A NOTE ON SOME STATISTICAL PROPERTIES OF RISE TIME PARAMETERS USED IN MUON ARRIVAL TIME MEASUREMENTS

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1. Introduction

Most investigations of the muon arrival time distribution in EAS during the past decade made use of parameters which can collectively be called rise time parameters. We will follow the definition of Blake et al. (1981) and define the rise time parameter $T_{A/B}$ as the time taken for the integrated pulse from a detector to rise from A% to B% of its full amplitude. The use of these parameters are usually restricted to the determination of the radial dependence thereof. This radial dependence of the rise time parameters are usually taken as a signature of the particle interaction characteristics in the shower. As these parameters have a stochastic nature, it seems reasonable to us that one should also take notice of this aspect of the rise time parameters. The aim of this paper is therefore to present a statistical approach to the rise time parameters, as this has not been done in the past.

2. Order statistics and rise time parameters

From the definition of $T_{A/B}$ it is reasonable to assume that on the average A% of the total number of particles which gave rise to the output pulse from the detector have arrived in the time $T_{0/A}$. A sample quantile of order p is defined as follows: Let $(X_1, X_2 \ldots X_n)$ be a random sample of size n of a random variable X with probability density function (pdf) f(t). Let $(X_1^{(n)}, X_2^{(n)} \ldots X_n^{(n)})$ be an arrangement of $(X_1, X_2 \ldots X_n)$ such that $X_1^{(n)} \leq X_2^{(n)} \leq \ldots X_n^{(n)}$. $X_i^{(n)}$ is called the i-th order statistic. The sample quantile of order p, $0<p<1$, is defined to be the order statistic $X_k^{(n)}$ for which $k = [np] + 1$, where $[np]$ is the greatest integer not larger than np. This definition states that a fraction p of the sample values is less than $X_k^{(n)}$. This corresponds exactly to the definition of $T_{0/A}$. The rise time parameter $T_{A/B}$ corresponds therefore to the difference between two order statistics of the sample. The statistical properties of $T_{A/B}$ can therefore be determined from the properties of the difference between two order statistics.

We now give a number of properties which are very useful. The following notation is used: $f_k^{(n)}(t)$ is the pdf of $X_k^{(n)}$, $W_{k\ell}^{n}(t)$ is the pdf of $X_\ell^{(n)} - X_k^{(n)}$, F(t) is the distribution function of the random variable X, i.e. $F(t) = P(X \leq t)$, and $g^{(n)}(t)$ is the pdf of the arrival times of muons with respect to the first detected muon.

(i) $f_k^{(n)}(t) = C\, f(t)\, [F(t)]^{k-1} [1-F(t)]^{n-k}$ with $C = n!/(k-1)!(n-k)!$

(ii) $W_{k\ell}^{(n)}(t) = C \int_0^{\infty} f(t) f(x+t)\, [F(t)]^{k-1}\, [F(x+t)-F(x)]^{\ell-k-1} [1-F(x+t)]^{n-k} dx$

with $C = n!/(k-1)!(n-k)!(\ell-k-1)!$

(iii) $\lim_{n\to\infty} P(|X_k^{(n)} - a_p| \geq \varepsilon) = 0$ for $\varepsilon>0$, $k = [np] + 1$, and a_p is the population quantile of order p.

(iv) $\lim_{n\to\infty} f_k^{(n)}(t) = N\left(a_p, \frac{p(1-p)}{nf^2(a_p)}\right)$ (Fisz(1963))

(v) $\lim_{n\to\infty} W_{k\ell}^{(n)}(t) = N\left(a_q - a_p; \frac{p(1-p)}{nf^2(a_p)} + \frac{q(1-q)}{nf^2(a_q)} - \frac{a_p a_q}{nf(a_p)f(a_q)}\right)$ (Swanepoel (1985)).

(vi) $\lim_{n\to\infty} g^{(n)}(t) = f(t)$ (Van der Walt (1984)).

Property (iii) states that the sample quantile converges stochastically to the population quantile. Properties (iv), (v) and (vi) give the limiting distributions of $f_k^{(n)}(t)$, $W_{k\ell}^{(n)}(t)$ and $g^{(n)}(t)$. The rise time parameters will then also have the above properties.

The significance of these results is further that it is possible to examine specific properties of the rise time parameters without having to simulate air showers. It must be made clear at this point that points (i) to (vi) does not add directly to the understanding of the physics involved in shower development. We feel, however, that it gives one a firm basis from which one can understand the properties of the measured rise time parameters.

Discussion

Consider, for example, property (iii). For the case of EAS, the arrival of the hypotherical shower front can be considered as the time t=0 from which timing measurements can be made. In this case $f(t) = 0$ for all $t \leq 0$, and we can allow p to be equal to zero. Then $X_1^{(n)}$, corresponds to the zeroth order sample quantile while $a_p = a_o = 0$. We then have $\lim_{n\to\infty} P(|X_1^{(n)} - 0| \geq \varepsilon) = 0$, which means that the arrival time of the first detected muon with respect to the shower front converges stochastically to zero. This means that for large samples one can consider the first detected particle as the time t = 0. Property (vi) is equivalent to what has just been said.

In figure 1 we show the arrival time distribution of muons together with the arrival time distribution of muons when the first detected muon was taken as the time t = 0. The difference at small delays is due to the fact that the sample size was only ten.

