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SHAPE DESIGN SENSITIVITY ANALYSIS OF BUILT-UP STRUCTURES

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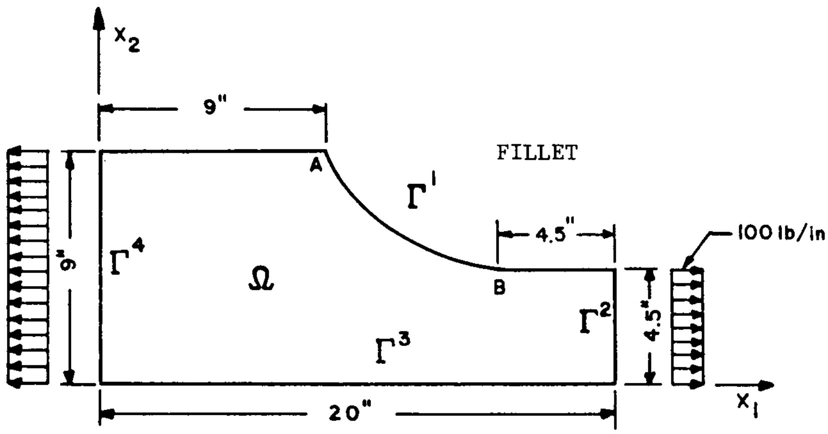
SHAPE DESIGN OF A FILLET

Selection of the best shape of a fillet in a tension bar such that no yielding occurs has long attracted the attention of engineers. Dimensions and notations for the bar and fillet are shown in figure 1. With symmetry, only the upper half of the bar is considered. The boundary segment Γ^1 is to be varied, but with fixed points at A and B. The segment Γ^3 is the central line of the fillet and Γ^4 and Γ^2 are uniformly loaded edges.

The optimal design problem is to find a boundary shape Γ^1 to minimize the total area of the fillet such that no yielding occurs. Constraints are placed on von Mises yield stress, averaged over small regions or finite elements Ω_k on which m_k is a characteristic function with value 1/(area of Ω_k) and $\phi(\sigma(z))^k$ is normalized von Mises yield stress.

The classical boundary value problem is reduced to a variational or energy related problem which not only has excellent properties of existence and uniqueness but also provides the mathematical foundation for finite element analysis. The variational formulation may be viewed as the principle of virtual work and the finite element method as an application of the Galerkin method to the variational equation for approximate solution of the boundary value problem.



Design Variable: Shape of Γ^1

Cost: $\psi_0 = \iint_\Omega d\Omega$

Constraint: $\psi_k = \iint_\Omega \phi(\sigma(z))m_k \, d\Omega \leq 0 \quad , \qquad k = 1,2,\ldots,NE$

Virtual Work Equation:

$$a(z,\bar{z}) \equiv \iint_\Omega \sum_{i,j=1}^{2} [\sigma^{ij}(z)\varepsilon^{ij}(\bar{z})]d\Omega = \int_{\Gamma^2} \sum_{i=1}^{2} T_i\bar{z}_i \, d\Gamma \quad ,$$

for all kinematically admissible virtual displacements $\bar{z}$

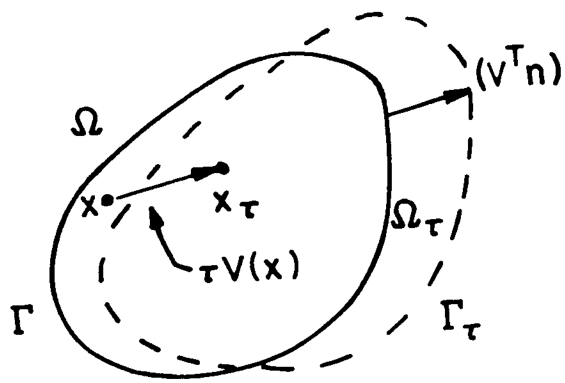
Figure 1

MATERIAL DERIVATIVE AND ADJOINT VARIABLE METHOD

Since shape of the domain is treated as the design variable, it is convenient to think of Ω as a continuous medium and utilize the material derivative idea from continuum mechanics. The process of deforming Ω to a new domain Ω_τ may be viewed as a dynamic process as shown in figure 2. One can define a transformation as $x_\tau = x + \tau V(x)$ where τ plays the role of time and x is a point in initial domain Ω that moves to point x_τ in the deformed domain Ω_τ. Note that the "shape design velocity" V(x) of point x can be considered as a perturbation of design variable. A detailed discussion of this method can be found in references 1 and 2.

