived. A special case of A1 is shown to be equivalent to an approach currently in use in structural dynamics. Second, accuracy indicators are developed to quantify the participation of system modes and noise modes in the realized system model. In other words, degree of controllability and observability for each participated mode is determined. A model reduction can then be made for controller design.

Important features of the ERA algorithm are summarized as follows. (1) From the computational standpoint, the algorithm is attractive, since only simple numerical operations are needed; (2) the computational procedure is numerically stable; (3) the structural dynamics requirements for modal parameter identification and the control design requirements for a reduced state space model are satisfied; (4) data from more than one test can be used simultaneously to efficiently identify the closely spaced eigenvalues; (5) no restrictions on number of measurements are imposed.

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Table 1 : Comparison of the ERA results with the NASTRAN Model

MODE SHAPE CORRELATION	ANALYTICAL FREQUENCY (NASTRAN), HZ										DECAY RATE %		μ
	.362	.612	1.425	2.331	2.826	4.936	5.476	6.429	7.047	9.841	γ	μ	
(ERA)	99	1	21	41	1	42	17	1	0	1	23.1	71	100
F	3	99	3	0	1	1	2	98	43	9	0.33	100	99.9
I	22	1	96	36	3	12	45	2	2	3	0.64	99.8	99.9
D	38	0	8	94	7	1	1	1	0	7	0.33	100	100
E	2	6	4	3	96	2	9	3	15	96	0.26	100	99.9
N	48	11	22	35	3	70	65	10	1	3	0.19	100	99.9
T	9	9	29	19	2	46	81	8	6	2	0.61	99.5	97.5
I	4	73	1	5	3	8	0	79	91	10	0.12	100	99.4
F	0	19	8	0	90	2	9	8	22	91	0.24	97.3	99.9

