LONGITUDINAL TRIAL FUNCTIONS AND THE COSMIC RAY ENERGY SCALE

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ABSTRACT

Formulae which have been proposed for representing the longitudinal profiles of cosmic ray air showers are compared, and the physical interpretation of the parameters they contain is examined. Applications to the problem of energy calibration are pointed out. Adoption of a certain especially simple formula is recommended, and its use is illustrated by means of examples.

1. Introduction. At all primary energies above $\sim 10^3$ GeV, incoming cosmic ray nuclei deposit more than half of their energy via the soft component. At the highest observed energies the fraction E_{EM}/E approaches 90-95%, where E_{EM} is the energy deposited by the composite electron-photon cascade resulting primarily from π_o decay γ-rays. The single most important step, therefore, in determining the cosmic ray energy scale for $E > 10^6$ GeV, where experiments above the earth's atmosphere currently give way to experiments using air showers, is measurement of E_{EM}.

The use of simple mathematical formulae to represent the *lateral* structure of extensive air showers began at an early stage in the study of shower phenomena (cf. Bethe quoted by Williams 1948), and such formulae continue to be useful tools. Analogous formulae for the *longitudinal* structure developed much later, possibly because accurate experimental data have taken much longer to obtain, and possibly because analytical methods have tended to be supplanted in recent years by Monte Carlo calculations.

With the advent of techniques for directly measuring the longitudinal profiles of individual air showers (Hammond *et al.* 1978, Grigoriev *et al.* 1979, Cassiday 1981, Cady *et al.* 1983) it seems an appropriate time to re-examine formulae that have been used to describe these profiles (Greisen 1956, Linsley 1967, Longo and Sestili 1975, Gaisser 1976 and 1979, Gaisser and Hillas 1977, Sass and Spiro 1978, Dyakonov *et al.* 1981). By analogy to formulae used for describing the lateral structure of various air shower components I will call them 'trial functions'. In case of lateral structure such functions have had an important role in comparing results of different experiments. One expects longitudinal trial functions to be useful in the same way. In case of the lateral structure of electrons, trial functions are also used in the conduct of experiments, to derive from raw data on shower density a global measure, the shower size N, as well as core location, age, and measures such as S_{600}. The corresponding step in dealing with profile data is to find N_m, long recognized as being one of the most reliable estimators of primary energy (Clark 1962), from data on N vs x. This step also yields x_m, the shower elongation, and may also yield the profile width σ_x. In case of the electronic lateral structure, trial functions are formulated in such a manner as to reflect the physical processes that govern cascade development. Thus core distances are expressed in Moliére units, and steepness is controlled by a parameter relating plausibly to shower age. One aspires to achieve similar success in case of the longitudinal structure.

2. Electromagnetic cascades. The longitudinal trial function proposed by Greisen (1956) provides a good illustration:

$$N = (0.31/\beta_o^{\frac{1}{2}})\exp[t(1 - \frac{3}{2}\ln s)], \text{ where } s = 3t/(t + 2\beta_o) \ . \qquad (1)$$

Clearly its structure was influenced in several ways by theoretical results obtained with diffusion equations (see Rossi 1952):

1) Thickness is measured in units of the radiation length ($t = x/x_o$).

2) Primary energy is measured in units of the critical energy E_c ($\beta_o = \ln(E/E_c)$).

3) There is a built-in elongation-energy relation, $x_m = x_o\beta_o$, which is approximately correct.

4) There is a built-in N_m relation, $N_m = (0.31E/E_c)/[\ln(E/E_c)]^{\frac{1}{2}}$, which is also approximately correct.

5) The shower age s is incorporated in an approximately correct manner.

The price paid for (5) is mathematical inconvenience. Expressions for N, Nt and Nt^2 cannot be integrated in closed form, so simple formulae for the track length integral, the average depth <x> and the profile width σ_x, provided by the exact theory, must be patched on *ad hoc*.