The adjoint variable method of design sensitivity analysis (refs. 1, 2, and 3) is applied by defining an adjoint equation for an adjoint displacement field λ^k to obtain the variation ψ_k' where Ω_k is the small region or the finite element considered, m_k is a characteristic function for the corresponding Ω_k, and ϕ is the normalized von Mises yield stress.

Note that only boundary integrals appear in the expression for ψ_k'. The normal movement $(V^T n)$ plays the role of shape design perturbation and can be expressed in terms of shape design parameters.



$$x_\tau = x + \tau V(x)$$

$$V(x) = \frac{dx_\tau}{d\tau}$$

Material derivative of cost: $\psi_0' = \int_\Gamma (V^T n) d\Gamma$

Material derivative of constraint $\Longrightarrow$ Adjoint equation

$$a(\lambda^k, \bar{\lambda}) = \iint_\Omega \sum_{i,j=1}^{2} \left[\frac{\partial \phi}{\partial \sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda})\right] m_k \, d\Omega$$

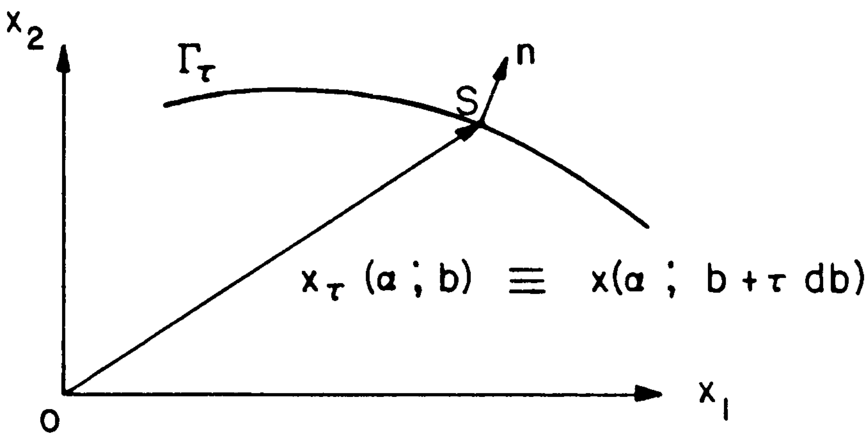
$$\psi_k' = - \int_{\Gamma^1} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^k)\right] (V^T n) d\Gamma - \int_{\Gamma_k} m_k(\psi_k - \phi)(V^T n) d\Gamma$$

Figure 2

PARAMETRIZATION OF BOUNDARY Γ

In order to compute design sensitivity ψ_k', the variable boundary should be parameterized in terms of a design variable vector b (refs. 2 and 3). Presume that points on the boundary Γ_τ are specified by a vector $x_\tau(\alpha;b)$ from the origin of the coordinate system to the point S on the boundary, as shown in figure 3, where α is a parameter vector.

When the vector b of design variables, $b = [b_1,\ldots,b_m]^T$, has been defined, the domain optimization problem reduces to selection of the finite dimensional vector b to minimize a cost function, subject to the constraints. By defining $x_\tau(\alpha;b) \equiv x(\alpha;b + \tau\delta b)$, one can define the velocity field at the boundary by taking the derivative of x_τ with respect to τ. Taking the scalar product of V with the unit outward normal to the boundary Γ and substituting the result into the analytical expressions for ψ_0' and ψ_k' yields numerically computable sensitivity formulas.



