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THE UNIVERSAL PROPAGATOR

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Abstract

For a general Hamiltonian appropriate to a single canonical degree of freedom, we characterize and define a universal propagator with the property that it correctly evolves the coherent-state Hilbert space representatives for an arbitrary fiducial vector. The universal propagator is explicitly constructed for the harmonic oscillator, with a result that differs from the conventional propagators for this system.

1 Introduction

Canonical coherent states, and the coherent state propagator they engender, have been around for over three decades.[1-3] In essence, their construction is simplicity itself. Let P and Q denote an irreducible pair of self-adjoint Heisenberg operators satisfying $[Q, P] = i(\hbar = 1)$, and let

$$|p, q; \eta\rangle = e^{-iqP} e^{ipQ} |\eta\rangle$$

denote a family of normalized states defined for a fixed fiducial vector $|\eta\rangle, \langle\eta|\eta\rangle = 1$, and for all $(p, q) \in \mathbf{R}^2$. These states are the canonical coherent states and they admit a resolution of unity in the form

$$\int |p, q; \eta\rangle\langle p, q; \eta| dpdq/2\pi = I\,,$$

for any $|\eta\rangle$, when integrated over all phase space.[2] These states lead to a representation of Hilbert space $\mathbf{H}$ by bounded, continuous functions,

$$\psi_\eta(p, q) \equiv \langle p, q; \eta|\psi\rangle\,,$$

defined for all $|\psi\rangle \in \mathbf{H}$, that evidently depend on the choice of $|\eta\rangle$, although that dependence is often left implicit. An inner product in this representation is afforded by

$$\langle\phi|\psi\rangle = \int \phi_\eta^*(p, q)\psi_\eta(p, q) dpdq/2\pi\,,$$

an integral which removes all trace of the fiducial vector $|\eta\rangle$.

1.1 Propagators

The abstract Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle\,,$$

involving the self-adjoint Hamiltonian $\mathcal{H}$, is formally solved with the aid of the evolution operator $U(t) = \exp(-it\mathcal{H})$, namely

$$|\psi(t'')\rangle = e^{-i(t''-t')\mathcal{H}}|\psi(t')\rangle\,.$$

In a coherent-state representation the evolution is effected by an integral kernel

$$K_\eta(p'',q'',t'';p',q',t') \equiv \langle p'',q'';\eta|e^{-i(t''-t')\mathcal{H}}|p',q';\eta\rangle$$

in the form

$$\psi_\eta(p'',q'',t'') = \int K_\eta(p'',q'',t'';p',q',t')\psi_\eta(p',q',t')dp'dq'/2\pi\,.$$

Clearly, K_η depends strongly on the fiducial vector as does ψ_η.

Our goal in this paper is to formulate a *universal propagator* $K(p'',q'',t'';p',q',t')$, a single function independent of any particular fiducial vector, which, nevertheless, has the property that

$$\psi_\eta(p'',q'',t'') \;=\; \int K(p'',q'',t'';p',q',t')\psi_\eta(p',q',t')dp'dq'/2\pi \qquad (1)$$

holds just as before for any choice of fiducial vector.

The functions K_η and K are qualitatively different as is clear from their behavior as $t'' \to t'$. In particular

$$\lim_{t''\to t'} K_\eta(p'',q'',t'';p',q',t') \;=\; \langle p'',q'';\eta|p',q';\eta\rangle\,, \qquad (2)$$

which clearly retains a strong dependence on the fiducial vector. On the other hand, if (1) is to hold for any η, we must require that

$$\lim_{t''\to t'} K(p'',q'',t'';p',q',t') = 2\pi\delta(p''-p')\delta(q''-q')\,. \qquad (3)$$

Next let us turn our attention to a suitable differential equation satisfied by K_η and K. It is straightforward to see that

$$\begin{aligned}\left(-i\frac{\partial}{\partial q}\right)\langle p,q;\eta|\psi\rangle \;&=\; \langle p,q;\eta|P|\psi\rangle\,,\\ \left(q+i\frac{\partial}{\partial p}\right)\langle p,q;\eta|\psi\rangle \;&=\; \langle p,q;\eta|Q|\psi\rangle\end{aligned}$$

hold quite independently of $|\eta\rangle$. Thus if $\mathcal{H} = \mathcal{H}(P,Q)$ denotes the Hamiltonian it follows that Schrödinger's equation takes the form

