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ADVANCED ALGORITHM FOR ORBIT COMPUTATION

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ABSTRACT

This paper is a report on the continuation of the author's work with V. Bond of NASA-JSC performed in 1981-82. The subject is the formulation of computational and analytical techniques which simplify the solution of complex problems in orbit mechanics, Astrodynamics and Celestial Mechanics.

The major tool of the simplification is the substitution of transformations in place of numerical or analytical integrations. In this way the rather complicated equations of orbit mechanics might sometimes be reduced to linear equations representing harmonic oscillators with constant coefficients.

The first part of this work was reported in several papers and reports by V. Bond and V. Szebehely, which are listed and discussed in the body of this paper. One outcome of the previous work was the derivation of an equation from which the transformations may be computed for a given problem. This equation is known today in the literature as the "Szebehely-Bond-Equation."

The recently performed work reported here, generalizes the previous results to multi-dimensional problems, investigates the role of integrals in conjunction with the transformations and discusses some of the, as yet unsolved problems.

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INTRODUCTION

The method of transforming non-linear differential equations of orbit mechanics into linear differential equations is one of the major problems in celestial mechanics. Hamiltonian canonical transformations in the extended phase-space attack this problem and the relation of this approach to regularization is discussed in detail in several reference books, see for instance Szebehely, 1967. Regularization is the technique to eliminate the singularities of the differential equations of motion and the associated transformations often lead to linearization, see for instance Levi-Civita, 1903 or Stiefel and Scheirele, 1971.

One of the first general investigation of regularization was performed by Sundman, 1912 who introduced a new independent variable and regularized binary collisions in the general problem of three bodies. The purpose of Sundman's work was not to linearize the equations of motion but to show the existence of solutions of the non-linear but regular differential equations of motion. For this reason Sundman's work was not generally accepted and was seldom used by workers in orbital mechanics until close-approach trajectories had to be computed in connection with lunar and planetary missions. Regularization and linearization were rediscovered and were described usually as "transformations" since, as it will be shown, new independent and dependent variables are to be introduced to linearize.

THEORY

The basic idea of this project is deceivingly simple. The execution is, on the other hand, extremely complicated and difficult.

The equations of motion in Astrodynamics are second order differential equations with non-linear terms and with singularities. All three difficulties (second order, non-linear, singular) were already known to Sir Isaac Newton; they are due to Newton's second law of motion (according to which the accelation, $\ddot{x}$ is related to the force), and are connected with Newton's law of gravity (according to which the force is inversely proportional to the square of the distance). The simplest demonstration of the problem uses a one-dimensional example which is represented by the equation

$$\ddot{x} = -\frac{\mu}{x^2} \quad , \tag{1}$$

where dots denote derivatives with respect to time, x is the distance between the participating bodies and μ is a constant depending on the masses of the participating bodies and on the constant of gravity. Equation (1) is a second order, non-linear differential equation with a singularity at x=0. In this, simplest of all cases, linearization is easily accomplished by introducing a new independent variable and consequently measuring time (t) with a new clock. The two times (s and t) are connected (Sundman, 1912) by:

$$ds = \frac{dt}{x} \tag{2}$$

It might be seen that this equation introduces a new time s. As x becomes small and the usual time step Δt is reduced during numerical integration, the new time step Δs remains approximately constant. This might be looked upon as a built-in time step control, popular in numerical integration techniques. If we rewrite Equation (1) in terms of s instead of t we obtain another non-linear equation, which, however, with the use of the integral of the energy, might be written in a linear form. The analysis is simple, nonetheless, it reveals some fundamental aspects of the problem and, therefore, it will be reproduced here. The two "time derivatives", i.e. the two velocities

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad x' = \frac{dx}{ds}$$

are related by $\dot{x} = \dfrac{x'}{x}$ (3)

and the two accelerations by

$$\ddot{x} = \frac{x''}{x^2} - (x')^2/x^3 \tag{4}$$

So the new equation of motion becomes

$$x'' = -\mu + \frac{(x')^2}{x} \tag{5}$$

which is just as unpleasant as the original equation was, see Equation (1). So the famous Sundman transformation (Equation 2) neither regularizes nor linearizes the equation of motion. Nevertheless, it might be shown that the term

$$(x')^2/x$$

becomes linear when the principle of energy conservation is used.