$$V = \frac{d}{d\tau}[x(\alpha;b + \tau\delta b)] = \frac{\partial x}{\partial b}\,\delta b$$

$$(V^T n) = \left[n^T \frac{\partial x(\alpha;b)}{\partial b}\right]\delta b$$

$$\psi_0' = \left[\int_\Gamma n^T \frac{\partial x}{\partial b}\, d\Gamma\right]\delta b$$

$$\psi_k' = \left[\int_\Gamma G(z,\lambda^k)\left(n^T \frac{\partial x}{\partial n}\right)d\Gamma\right]\delta b$$

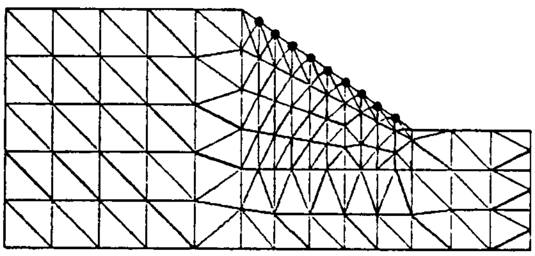
Figure 3

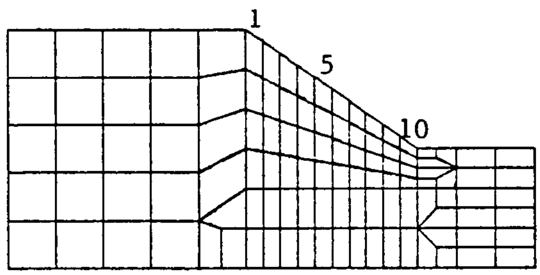
COMPARISON OF FINITE ELEMENT METHODS

Since shape design sensitivity information is given as a boundary integral, one has to check the accuracy of the numerical analysis results on the boundary. For comparison of accuracy, constant stress triangular (CST), linear stress triangular (LST), and 8-noded isoparametric (ISP) elements with optimal stress (refs. 4 and 5) are used to calculate design sensitivity. That is, stress values are evaluated at Gauss points and linearly extrapolated to obtain boundary stresses and strains.

For boundary parameterization, piecewise linear and cubic spline representations are used. In order to compare accuracy of results obtained with different finite elements, the same small region should be used to average stress. The small regions selected are shown in figure 4, located next to the variable boundary where it is most difficult to obtain accurate design sensitivity results (ref. 2).

Define $\Delta\psi_k \equiv \psi_k(b + \delta b) - \psi_k(b)$. The ratio of ψ' and $\Delta\psi$ times 100 is used as a measure of accuracy; i.e., 100% means that the predicted change ψ' is exactly the same as actual change. Numerical results with $\delta b = 0.001b$ are shown in figure 4.





Sensitivity Check $(\psi'/\Delta\psi \times 100)\%$

	Piecewise Linear			Cubic Spline
Region No.	CST	LST	ISP	ISP
5	67.5	99.2	102.8	102.6
6	68.6	99.2	101.8	101.7
7	68.3	99.1	100.0	100.4
8	70.1	99.1	98.4	97.4
9	79.3	98.3	105.2	104.9
10	183.6	87.0	102.8	104.1

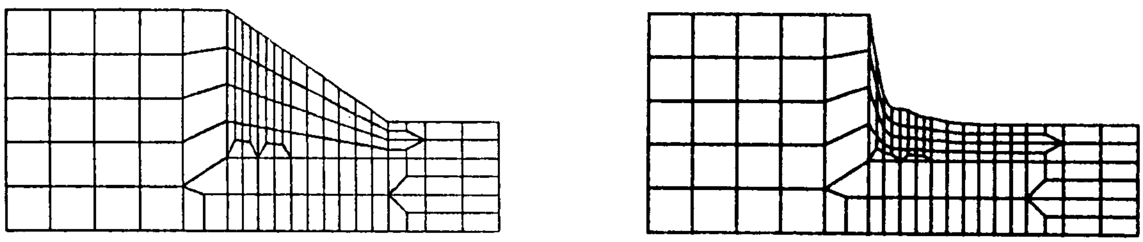
Figure 4

OPTIMIZATION OF FILLET

The cubic spline function, which has two continuous derivatives everywhere and possesses minimum mean curvature, is employed here to define the moving boundary and 8-noded isoparametric finite elements shown in figure 5 are used for analysis. The finite element model contains 131 elements, 458 nodal points, and 846 degrees of freedom.

