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SOME APPLICATIONS OF LIE GROUPS IN ASTRODYNAMICS

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ABSTRACT

Differential equations that arise in astrodynamics are examined from the standpoint of Lie group theory. A summary of the Lie method is given for first degree differential equations. The Kepler problem in Hamiltonian form is treated by this method. Extension of the Lie method to optimal trajectories is outlined.

Introduction

The study of problems in astrodynamics, as well as applied mathematics in general, leads to differential equations. When the dynamic of a problem leads to non-linear differential equations there is usually no systematic method of attack for their solution. The use of continuous transformation groups, in particular Lie groups, provides a general method to approach to global integration.

A study was made of some elementrary problems in astrodynamics : the one and three dimensional Kepler problems. These problems are important because they have a formulation by the variational methods of mechanics and the resulting invariants of the Lie groups are constants of the motion. It is thus tempting to use Lie theory to study optimal control theory in astrodynamics since the this variational theory may be cast into a form that is an analog of Hamiltonian theory in particle dynamics.

Theory:The Lie method of extended groups

The practical application of Lie groups to the solution of differential equations(Bluman and Cole,1974) stems from the fact that if the differential equation

$$\frac{dy}{dx} = F(x,y) \quad \text{or} \quad \Omega(x,y,y') = 0$$

remains invariant under certain transformations, given by

$$x_1 = f(x,y) \qquad y_1 = g(x,y)$$

then the equations can be arranged in such a way that the independent variables do not occur explicitly. The re-arrangement involves a change of variable which are well defined by the invarance property. That is the invariance implies a change of variable

$$u = u(x,y) \quad ; \quad V = V(x,u)$$

such that the differential equation becomes

$$\frac{du}{dV} = \Phi(u)$$

The kind of transformation required defines a continuous one-parameter group. That is a mathematical entity with the properties:(1) the the product of two elements is uniquely defined and is an element of the group,(2) the associative law holds for elements of the group, (3) there is a unit element,(4) and there is an inverse element.

The basic proceedure for solving ordinary differintial equations starts from the infinitesmal parametric transformation

$$x_1 = x + \epsilon \xi(x,y)$$

and

$$y_1 = y + \epsilon \eta(x,y)$$

where ϵ is a parameter and ξ and η are the generators of the transformation. Demanding invariance of the differential equation and the integral invariants (that is constants of motion) u and V leads to the operational equations

$$\mathcal{L}^{(1)}\Omega = 0 \qquad (1)$$

$$\mathcal{L} u = 0 \qquad (2)$$

$$\mathcal{L} V = 1 \qquad (3)$$

where $\mathcal{L}$ is the Lie operator defined by .

$$\mathfrak{X} = \xi \partial_x + \eta \partial_y$$

and $\mathfrak{X}^{(1)}$ is its extension

$$\mathfrak{X}^{(1)} = \xi \partial_x + \eta \partial_y + \eta^{(1)}(\xi, \eta) \partial_{y'}$$

The generators are determined from (1) and the invariants u and v (in dynamics called canonical coordinates are calculated from the first order partial differential equations given by (2) and (3). The calculation process can be summerized as, given the differential equation Ω apply the Lie operators $\mathfrak{X}^{(1)}$ and $\mathfrak{X}$ to get

$$\mathfrak{X}^{(1)} \Omega = 0$$

linear partial differential equations in the generators. Use

$$\mathfrak{X} u = 0 \qquad \mathfrak{X} v = 1$$

and solve the resulting characterstic equations to get the invariants u and v. Form the equation

$$\frac{du}{dv} = \Phi(u)$$

gaining a reduction in order and a final quadruture if possible.

Kepler Motion

In the dynamics problems that follow focus is on the Hamiltonian formulation of the euqations of motion (Leach, 1981), that is consider the first order system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

where p and q are the canonical momenta and coordinates and H is the Hamiltonian.

As an example of a Lie group solution to a differential equation consider the one dimensional Hamiltonian defined by

$$H = \frac{p^2}{2m} - \mu/x$$

where p is the momentum, x the position of a particle and μ the gravitational constant. The equations of motion are given by

$$\Omega_1 = \dot{x} - p = 0$$

and

$$\Omega_2 = \dot{p} + \frac{\mu}{x^2} = 0 .$$

Impose invariance under one-parameter group transformations, that is under the extended Lie operator

$$\mathcal{L}^{(1)}\Omega_1 = 0 \qquad \mathcal{L}^{(2)}\Omega_2 = 0$$

The only surviving derivitives are

$$\frac{\partial \Omega_1}{\partial p} = -1 \qquad \frac{\partial H}{\partial x} = \mu x^{-2} \qquad \frac{\partial \Omega_2}{\partial x} = -2\mu x^{-3}$$

$$\frac{\partial H}{\partial p} = p \qquad \frac{\partial \Omega_1}{\partial \dot{x}} = 1 \qquad \frac{\partial \Omega_2}{\partial \dot{p}} = +1$$