Equation (1) has an energy integral which might be written as

$$\dot{x}^2 = \frac{2\mu}{x} + 2\,h \qquad (6)$$

or as

$$\frac{(x')^2}{x} = 2\mu + 2\,h\,x\,. \qquad (7)$$

Consequently, Equation (5) becomes

$$x'' - 2\,h\,x = \mu \qquad (8)$$

This simple example reveals several difficulties of fundamental importance in the theory of linearization, some of which have still not been overcome in the case of the n-dimensional perturbed motion.

As we have seen, the use of Sundman's transformation was not sufficient to linearize the equation of motion and an integral of the system had to be used to accomplish linearization. Furthermore, the dependent variable x was not transformed. The fact is that with a proper x=f(y) transformation, Equation (1) may be linearized <u>without</u> the use of the energy integral.

Consequently, in principle, non-linear equations without energy conservation might be linearized when the proper dependent variable is selected. This is discussed in considerable detail in Szebehely (176a,b) ; Schrapel (1978); Szebehely and Bond (1982 and 1983); Bond and Szebehely (1982); Mittleman and Jezewski (1982); Belen'kii (1981), etc.

Now that the basic approach and some fundamental problems have been presented, we are ready to increase the dimensionality from one to two. One of the recent papers on this problem is by Szebehely and Bond (1983) in which the Szebehely-Bond equation is derived in the form:

$$A\rho + B = 1/2 \frac{d}{d\rho} \left[\left(\frac{g}{F^*} \right)^2 G \right], \tag{9}$$

where $r=F(\rho)$ is the transformation of the old radial cordinate r to the new one ρ, $g(r)$ is the function controlling the time transformation which now becomes

$$ds = \frac{dt}{g(r)}, \tag{10}$$

the function

$$G = 2h + \frac{2\mu}{r} - \frac{c^2}{r^2} \tag{11}$$

represents the energy integral and

$$F^* = \frac{dF}{d\rho}. \tag{12}$$

Those functions (F and g) which satisfy Equation (9) will linearize the two-dimensional equations of motion. Various combinations of these functions were and are discussed in the literature (beginning with Kepler).

The most recent is by Ferrer, to be published in Celestial Mechanics, [Ferrer, 1983].

Similar techniques are available and applicable to accurate orbit calculations for relative motion of satellites for docking of space probes. etc. [Nacozy and Szebehely, 1976; Szebehely, 1975 and 1976c; etc.]

Transformations leading from unsolved non-linear differential equations to solved non-linear equations are also popular in the mathematical literature. These transformations do not transform the independent variable and, consequently, might not be ideal for problems pertinent to celestial mechanics, nevertheless, they are mentioned here since they may open up new avenues of research [Dasarathy and Srinivasan, 1968; de Spautz and Lerman, 1967 and 1969].

RESULTS

The generalization to n-dimensional motion was performed during the period of May 9 - July 15, 1983. The results will be summarized in this section using analytical description. The verbal evaluation of these results is in the section entitled CONCLUSIONS.

The two-pronged attack may be described as using the direct and the inverse approaches.

The analytical formulation may be represented by matrix notation or by subscript notation.

Consequently, four basic equations represent the results.

The direct approach starts with given non-linear differential equations and attempts to find the transformations which result in linear differential equations. The transformations of the independent and dependent variables are

$$ds = \frac{dt}{g} \tag{13}$$

and

$$y_i = F_i\ (x_j)\ ;\ x_i = F_i\ (y_j) \tag{14}$$

Using matrix notation these become

$$P = F\ (R)\ ;\ R = \Phi\ (P)\ . \tag{15}$$

Here $g=g(x_i)$ or $g=g(R)$ is the function which controls the transformation of the independent variable t. Furthermore x_i is the i-th component of the position vector appearing in the original, nonlinear differential equation. The corresponding vector in matrix notation is R. The dependent variable of the transformed linear equation is y_i or in matrix notation P. Consequently, Equations(14) represent the coordinate transformations in subscript notation and Equations (15) in matrix notation. All symbols represent vectors (R, F, R, Φ) or components of vectors (y_i, F_i, x_i, f_i) excepting the function g which is a scalar depending on the original dependent variable. The function g in the literature is often called a scalar - vector function.