Heights of nodes that define the varied boundary are chosen as the design variables, as shown in figure 5. The fillet is optimized using the Linearization Method (ref. 6). Convergence criteria require the L-2 norm of direction vector p to be zero at the optimum point, where p is obtained by solving a quadratic programming problem. For numerical data, Young's modulus, Poisson's ratio, and allowable yield stresses are 30×10^6 psi, 0.293, and 120 psi respectively.

The initial design is $b = [5.55, 5.1, 4.65, 4.2, 3.75, 3.3, 2.85, 2.4, 1.95]^T$. Initially, cost, maximum stress violation, and $||p||$ are 145.1 in^2, 2.1×10^{-1}, and 2.0 respectively. After optimization, they are reduced to 133.4 in^2, 6.0×10^{-4}, and 8.8×10^{-4}, respectively. The final design is shown in figure 5 with design variable $b = [2.64, 2.13, 1.90, 1.74, 1.61, 1.55, 1.5, 1.5, 1.5]$.



(a) initial design

(b) final design

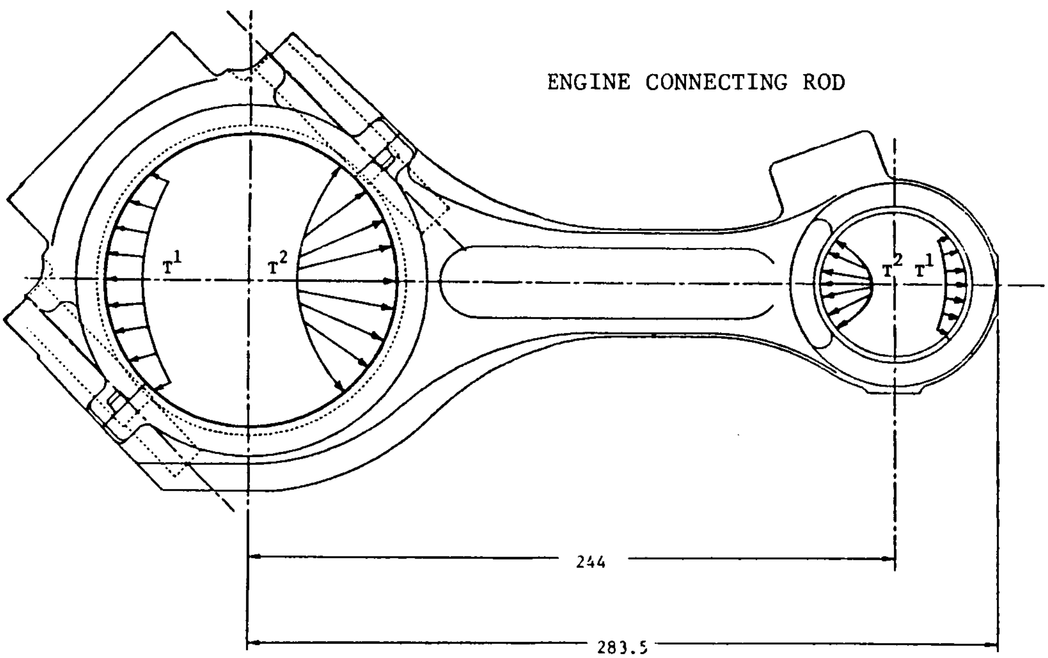
$\psi_0 = 145.1 \text{ in.}^2 \rightarrow 133.4 \text{ in.}^2$

$\max \psi_k = 2.1 \times 10^{-1} \rightarrow 6.0 \times 10^{-4}$

Figure 5

DESIGN OF AN ENGINE CONNECTING ROD

An engine connecting rod connects the crankshaft and piston pin of an engine, transmitting axial compressive load during firing and axial tensile load during the suction cycle of the exhaust stroke. The geometry of the connecting rod considered is shown in figure 6. Considering that the loads acting on the rod are in a plane and that the rod is nearly symmetric about this plane, one can reasonably assume that the rod is in a plane stress state. With the main interest in the shank and neck regions, the shape of the shank and neck regions of the rod are to be determined through the optimization process. The optimum thickness distribution, which varies independently from the domain variation, is to be determined in the optimization process. To satisfy the condition that the distance between the piston pin and the crankshaft is prescribed, it is required that the length of the rod not be changed.