These values lead to the equations for the generators

$$\varsigma - \partial_t \eta + p\,\partial_x \eta - p(\partial_t \xi + p\,\partial_x \xi) = 0 \qquad (4)$$

$$-\frac{2\mu}{x^3}\eta + \partial_t \varsigma + p\,\partial_x \varsigma - \mu/x^2 \frac{\partial \varsigma}{\partial p} + \frac{\mu}{x^2}\left(\partial_t \xi + p\,\partial_x \xi\right) = 0 \quad (5)$$

Solving equation(4)for ς and putting into(5)leads to solutions for the generators given by

$$\xi = \text{constant} \qquad \eta = 0 \qquad \varsigma = 0$$

The integral surfaces of the partial differential implied by are generated by the integral curves of the equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dp}{0} = \frac{du}{0} = \frac{dv}{0} = \frac{dw}{1}$$

there are the solutions

$$u = x \qquad v = p \qquad t = w$$

use of these coordinates and the equations of motion gives a relationship between u and v

$$\frac{du}{dw} = \Phi(u,v) = -vu^2/\mu$$

which integrates to

$$\mu u^{-1} = v^2/2 + \text{const}$$

or that

$$\frac{p^2}{2} - \mu/x = E,$$

which means that E the energy is an invariant under translations in time.

From

$$\frac{du}{dw} = v$$

there results

$$\frac{\mu\, du}{[2\mu u - cu^2]^{1/2}} = dt$$

which can be integrated to

$$t - t_0 = (\mu/c^{3/2}) \sin^{-1}\left[\frac{2\mu - cx}{2\mu}\right] - \frac{\sqrt{2\mu x - cx^2}}{c}$$

a known result for one-dimensional Kepler motion.

It is also possible to outline the three dimensional Kepler problem. The Hamiltonian will have the form

$$H = \frac{P \cdot P}{2} - \frac{\mu}{|Q|}$$

where P and Q are vectors. The extended Lie operator

$$\mathcal{L}^{(1)} = \xi \partial_t + \eta \cdot \nabla_Q + \zeta \cdot \nabla_P + \eta' \cdot \nabla_{\dot{Q}} + \zeta' \cdot \nabla_{\dot{P}}$$

acting on the equations of motion

$$\Omega_1 = \dot{Q} - \nabla_P H = 0 \qquad \Omega_2 = \dot{P} - \nabla_Q H$$

gives the vector partial differential equations

$$-\zeta + (P \cdot \nabla_Q)\eta + \partial_t \eta - P\,(P \cdot \nabla_Q \xi + \partial_t \xi) = 0 \qquad (6)$$

and

$$\eta \cdot \nabla_Q\left(\frac{\mu Q}{|Q|^3}\right) + P \cdot \nabla_Q \eta - \frac{\mu}{|Q|^3}\, Q \cdot \nabla_P \zeta + \partial_t \zeta + \frac{\mu Q}{|Q|^3}\left[P \cdot \nabla_Q \xi + \partial_t \xi\right] = 0 \qquad (7)$$

Put ζ from (6) into (7) and solve for ξ and η . The process leads to 3 groups:translations in time, rotations in space, and affine translations in space and time. In turn these supply the 3 invariants of motion, energy, vector angular momentum and the Laplace vector.

Optimal Trajectories

Consider the problem of extremizing a scalar performance index

$$J[x_{final}, t_{final}]$$

subject to the constraints

$$\dot{x} = f(x, u, t)$$

where u is a vector of control variables. This state equation conbined with boundry conditions defines the problem. For an extremal the Hamiltonian defined by

$$H = \Sigma_1' \lambda_i f_i$$

must satisfy the necessary conditions

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad , \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \qquad (8)$$

The Lie approach to (8) for the Kepler coasting flight car be outlined as follows: from the Hamiltonian

$$H = \dot{R} \cdot \dot{\Lambda} + \frac{\mu \Lambda \cdot R}{R^3}$$

Writing the vector R, Λ as

$$Q_1 = \Lambda, \quad Q_2 = R, \quad P_1 = \dot{R}, \quad P_2 = \dot{\Lambda}$$

consider point transfromations in R and space

$$R_1 = R + \epsilon \eta, \quad \Lambda_1 = \Lambda + \epsilon \xi \quad \text{and} \quad t_1 = t + \epsilon \zeta$$

where η, ξ are vectors. Apply the Lie and extended Lie Operators to find the generators and the invariants.

Conclusion

It has been shown that the use of the method of the Lie theory of xtended groups in the context of Hamiltons equations of motion yield the constants of motion. A method had been presented for use of Lie groups in optimal control theory, where hopefully with further work the non-linear equations will lead to invariants of the problem.

References

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