The original nonlinear equation to be transformed is

$$\ddot{x}_i + H_i(x_j, \dot{x}_j, t) = 0 \tag{16}$$

or

$$\ddot{R} + H(R, \dot{R}, t) = 0 \tag{17}$$

where dots represent derivatives with respect to time.

The desired result of the transformation is

$$y_i'' + a_{ij} y_j' + b_{ij} y_j = 0 \tag{18}$$

or

$$P'' + AP' + BP = 0 , \tag{19}$$

where primes are derivatives with respect to the transformed time s and $A(a_{ij})$, $B(b_{ij})$ are matrices with constant or s -dependent elements.

Equations (16) and (17) are the equations to be transformed and are represented here in their most general forms. After the transformations, Equations (16) and (17) become

$$y_1'' + g F_{1,i} (f_{i,j} g^{-1})_{,k} y_k' y_j' - g^2 F_{1,i} H_i = 0 \tag{20}$$

and

$$P'' + (\Phi^*)^{-1} [(\Phi^{**} - \frac{\Phi^*}{g} I \frac{dg}{d\Phi} \Phi^*) P'] P' - g^2 (\Phi^*)^{-1} H = 0 \tag{21}$$

Here

$$y_1' = \frac{dy_1}{ds}, \quad P' = \frac{dP}{ds},$$

$$F_{1,i} = \frac{\partial F_1}{\partial x_i} \quad \text{and} \quad \Phi^* = \frac{d\Phi}{dP}$$

Consider Equations (19) and (21). In order to accomplish linearity the two last terms of Equation (19) must be equal to the two last terms of Equation (21). Similarly for Equations (18) and (20). These are the conditions to be satisfied by the transformation functions, g and F in order to obtain linearization.

The problems associated with these requirements will be discussed in the section entitled CONCLUSIONS. Several examples were investigated and interesting and unexpected results were obtained.

The inverse approach starts with the linear Equations (18) and (19), then the transformations, given by Equations (14) and (15), are applied and the following results are obtained:

$$\ddot{x}_l + \alpha_{ljk} \dot{x}_j \dot{x}_k + \beta_{lk} x_k + \gamma_l = 0 \qquad (22)$$

and

$$\ddot{R} + [A \dot{R}] \dot{R} + B\dot{R} + \Gamma = 0 \qquad (23)$$

where

α_{ljk}, β_{lk}, γ_l, A, B and Γ depend on

x_i, R, F, g, a_{ij}, A, b_{ij}, B .

Equations (22) and (23) describe the type of non-linear equations which might be linearized by properly selected transformations. Once again, the requirements placed on the transformations will be discussed in the next section. It is noted here that Equations (22) and (23) (or in other words α, β, γ, A, B, Γ) are available in forms similar to the details given in Equations (20) and (21).

CONCLUSIONS

(1) The two major approaches, using two different notations were formulated and a thorough study of the comparison revealed complete agreement. From this it may be concluded that the main results, i.e. Equations (20) - (23) are reliable.

(2) Specific examples pertinent to orbit mechanics have shown that integrals of the motion play an important role in addition to the transformations selected.

(3) Transformations given in the literature were substituted and the requirements mentioned in the previous section were satisfied.

(4) It was found that the linearized systems did not necessarily represent the final solutions of the problems and presently diagonalization and triangularization requirements of the matrices A and B are investigated.

(5) The literature concerning transformations of nonlinear differential equations is impressive, to say the least, and the number of references given here could be easily tripled.

(6) The transformations described in this report are restricted and their generalizations might be of considerable interest.

(7) Linear differential equations do not necessarily have Lyapunov-stable solutions. This should influence the selection of the transformation functions.

(8) There are several dynamical systems of considerable importance in orbit mechanics which represent so-called non-integrable systems. If

these systems are investigated in the light of the present report we arrive at one of the following conclusions:

(i) Non-integrable dynamical systems cannot be transformed to linear systems since linear systems are integrable and their integrals, transformed back into the system of the original variables, would produce integrals of the system. The contradiction might be resolved by claiming that the transformations do not exist.

(ii) Another resolution is that <u>non-integrable</u> systems are in reality <u>not-integrated</u> systems, meaning that the non-integrability condition exists only under certain conditions, see Poincaré's and Bruns' assumptions concerning the non-integrability of the restricted and of the general problems of three bodies. Accordingly, transcendental transformation functions might result in linearization and consquently in showing integrability of these famous "non-integrable" dynamical systems since some of the above-quoted conditions claim non-integrability in terms of algebraic functions.