$$\begin{aligned}i\frac{\partial}{\partial t}\psi_\eta(p,q,t) \;&=\; \langle p,q;\eta|\mathcal{H}(P,Q)|\psi(t)\rangle\\ &=\; \mathcal{H}(-i\frac{\partial}{\partial q}, q+i\frac{\partial}{\partial p})\psi_\eta(p,q,t)\end{aligned}$$

valid for any $|\eta\rangle$.[3] The propagators are also solutions of Schrödinger's equation so it follows that

$$i\frac{\partial}{\partial t}K_{\#}(p,q,t;p',q',t') \;=\; \mathcal{H}(-i\frac{\partial}{\partial q}, q+i\frac{\partial}{\partial p})K_{\#}(p,q,t;p',q',t')\,, \tag{4}$$

where $K_{\#}$ denotes either K_η or K. What distinguishes which function is under consideration is the *initial condition* (at $t=t'$) of the solution, namely, either (2) or (3).

When K_η is under consideration, the operators $-i\frac{\partial}{\partial q}$ and $q+i\frac{\partial}{\partial p}$ refer to a *single* degree of freedom made irreducible by confining attention to the subspace of $L^2(\mathbf{R}^2, dpdq/2\pi)$ spanned by $\psi_\eta(p,q)$ for a fixed $|\eta\rangle$ and for all $|\psi\rangle \in \mathbf{H}$. This restriction is implicit in K_η because as $t''\to t'$ the resultant integral kernel $\langle p'',q'';\eta|p',q';\eta\rangle$ is a *projection operator* onto the subspace of an irreducible representation.

1.2 The Universal Propagator

In contrast to the former case, when the universal propagator K is under consideration the resultant Schrödinger equation (4) is interpreted as one appropriate to *two* degrees of freedom. In this view $y_1=q$ and $y_2=p$ denote two "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of two-variable Hamiltonians, ones where the classical Hamiltonian is restricted to have the form $H_c(p_1, y_1-p_2)$, rather than the most general form $H_c(p_1,p_2,y_1,y_2)$.

In the case of K, and based on the interpretation described above, a standard phase-space path integral solution may be given for the universal propagator. In particular, and for a sufficiently wide class of Hamiltonians, it follows that

$$K(p'',q'',t'';p',q',t') = \mathcal{M}\int e^{i\int[x\dot{p}+k\dot{q}-\mathcal{H}(k,q-x)]dt}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x\,.$$

Note that "x" and "k" are "momenta" conjugate to the "coordinates" "p" and "q", and also that the special form of the Hamiltonian has been used. In the standard phase-space path integral there is always one more (k,x) pair of integrals compared to the (p,q) family, and the (k,x) integrals are unrestricted. This situation is made explicit in the regularized prescription for the path integral given, in standard notation, by

$$\begin{aligned}
&K(p'',q'',t'';p',q',t')\\
&\quad= \lim_{\substack{\epsilon\to 0\\ L\to\infty}} 2\pi\int\cdots\int e^{i\sum_{\ell=0}^{L}\left[x_{\ell+\frac12}(p_{\ell+1}-p_\ell)+k_{\ell+\frac12}(q_{\ell+1}-q_\ell)-\epsilon\mathcal{H}(k_{\ell+\frac12},(q_{\ell+1}+q_\ell)/2-x_{\ell+\frac12})\right]}\\
&\qquad\times\prod_{\ell=1}^{L}dp_\ell dq_\ell\prod_{\ell=0}^{L}dk_{\ell+\frac12}dx_{\ell+\frac12}\;/\;(2\pi)^2\,,
\end{aligned}$$

where $p_{L+1},q_{L+1}=p'',q'', p_0,q_0=p',q'$, and where $(L+1)\epsilon=(t''-t')$ is held fixed. Let us first change the variables $x_{\ell+\frac12}\to x_{\ell+\frac12}+(q_{\ell+1}+q_\ell)/2$, followed by a second change $x_{\ell+\frac12}\to -x_{\ell+\frac12}$. The resultant regularized path integral reads

$$K(p'',q'',t'';p',q',t')$$

$$= \lim_{\substack{\epsilon\to 0\\ L\to\infty}} 2\pi \int\cdots\int e^{i\sum_{\ell=0}^{L}[\frac{1}{2}(q_{\ell+1}+q_\ell)(p_{\ell+1}-p_\ell)+k_{\ell+\frac{1}{2}}(q_{\ell+1}-q_\ell)-x_{\ell+\frac{1}{2}}(p_{\ell+1}-p_\ell)-\epsilon\mathcal{H}(k_{\ell+\frac{1}{2}},x_{\ell+\frac{1}{2}})]}$$