Design variables: Thickness h and shape of shank area

Figure 6

DESIGN SENSITIVITY ANALYSIS OF CONNECTING ROD

The optimal design problem is to find a boundary shape and shank thickness to minimize total volume of the rod, with stress constraints. For stress constraints, lower and upper bounds are imposed on averaged principal stresses of inertia and firing loads.

As in the fillet design problem, one can use the principle of virtual work to derive a variational equation of elasticity. One can then employ the material derivative idea from continuum mechanics and an adjoint variable technique to calculate the shape design sensitivity formulas (ref. 7). The sensitivity expression resulting from thickness variation can also be found using the same adjoint variable method (ref. 2).

To use the sensitivity formulas computationally, the thickness function h is selected to be piecewise constant over strips of finite elements that run along the shank. Also, a cubic spline function is used to parameterize the boundary. (See fig. 7.)

Cost: $\psi_0 = \iint_\Omega h \, d\Omega$

Constraint: $\psi_k = \iint \phi(\sigma(z)) m_k \, d\Omega \leq 0 \quad , \qquad k = 1,2,\cdots,2NE$

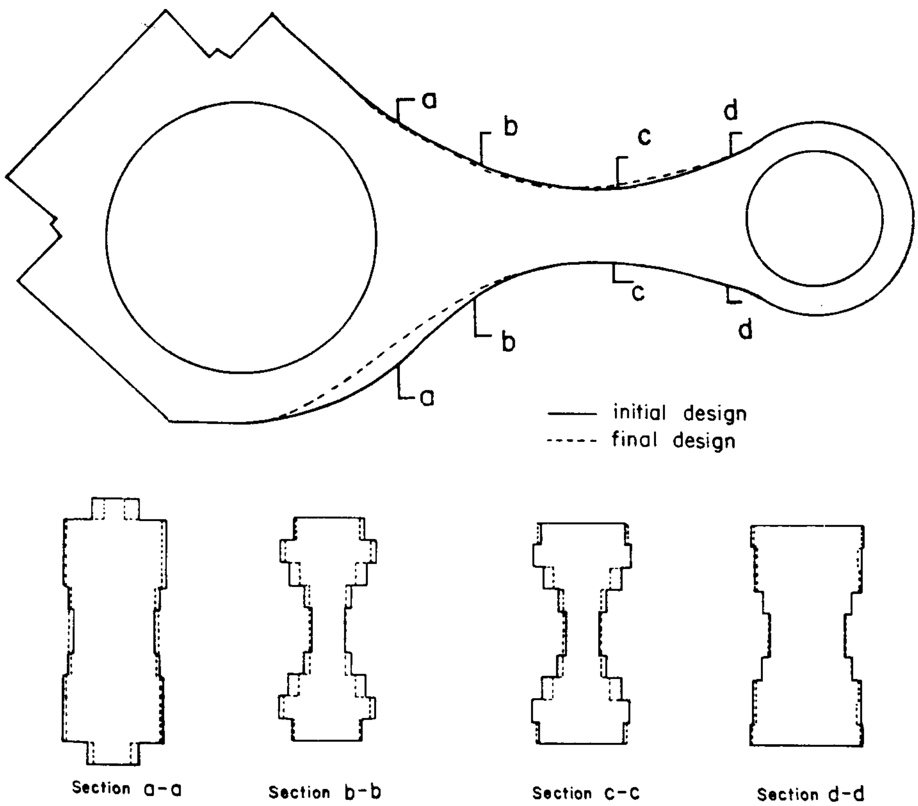
$$\psi_0' = \int_\Gamma h(V^T n) d\Gamma + \iint_\Omega \delta h \, d\Omega$$

$$\psi_k' = \iint_\Omega F_1(z,\lambda^k)\delta h \, d\Omega + \int_\Gamma F_2(z,\lambda^k)(V^T n) d\Gamma$$

Figure 7

With the sensitivity coefficient obtained, one can apply the Linearization Method (ref. 6) to obtain the optimum shape and thickness distribution. An 8-noded isoparametric element is used for analysis. A finite element model including 422 elements, 1493 nodal points, and 2983 degrees of freedom is employed. For numerical data, Young's modulus and Poisson's ratio are 2.07×10^{5} MPa and 0.298 respectively. Upper and lower bounds of principal stresses of inertia are 136 MPa and -80 MPa, whereas they are 37 MPa and -279 MPa for the firing case.