γ : Modal Amplitude Coherence

μ : Modal Phase Collinearity

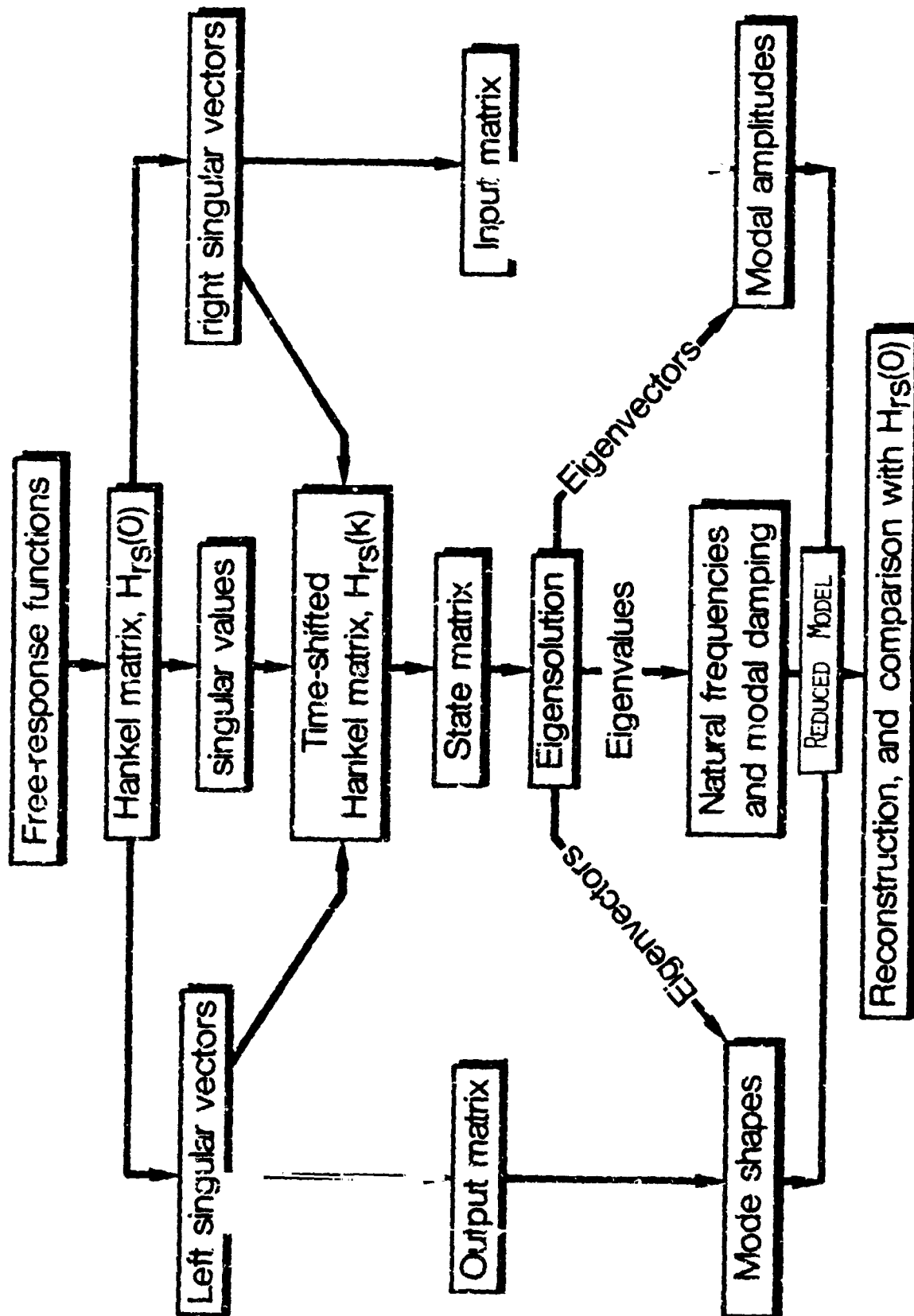


FIGURE 1. FLOW CHART OF ERA

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OF POOR QUALITY

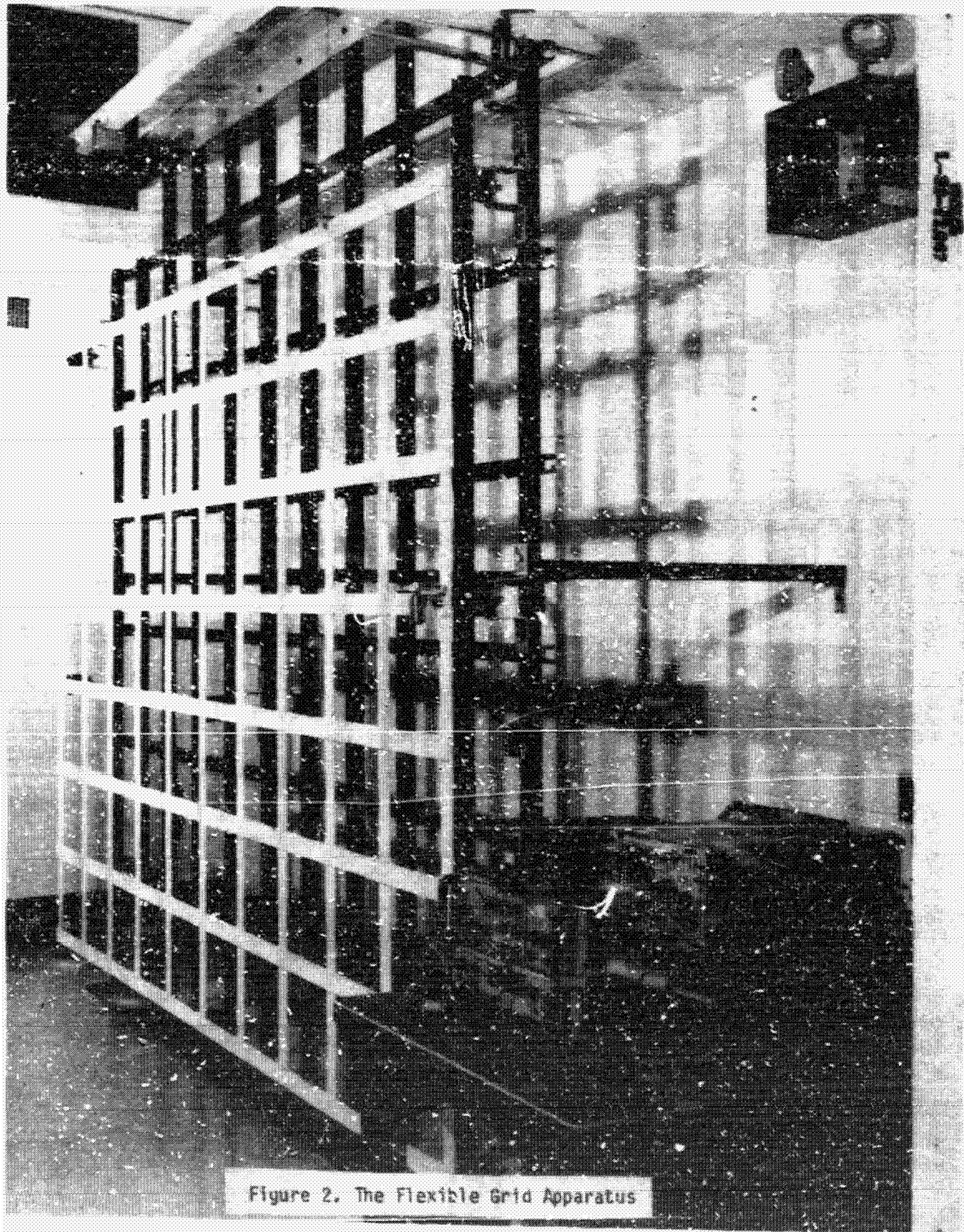


Figure 2. The Flexible Grid Apparatus