$$\times \prod_{\ell=1}^{L} dp_\ell dq_\ell \prod_{\ell=0}^{L} dk_{\ell+\frac{1}{2}} dx_{\ell+\frac{1}{2}} \;/\; (2\pi)^2 ,$$

or the formal path integral is given by

$$K(p'',q'',t'';p',q',t') = \mathcal{M}\int e^{i\int[q\dot p+k\dot q-x\dot p-\mathcal{H}(k,x)]dt}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x\,, \tag{5}$$

which is our final expression for the universal propagator in the present case. From this formula it is clear that the dependence on p'' and p' is always of the form $p''-p'$, and the dependence on q'' and q' is always of the form $q''-q'$ save for the universal phase factor $\frac{1}{2}(q''+q')(p''-p')$. In other words,

$$K(p'',q'',t'';p',q',t') = F(p''-p',q''-q',t''-t')e^{i\frac{1}{2}(q''+q')(p''-p')}$$

for some function F. Of course, if $\mathcal{H}$ depends explicitly on time then F is not simply a function of the time difference $t''-t'$.

2 Examples of the Universal Propagator

2.1 Vanishing Hamiltonian

Let us evaluate the universal propagator in three soluble examples. The simplest case is that of a vanishing Hamiltonian which leads to

$$\begin{aligned} K(p'',q'',t'';p',q',t') &= \mathcal{M}\int e^{i\int[q\dot p+k\dot q-x\dot p]dt}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \\ &= \mathcal{N}\int e^{i\int q\dot p dt}\delta\{\dot q\}\delta\{\dot p\}\mathcal{D}p\mathcal{D}q \\ &= 2\pi\delta(p''-p')\delta(q''-q')\,, \end{aligned}$$

where the normalization follows from the initial condition. Evidently this is the correct result.

2.2 Free Particle

The next case is the free particle where $\mathcal{H}(k,x) = k^2/2m$. In this event

$$\begin{aligned} K(p'',q'',t'';p',q',t') &= \mathcal{M}\int e^{i\int(q\dot p+k\dot q-x\dot p-k^2/2m)dt}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \\ &= \mathcal{N}\int e^{i\int(q\dot p+m\dot q^2/2)dt}\delta\{\dot p\}\mathcal{D}p\mathcal{D}q \\ &= \sqrt{\frac{2\pi m}{i(t''-t')}}\delta(p''-p')e^{im(q''-q')^2/2(t''-t')}\,. \end{aligned}$$

2.3 Harmonic Oscillator

The last case we consider is the harmonic oscillator where $\mathcal{H}(k,x) = (k^2+\omega^2x^2)/2$. Now

$$
\begin{aligned}
&K(p'',q'',t'';p',q',t') \\
&= \mathcal{M}\int e^{i\int[q\dot p + k\dot q - x\dot p - (k^2+\omega^2x^2)/2]dt}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \\
&= \mathcal{N}\int e^{i\int[q\dot p + (\dot q^2+\dot p^2/\omega^2)/2]dt}\mathcal{D}p\mathcal{D}q \\
&= (2i)^{-1}\csc(\omega T/2)\exp\left(i\left\{\frac{1}{2}(q''+q')(p''-p') + \frac{1}{4}\cot(\omega T/2)\left[\frac{1}{\omega}(p''-p')^2+\omega(q''-q')^2\right]\right\}\right)
\end{aligned}
\tag{6}
$$

where $T \equiv t''-t'$. Observe that this result is rather different from conventional propagators for harmonic oscillator Hamiltonians. Indeed, (6) is more like the propagator for a two-dimensional free particle in a uniform magnetic field.[4] This result also applies even when $\omega \to i\omega$, or with a suitable limit, even when $\omega \to 0$ leading to the free particle solution.