The manufacturer's design is taken as an initial design, where the cost functional, maximum constraint violation, and $||p||$ were initially 726050 mm^3, 2.7×10^{-1}, and 5.9, respectively, and two constraints were active or violated around the neck area (near section a-a). After optimization, they are reduced to 697182 mm^3, 1.0×10^{-3}, and 6.5×10^{-1}, respectively, with 50 stress constraints active. The shape of the initial and final designs and several cross sections are illustrated in figure 8.



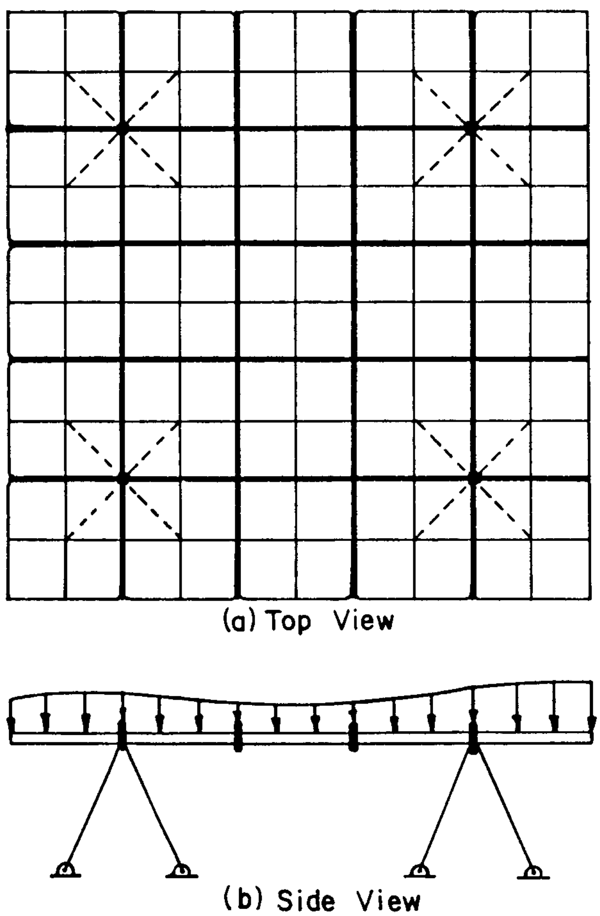
$\psi_0 = 726050 \text{ mm}^3 \rightarrow 697182 \text{ mm}^3$, $\max \psi_k = 2.7 \times 10^{-1} \rightarrow 1.0 \times 10^{-3}$

Figure 8

DESIGN OF A BEAM-PLATE-TRUSS

Figure 9 shows a truss-beam-plate built-up structure in which thin flat plates, stiffened by longitudinal and transverse beams, are supported by four 4-bar trusses. A uniformly distributed load is applied to the plates. The points supported by trusses are at the intersection of two crossing beams nearest the free edges of the structure. The plates and beams are assumed to be welded together. The design variable in this problem is the combination of plate thickness, width and height of the rectangular cross sections of beams, and positions of beams. The design problem is to minimize the volume of the built-up structure, subject to constraints on displacement, stress, natural frequency, and bounds on design variables.

The state variable for this built-up structure consists of the plate displacements, beam displacements and torsion angles, and nodal displacements of the trusses, which satisfy kinematic interface conditions (kinematically admissible displacement fields). Hamilton's principle results in a variational formulation of the governing structural equilibrium and eigenvalue (free vibration) equations.