(iii) Furthermore, it is known that certain non-integrable systems have locally valid integrals. These should correspond to locally valid transformations which should satisfy the requirements mentioned in the previous section.

(9) The two, seemingly most significant conclusions are left to items (9) and (10). From a practical point of view, transformations which reduce or eliminate numerical integrations are of the utmost importance. Numerical accuracy is increased and the time requirement for computations is reduced. Autonomous operations require such improvements and their executions are associated closely with the success of establishing the proper transformations.

(10) Establishing transformations either to linear systems or to integrated non-linear systems might be considered one of the greatest accomplishments of modern celestial mechanics. Accurate long-time predictions would be possible for any length of time. This is intimately associated with the study of the stability of the solar system and of the origin and evolution of the Universe. To integrate "non-integrable" systems would show that these systems should have been called "not-integrated" systems to begin with and would challenge the foundation and the famous and classical results of celestial mechanics.

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REFERENCES

1. Belen'kii, I.M., "A Method of Regularizing the Equations of Motion in the Central Force-Field," Celestial Mechanics, Vol. 23, pp 9-23, 1981.

2. Bond V. and Szebehely V., "Transformations of the Two-Body Problem to Harmonic Oscillators," NASA-JSC-17933, March 1982.

3. Dasarathy, B.V. and Srinivasan, P., "On the Use of Certain Transformation Techniques in the Study of Non-linear Systems," International Journal of Control, Vol. 7, No. 4, pp 361-364, 1968.

4. de Spautz, J.F. and Lerman, R.A., "Equations Equivalent to Nonlinear Differential Equations," Proceedings of the American Mathematical Society, Vol. 18, pp 441-444, 1967.

5. de Spautz, J.F. and Lerman, R.A., "Nonlinear Differential Equations and the Terrestrial Brachistochrone," American Institute of Aeronautics and Astronautics, Journal, Vol. 7, No. 6, pp 1173-1174, June 1969.

6. Ferrer, S. and Sein-Echaluce, M.L., "On the Szebehely-Bond Equation," to be published in Celestial Mechanics, July, 1983.

7. Levi-Civita, T., "Traiettorie Singulari ed Urti nel Problema Ristretto dei Tre Corpi," Ann. Math., [3], Vol. 9., p. 1., 1903.

8. Mittleman, D. and Jezewski, D.J. "A Class of Problems in Celestial Mechanics Reducible to Perturbed Harmonic Oscillator," NASA-JSC-18344, September 1982.

9. Nacozy, P. and Szebehely, V., "The Computation of Relative Motion with Increased Precision," Celestial Mechanics, Vol. 13, pp 449-453, 1976.

10. Schrapel, H.D., "Lineare und nichtlineare Hauptschwingungen", Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 58, pp.T. 172-173, June 1978.

11. Stiefel, E. and Scheifele, G., "Linear and Regular Celestial Mechanics," Springer Verlag, Berlin and N.Y., 1971.

12. Sundman, K. "Memoire sur de Probleme des Trois Corps," Acta Math., Vol. 36, p. 105, 1912.

13. Szebehely, V., "Accurate Computation of Relative Motion by Orbit Smoothing," NASA-JSC-09834, August, 1975.

14. Szebehely, V. and Bond, V., "Equations for Autonomous Orbit Calculations," NASA-JSC-18280, June 1982.

15. Szebehely, V. and Bond, V., "Transformations of the Perturbed Two-Body Problem to Unperturbed Harmonic Oscillators, CELESTIAL MECHANICS Vol. 30, pp. 59-69, 1983

16. Szebehely, V., "Lectures on Linearizing Transformations of Dynamical Systems" in Long-Time Predictions in Dynamics (V. Szebehely and B. Tapley editors), pp 17-42. D. Reidel Publishing Company, Dordrecht-Holland, 1976a.

17. Szebehely, V. "Linearization of Dynamical Systems Using Integrals of the Motion," Celestial Mechanics, Vol. 14, pp 499-508, 1976b.

18. Szebehely, V. "Theory of Orbits," Academic Press, N.Y. Chapters 2,3, and 10, 1967.

19. Szebehely, V., "Time Transformations for Relative Motion," Celestial Mechanics, Vol 13, pp 465-470, 1976c.