3 Propagation with the Universal Propagator

In order to check our results for the universal propagator let us put them to the test. For ease of computation we choose as the initial state for our propagation the coherent-state overlap function

$$\langle p',q';\eta|p,q;\eta\rangle\,,$$

and additionally we choose the fiducial vector $|\eta\rangle$ to be the ground state of an oscillator with frequency Ω for which $\langle\eta|Q|\eta\rangle = 0 = \langle\eta|P|\eta\rangle$. In that case the initial state reads

$$
\begin{aligned}
\psi_\eta(p',q',t') &\equiv \langle p',q';\eta|p,q;\eta\rangle \\
&= \exp\left(i\left\{\frac{1}{2}(p'+p)(q'-q) - \frac{1}{4}[\Omega^{-1}(p'-p)^2+\Omega(q'-q)^2]\right\}\right)\,.
\end{aligned}
$$

3.1 Free Particle

For the free particle case we need to compute ($T = t''-t'$)

$$\frac{\sqrt{m}}{\sqrt{2\pi iT}}\int\exp\left\{\frac{i}{2}mT^{-1}(q''-q')^2 + \frac{i}{2}(p''+p)(q'-q) - \frac{1}{4}[\Omega^{-1}(p''-p)^2+\Omega(q'-q)^2]\right\}dq'\,,$$

which is readily found to be [$\overline{p} \equiv (p''+p)/2, \overline{q} \equiv (q''+q)/2, p^* \equiv p''-p, q^* \equiv q''-q$]

$$
\begin{aligned}
&\psi_\eta(p'',q'',t'') \\
&= \frac{\sqrt{m}}{\sqrt{m+i\Omega T/2}}\exp\left\{\frac{im}{2}\frac{(\Omega^2Tq^{*2}/4 - T\overline{p}^2 + 2m\overline{p}q^*)}{(m^2+\Omega^2T^2/4)} - \frac{p^{*2}}{4\Omega} - \frac{\Omega(mq^*-T\overline{p})^2}{4(m^2+\Omega^2T^2/4)}\right\}\,.
\end{aligned}
\tag{7}
$$

This result agrees with one obtained elsewhere[5] thereby establishing its validity.

In addition, as $\Omega \to \infty$ or $\Omega \to 0$, Eq. (7), apart from a suitable scale factor, yields the result in the sharp q or sharp p representation, respectively.[3,5] In particular, consider

$$\begin{aligned}\psi(q'',t'') &= \lim_{\Omega\to\infty} \frac{1}{2}\sqrt{\frac{\Omega}{\pi}}\psi_\eta(p'',q'',t'') \\ &= \sqrt{\frac{m}{2\pi i T}}e^{i\frac{m}{2T}(q''-q)^2},\end{aligned}$$

which is the appropriate Schrödinger representation solution for the free particle. Likewise, consider

$$\begin{aligned}\tilde{\psi}(p'',t'') &= \lim_{\Omega\to 0} \frac{1}{2\sqrt{\pi\Omega}}\psi_\eta(p'',q'',t'')e^{-i(p''q''-pq)} \\ &= \lim_{\Omega\to 0} \frac{1}{2\sqrt{\pi\Omega}}e^{-i\frac{T\overline{p}^2}{2m}-\frac{1}{4\Omega}p^{*2}-ip^*\overline{q}} \\ &= \delta(p''-p)e^{-i\frac{Tp^2}{2m}},\end{aligned}$$

which is the proper answer in momentum space for the free particle.

3.2 Harmonic Oscillator

Finally, let us consider the time evolution for the harmonic oscillator as given by

$$\begin{aligned}(4\pi i)^{-1}\csc(\omega T/2)\int \quad & \exp\left(i\left\{\frac{1}{2}(q''+q')(p''-p')+\frac{1}{4}\cot(\omega T/2)\left[\frac{1}{\omega}(p''-p')^2+\omega(q''-q')^2\right]\right\}\right) \\ \times \quad & \exp\left\{i\frac{1}{2}(p'+p)(q'-q)-\frac{1}{4}[\Omega^{-1}(p'-p)^2+\Omega(q'-q)^2]\right\}dp'dq',\end{aligned}$$

which is evaluated as $[s \equiv \sin(\omega T/2),\ c \equiv \cos(\omega T/2)]$.

$$\begin{aligned}\psi_\eta(p'',q'',t'') &= C(T)^{-\frac{1}{2}}\exp\left(i\overline{q}p^*\frac{\omega^2 s^2}{\Omega^2 c^2+\omega^2 s^2}+i\overline{p}q^*\frac{\omega^2 c^2}{\omega^2 c^2+\Omega^2 s^2}\right) \\ &\quad\times\exp\left[-\frac{i\omega sc}{4(\omega^2c^2+\Omega^2s^2)}(4\overline{p}^2-\Omega^2q^{*2})-\frac{i\omega\Omega^2 sc}{4(\Omega^2c^2+\omega^2s^2)}(4\overline{q}^2-\Omega^{-2}p^{*2})\right] \\ &\quad\times\exp\left[-\frac{\Omega(cp^*+2\omega s\overline{q})^2}{4(\Omega^2c^2+\omega^2s^2)}-\frac{\Omega\omega^2(cq^*-2\omega^{-1}s\overline{p})^2}{4(\omega^2c^2+\Omega^2s^2)}\right], \qquad (8)\end{aligned}$$

where

$$C(T)=\cos\omega T+i\frac{1}{2}\left(\frac{\omega}{\Omega}+\frac{\Omega}{\omega}\right)\sin\omega T.$$