BEAM-PLATE-TRUSS

Design Variables:

- Beam cross-sectional area
- Plate thickness
- Positions of Beams

Constraints:

- Displacement
- Compliance
- Eigenvalue
- Stress on beams and plates

Figure 9

DESIGN SENSITIVITY ANALYSIS OF BEAM-PLATE-TRUSS

Design sensitivity analysis with respect to conventional design variable and shape using material derivative and adjoint variable method may be extended directly to the built-up structure problems. For conventional design variation, the general sensitivity formula contains contributions from each structural component directly. For shape variation, contributions from each component appear as integrals over common boundaries, using interface conditions on the common boundaries.

In figure 10, comparison between actual changes and predictions for constraints with 5% changes in all conventional design variable are presented. A finite element model of 100 plate elements, 80 beam elements, and 16 truss elements is used, with 363 degrees of freedom for total structure. For numerical data, Young's modulus, Poisson's ratio, and material density are 3.0×10^7 psi, 0.3, and 0.1 lb/in^3 respectively. Results shown in figure 10 indicate that sensitivity accuracy is very good for conventional design.

$$\psi_k' = \sum_{i,j} \iint_{\Omega^{ij}} F_1(z,\lambda^k)dh\, d\Omega + \sum_{i,j} \int_{\Gamma^{ij}} F_2(z,\lambda^k)(V^T n)d\Gamma$$

SENSITIVITY CHECK FOR CONVENTIONAL DESIGN

Constraint	El. No.	$\psi'/\Delta\psi\times100$	Constraint	El. No.	$\psi'/\Delta\psi\times100$
Displacement	C	112.7	stress on plate element	1	95.1
				3	111.3
Stress on beam element	1	108.8		5	109.5
	3	109.7		12	109.7
	5	109.6		14	109.5
	11	106.8		23	109.1
	13	110.0		25	109.8
	15	109.2		35	115.1
				45	113.9
Eigenvalue		91.3			

Figure 10

SHAPE DESIGN SENSITIVITY CHECK FOR BEAM-TRUSS-PLATE

It is well known (ref. 8) that finite element results on interface boundaries, where abrupt changes in the boundary conditions occur (interface conditions), are far from being satisfactory. Based on this fact, a finer grid is used for shape design sensitivity calculations. Only one quarter of the entire structure is used for calculation, due to symmetry. A nonconforming 12 degrees-of-freedom finite element is used for plates. A finite element model of 400 rectangular plate elements, 80 beam elements, and 4 truss element is used, with total of 1281 degrees of freedom. The same numerical data that are used in conventional design sensitivity calculations are used.

In figure 11, sensitivity accuracy results are given for 5% uniform changes in all shape design variables (positions of beams). Results in figure 11 show reasonably good agreement between sensitivity predictions ψ_k' and actual changes $\Delta\psi_k$ for all except some stress constraints on plate elements. That is, the sensitivity results for the stress constraints on plate elements adjacent to the interface (marked by *) are poor, even with finer grid.

Constraint	El. No.	$\psi'/\Delta\psi\times 100$	Constraint	El. No.	$\psi'/\Delta\psi\times 100$
Displacement	C	97.5	Stress on plate element	1	100.7
				19	103.6
				44	109.9*
Compliance		92.3		58	98.7
				85	188.3*
Eigenvalue		95.5		128	97.8
				149	98.7
Stress on beam element	1	99.6		175	91.8
	21	99.7		198	109.6
	30	100.0		237	449.0*
	45	98.3		259	138.8*
	55	103.1		296	95.9*
	77	103.9		317	87.8
				358	105.3
				400	111.3