In figure 2 we present the relationship between $\langle X_5^{(5)} - X_1^{(5)} \rangle$ and $\langle X \rangle$ for the case when f(t) is a gamma density function. The same relationship was found to exist for larger samples. Is is also possible to show that this linear relationship exists for a wide variety of density functions. In figure 3 we present the relationship between $\langle X_{10}^{(10)} - X_1^{(10)} \rangle$ and $\langle X \rangle$ for shower simulation data. $X_i^{(10)}$ is the i-th order statistic for a sample of 10 muon arrival times measured in a detector and $\langle X \rangle$ is the mean muon arrival time. It can be seen that the relationship is also linear even though we do not know the parametric form of the muon arrival time distribution. This example also illustrates that it is not necessary to determine the properties of the rise time parameters through the simulation of air showers. It should also be noted from figure 2 that $<X_\ell^{(n)} - X_k^{(n)}>$ does not determine <X> uniquely. This is an example of some of the limitations of rise time parameters.

Conclusion

With the above examples we have tried to illustrate that there exist a statistical basis for the rise time parameters. We believe that the statistical properties of rise time parameters may be of use not only for the analysis of experimental data but also for the planning of experiments.

References

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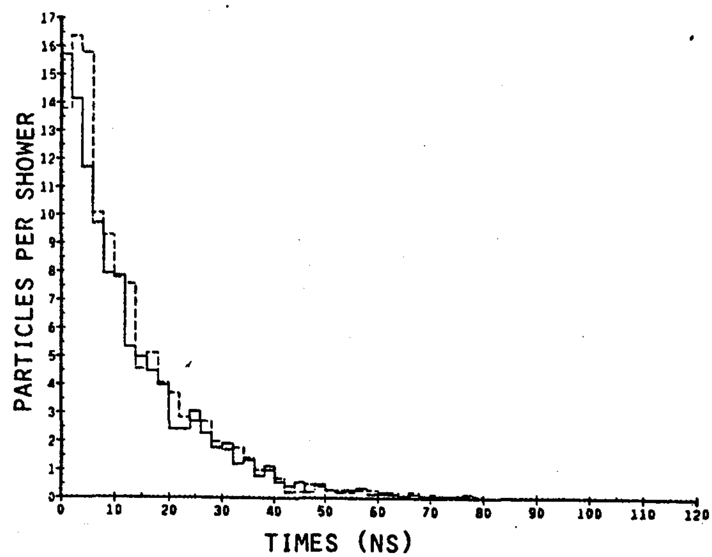

Figure 1 Comparison between $g^{(10)}(t)$ and f(t) for 14 Fe-initiated showers. —— f(t), --- $g^{(10)}(t)$. $110m \leq R \leq 120m$, $E_\mu \geq 1$ GeV

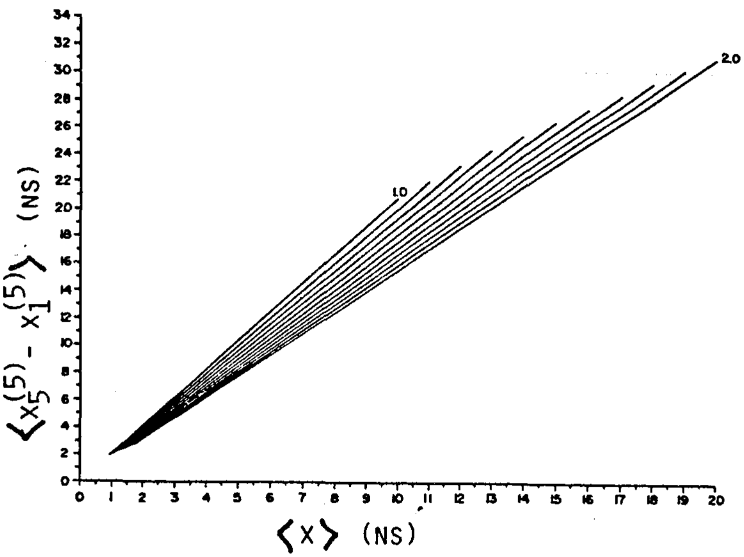

Figure 2 Relationship between $\langle X_5^{(5)} - X_1^{(5)} \rangle$ and $\langle X \rangle$ for a number of gamma random variables for which $1 \leq \langle T \rangle^2/\sigma_T^2 \leq 2$.

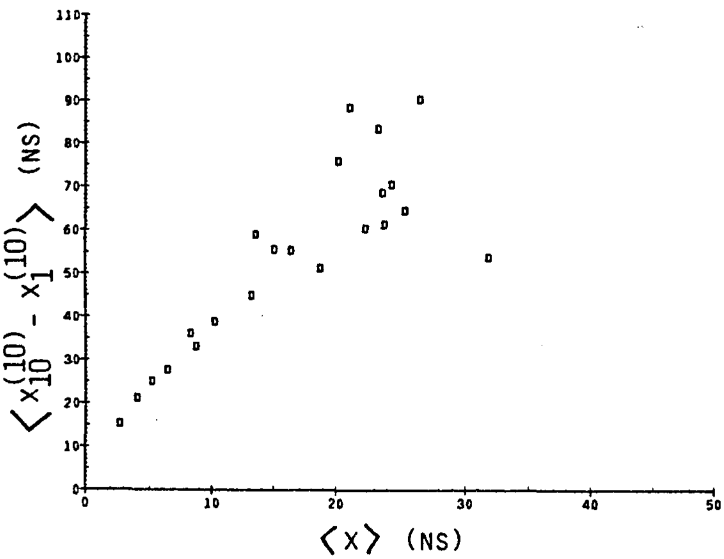

Figure 3 Relationship between $\langle X_{10}^{(10)} - X_1^{(10)} \rangle$ and $\langle X \rangle$ for 14 Fe-initiated showers. $E_\mu \geq 1$ GeV.