As in the free particle case, we can obtain the sharp q propagator by the same kind of limit, namely

$$\begin{aligned}\psi(q'',t'') &= \lim_{\Omega\to\infty}\frac{1}{2}\sqrt{\frac{\Omega}{\pi}}\psi_\eta(p'',q'',t'') \\ &= \sqrt{\frac{\omega}{2\pi i\sin\omega T}}\exp\left\{\frac{i\omega}{2\sin\omega T}\left[\cos\omega T(q''^2+q^2)-2q''q\right]\right\},\end{aligned}$$

which, of course, is the standard result. Likewise, the sharp p propagator is given by

$$\begin{aligned}\tilde{\psi}(p'',t'') &= \lim_{\Omega\to 0}\frac{1}{2}\frac{1}{\sqrt{\pi\Omega}}\psi_\eta(p'',q'',t'')e^{-i(p''q''-pq)}\\ &= \frac{1}{\sqrt{2\pi i\omega\sin\omega T}}\exp\left\{\frac{i}{2\omega\sin\omega T}\left[\cos\omega T(p''^2+p^2)-2p''p\right]\right\},\end{aligned}$$

which, again, is the standard result.

We may also observe that the harmonic oscillator evolution simplifies considerably when $\Omega=\omega$. In that case[1]

$$\psi_\eta(p'',q'',t'') = \exp\left[\frac{i}{2}(q''p''-qp)+\frac{i}{2}(q''p_T-p''q_T)-\frac{1}{4\omega}(p''-p_T)^2-\frac{\omega}{4}(q''-q_T)^2\right],$$

where

$$\begin{aligned}q_T &= q\cos(\omega T)+\omega^{-1}p\sin(\omega T),\\ p_T &= p\cos(\omega T)-\omega q\sin(\omega T),\end{aligned}$$

evolution equations that are seen to follow the classical solution.

3.3 Generalization

Although we have only shown that a limited set of fiducial vectors are correctly propagated by the universal propagator, it should be fairly clear that the stated properties of the universal propagator hold true. Indeed, the general case may be discussed by considering as initial condition

$$\begin{aligned}\langle\tilde{p},\tilde{q};\eta|e^{-ip'Q}e^{iq'P}|p,q;\eta\rangle &= e^{-i\tilde{q}p'}\langle p'+\tilde{p},q'+\tilde{q};\eta|p,q;\eta\rangle\\ &= e^{i(\tilde{p}q'-\tilde{q}p')+i\tilde{p}(\tilde{q}-q)}\langle p',q';\eta|p-\tilde{p},q-\tilde{q};\eta\rangle\end{aligned}$$

for just a *single* $|\eta\rangle$, say a Gaussian with $\Omega=1$. Then a suitable superposition over $\tilde{p},\tilde{q}$ leads to any fiducial vector of interest, while a second and independent suitable superposition over p,q leads to any initial state $|\psi\rangle$ of interest.

4 Classical Limit

Although the universal propagator has been derived by identifying the relevant Schrödinger equation as one for two degrees of freedom, it should nevertheless be true that the classical limit refers to a single degree of freedom. This is possible, in the present case, because of the limited form of the quantum or classical Hamiltonian.

Recall, under standard assumptions, that the classical action for a conventional coherent state path integral reads, in the limit $\hbar\to 0$, as

$$I=\int[p\dot{q}-\mathcal{H}(p,q)]dt\,.$$

Extremal variation of this expression holding the end points fixed leads to the usual Hamiltonian equations of motion,

$$\begin{aligned}\dot{q} &= \partial\mathcal{H}(p,q)/\partial p\,,\\ \dot{p} &= -\partial\mathcal{H}(p,q)/\partial q\,,\end{aligned}$$

appropriate to a single degree of freedom. Let us denote a generic solution of these equations by $q_c(t)$ and $p_c(t)$.