An alternative form is the gamma distribution, recommended in the current *Particle Properties Data Booklet* for representing results from EGS, a well known Monte Carlo program for simulating electronic cascades. Gamma distributions have been chosen independently by several workers to represent air shower profiles as well. Indeed, the choice has been unanimous, and this is the choice recommended here.

3. Mathematical properties of gamma distributions. I will write this distribution in the form

$$N = N_o \xi^q e^{-q\xi} \ , \qquad (2)$$

calling the numerical constant q the 'index'. The optimum values of ξ and N are given by

$$\xi_m = 1 \ , \qquad (3) \qquad\qquad N_m = N_o e^{-q} \ , \qquad (4)$$

while the mean value and variance are given by

$$\langle\xi\rangle = \frac{q+1}{q} \qquad (5) \qquad\qquad \sigma_\xi^2 = \frac{q+1}{q^2} \ . \qquad (6)$$

The normalization is given by

$$\int N d\xi = \Gamma(q+1)/q^{q+1} \ , \qquad (7)$$

which admits approximation using Stirling's theorem

$$\int N d\xi \cong N_o(2\pi/q)^{\frac{1}{2}}e^{-q} = N_m(2\pi/q)^{\frac{1}{2}} \ . \qquad (8)$$

I propose to determine the parameters of (2) by using

1) the elongation-energy relation, x_m vs lnE (note from (3) that $\xi=x/x_m$),

2) a relation due to Kraushaar (1957) between the typical profile of an *individual* proton shower and the *average* profile of proton showers,

3) experimental data on σ_x, and on E_{EM}/E, the fraction of primary energy given to electrons.

First, however, I will show that (2) provides an excellent fit to electromagnetic cascade profiles.

4. Electromagnetic cascades revisited. Results of cascade theory in approximation B are (Rossi 1952):

$$x_m = x_o(1.01\beta + A), \quad \langle x\rangle = x_o(1.01\beta + B), \quad \sigma_x^2 = x_o^2(1.61\beta + C) \ , \tag{9}$$

where $\beta = \ln(E/E_c)$ and A,B,C are constants ~ 1 whose exact values depend on whether the primary particle is an electron or photon. Using (3) and substituting in (6) I solve for the index

$$q = 0.63 + D \ , \tag{10}$$

where D = -0.4, 0.2 for primary electrons and photons, respectively. Substituting this result in (5), I find

$$\langle x\rangle - x_m = 1.6x_o \ , \tag{11}$$

expressing the fact that the profile is unsymmetrical. To conserve energy the track length integral must equal E. Thus

$$\int (E_c/x_o)N dx = E \ . \tag{12}$$

Using (8), which I solve for N_m, and again substituting for q and x_m,

$$N_m = (0.31E/E_c)/[\ln(E/E_c) - F]^{\frac{1}{2}} \ , \tag{13}$$

where F = 1.7, 1.0 for primary electrons and photons, respectively. Results (11) and (13) are almost exactly the same as given by the detailed theory.

5. Application to air showers. One begins as in the preceding example, by letting $\xi = x/x_m$, where x_m is given by the elongation-energy relation. A reasonable choice for this is

$$x_m = A + D_{10}\log E \tag{14}$$

with A = 159 g/cm^2 and D_{10} (elongation rate per decade) = 65 g/cm^2 (Linsley and Watson 1981; E is in GeV). The index q can be found as in the preceding example, using (6), but first I will show an alternative method illustrating a deep connection between gamma distributions and cascade processes.

It was pointed out by Kraushaar (1957) that the average number of electrons N at atmospheric depth x arising from a primary particle incident upon the top of the atmosphere is related as follows to the average number N_1 of electrons at thickness x' below a nuclear interaction of such a particle:

$$N(x) = \int_0^x N_1(x')\exp[-(x-x')/\lambda]dx'/\lambda \ , \tag{15}$$

$$N_1(x) = [1 + \lambda(\partial/\partial x)]N(x) \ , \tag{16}$$

where λ is the interaction mean free path of the particle, assuming that there are no fluctuations other than those in the starting level. Applying these, it is readily shown that *the condition for N and N_1 to be self similar*; meaning in this case for one to be a gamma distribution if the other one is, is the following:

$$q = x_m/\lambda \ . \tag{17}$$

Substituting (17) in (6) one obtains a very interesting result,

$$\sigma_x^2 = \lambda^2 + \lambda x_m \ , \tag{18}$$

which can be combined with (14), giving

$$\sigma_x^2 = (\lambda^2 + A\lambda) + \lambda D_{10}\log E \ . \tag{19}$$

This says that if the energy dependence of λ is neglected, the energy dependence of σ_x^2 has the same form as that of x_m, just as it does in case of electromagnetic cascades. The energy dependence is described by a *rate*, analogous to the elongation rate, which is in fact proportional to the elongation rate. This new rate, which might be called the 'width rate' or 'spreading rate', is equal to λD_{10}.

An experimental result on σ_x has recently been reported by the Fly's Eye group. The value given, 220±33 g/cm^2 at 10^9 GeV (Baltrusaitis *et al.* 1985), agrees very well with a value derived from an average profile published by Grigoriev *et al.* 1983 for a slightly lower energy. Substituting the average of these results, and values of A and D_{10}, in (19), one finds λ = 58 g/cm^2.

This is appreciably greater than the mean free path for proton-air inelastic collisions found by other methods (see conference paper HE1.1 -1). It is reasonable to assume that λ is greater than $\lambda_{pa,inel.}$ for the same reason that holds in similar cases: neglect of development fluctuations. This needs further study. Pending results of such a study, a reasonable estimate of σ_x^2 for energies other than 10^9 GeV is obtained by assuming that λ is constant with the value found above. Then

$$q = 3.36 + 1.12\log E \,, \quad (20) \quad \text{and} \quad \sigma_x^2 = 1.5\cdot 10^4 + 3.8\cdot 10^3 \log E \quad (21)$$

The remaining parameter in (2), N_O, is found in the same manner as for electromagnetic cascades, using (7) or (8), keeping in mind, of course, that the track length integral equals E_{EM}, not E. The relation between E_{EM} and E is discussed in conference paper OG5.1-5 (or see Linsley 1983). A convenient formula representing results of that work is

$$E_{EM}/E = 1 - 2.8E^{-0.17} \text{ (E in GeV)} \,. \quad (22)$$

References. BALTRUSAITIS *et al.* 1985, Phys. Rev. Lett. 54, 1875; CADY *et al.* 1983, Proc. 18th ICRC 11, 412; CASSIDAY 1981, *Proton-Antiproton Collider Physics* (Am. Inst. Phys.: New York) p. 528; CLARK 1962, J. Phys. Soc. Japan 17, Suppl. A-III, 286; DYAKONOV *et al.* 1981, Proc. 17th ICRC 6, 106; GAISSER 1976 private communication; 1979, *Proc. Air Shower Workshop Univ. Utah* (Bartol Foundation, Newark NJ) p. 57; GAISSER and HILLAS 1977, Proc. 15th ICRC 8, 353; GREISEN 1956, *Progress in Cosmic Ray Physics* (North-Holland: Amsterdam) Vol. 3; GRIGORIEV *et al.* 1979 ZhETP 30, 747; 1983, Proc. 18th ICRC 6, 204; HAMMOND *et al.* 1978, *Nuovo Cimento* 1C, 315; KRAUSHAAR 1957, *Nuovo Cimento Suppl.* 8, 623; LINSLEY 1967, Rev. Sci. Instrum. 38, 1268; 1983, Proc. 18th ICRC 12, 135 (see also paper OG5.1-4, this conference); LINSLEY and WATSON 1981, Phys. Rev. Lett. 46, 459; LONGO and SESTILI 1975, Nucl. Instrum. Meth. 128, 283; ROSSI 1952, *High Energy Particles* (Prentice-Hall: Englewood Cliffs NJ); SASS and SPIRO 1978, CERN $\bar{p}p$ Tech. Note 78-32; WILLIAMS 1948, Phys. Rev. 74, 1689.