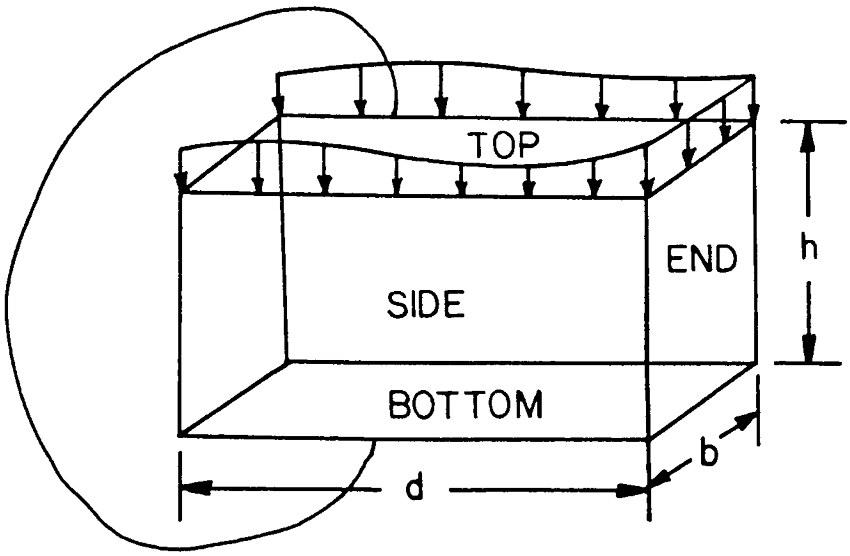
Figure 11

DESIGN OF A SIMPLE BOX BUILT-UP STRUCTURE

A simple box built-up structure, in which five plane elastic solid plates are welded together, attached to a wall is shown in figure 12. A uniformly distributed line load is applied on top of the two side plates and the end plate. The shape design variable in this problem is the length d, width b, and height h of the box.

As in the beam-plate-truss case, the principle of virtual work results in a variational formulation of the governing structural equations. Then, one can use the material derivative idea and an adjoint variable method to obtain the shape sensitivity formula.

In view of beam-plate-truss shape sensitivity results, an equivalent but alternate form of shape sensitivity formula is used for this problem. Since finite element results are accurate on the domain and not on the boundary, the shape sensitivity formula is expressed in terms of domain integral (refs. 1 and 2). Hence, instead of ($V^T n$), one has terms V and (div V) in the sensitivity formula.



Design Variables: d, b, and h

Constraints: $\psi_k = \iint_\Omega \phi(\sigma(z))m_k \, d\Omega \leq 0 \quad , \quad k = 1,2,\cdots,NE$

$$\psi_k' = \sum_i \iint_{\Omega^i} \left[g(\lambda,z)^T V + F(\lambda,z)\mathrm{div}\, V\right]d\Omega$$

Figure 12

SHAPE DESIGN SENSITIVITY FOR SIMPLE BOX

Since shape design variables are given as d, b, and h, one can assume the velocity field to be linear on each plate and thus (div V) is constant. An 8-noded isoparametric element is used for analysis. A finite element model of 320 elements, 993 nodes, and 1886 degrees of freedom is used. For numerical data, Young's modulus and Poisson's ratio are 1.0×10^7 psi and 0.316 respectively. The dimension of the structure is b = d = h = 8 in. and the thickness of the plates is 0.1 in. Uniform external load is 4.77 lb/in.

In figure 13, the sensitivity accuracy result is given separately for 3% change in d and h. Results given in figure 13 show excellent agreement between predictions ψ_k' and actual changes $\Delta\psi_k$. The boundary method that is applied to the beam-plate-truss built-up structure is tested to the same box problem with unacceptable results. The domain method of shape design sensitivity for built-up structure has a promising future. Work continues in evaluating the method on larger scale examples.

Region	El. No.	$\psi'/\Delta\psi \times 100$ $\delta d = 0.03d$	$\psi'/\Delta\psi \times 100$ $\delta h = 0.03h$	Region	El. No.	$\psi'/\Delta\psi \times 100$ $\delta d = 0.03d$	$\psi'/\Delta\psi \times 100$ $\delta h = 0.03h$
	1	97.6	103.6		129	96.2	103.4
	15	98.2	103.6		143	98.1	103.0
Top	29	98.8	103.5	Side	157	98.0	102.6
	43	100.4	103.6		171	95.6	103.0
	57	110.1	103.5		184	100.4	103.2
	72	97.9	103.3		264	99.9	103.7
	86	98.5	103.2		278	114.1	103.9
Bottom	100	98.5	103.1	End	291	95.1	101.1
	114	100.9	103.3		307	120.3	103.5
	128	100.9	103.2		320	99.9	103.7

Figure 13

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