Before proceeding it is important to reexamine the "standard assumptions" that lead to this result. For finite $\hbar$ the expression that represents the classical Hamiltonian in a coherent state path integral is traditionally given by

$$\begin{aligned}H(p,q) &= \langle p,q;\eta|\mathcal{H}(P,Q)|p,q;\eta\rangle\\ &= \langle\eta|\mathcal{H}(P+p,Q+q)|\eta\rangle\,.\end{aligned}$$

Normally, one restricts $|\eta\rangle$ so that $\langle\eta|Q|\eta\rangle = 0 = \langle\eta|P|\eta\rangle$, and $\langle\eta|Q^2|\eta\rangle \to 0$ and $\langle\eta|P^2|\eta\rangle \to 0$ as $\hbar \to 0$. In this case

$$\lim_{\hbar\to 0} H(p,q) = \mathcal{H}(p,q)\,.$$

However, in the present paper we want to deal with more general fiducial vectors $|\eta\rangle$ such that

$$\begin{aligned}\langle\eta|Q|\eta\rangle &= q_\eta\,,\\ \langle\eta|P|\eta\rangle &= p_\eta\,,\end{aligned}$$

are arbitrary real variables. We still insist on vanishing dispersion as $\hbar \to 0$, namely, that

$$\begin{aligned}\langle\eta|(Q-q_\eta)^2|\eta\rangle &\to 0\,,\\ \langle\eta|(P-p_\eta)^2|\eta\rangle &\to 0\,,\end{aligned}$$

as $\hbar \to 0$. This more general situation leads to the result

$$\lim_{\hbar\to 0} H(p,q) = \mathcal{H}(p+p_\eta, q+q_\eta)$$

as the representative of the classical Hamiltonian.

In this more general case the classical action appropriate to the coherent state path integral becomes

$$I = \int[(p+p_\eta)\dot{q} - q_\eta\dot{p} - \mathcal{H}(p+p_\eta, q+q_\eta)]dt\,.$$

In this expression $p = p(t)$ and $q = q(t)$, while p_η and q_η are time-independent constants. The term $\int(p_\eta\dot{q} - q_\eta\dot{p})dt = p_\eta(q'' - q') - q_\eta(p'' - p')$ is a pure surface term and will not affect the equations of motion; it could be eliminated simply by a phase change of the coherent states. Extremal variation leads to the equations of motion

$$\begin{aligned}\dot{q} &= \partial\mathcal{H}(p+p_\eta, q+q_\eta)/\partial p\\ \dot{p} &= -\partial\mathcal{H}(p+p_\eta, q+q_\eta)/\partial q\,,\end{aligned}$$

which have as their solutions

$$\begin{aligned} q(t) &= q_c(t) - q_\eta\,, \\ p(t) &= p_c(t) - p_\eta\,, \end{aligned}$$

where $q_c(t)$ and $p_c(t)$ denote a generic solution of Hamilton's equations when $q_\eta = p_\eta = 0$, as discussed above.

Finally, we note that although the *dispersion* of $|\eta\rangle$ vanishes as $\hbar \to 0$, the generally nonvanishing values of q_η and p_η are *vestiges of the coherent-state representation induced by* $|\eta\rangle$ *that remain even after* $\hbar \to 0$.

4.1 Classical Limit of the Universal Propagator

In the case of the universal propagator the expression that serves as the classical action is identified as [cf. (5)]

$$I = \int [q\dot{p} + k\dot{q} - x\dot{p} - \mathcal{H}(k,x)]dt\,. \tag{9}$$

Extremal variation of this expression holding the end points fixed leads to the set of equations

$$\begin{aligned} \dot{q} &= \dot{x}\,, \\ \dot{p} &= \dot{k}\,, \\ \dot{q} &= \partial\mathcal{H}(k,x)/\partial k\,, \\ \dot{p} &= -\partial\mathcal{H}(k,x)/\partial x\,. \end{aligned}$$

Consequently

$$\begin{aligned} \dot{x} &= \partial\mathcal{H}(k,x)/\partial k\,, \\ \dot{k} &= -\partial\mathcal{H}(k,x)/\partial x\,, \end{aligned}$$

which show that (k,x) satisfy exactly the same equations of motion as do (p_c, q_c) in the usual classical theory. Thus we may identify the solution $k(t), x(t)$ with $p_c(t), q_c(t)$. In addition, we have

$$\begin{aligned} q(t) &= q_c(t) - c_q\,, \\ p(t) &= p_c(t) - c_p\,, \end{aligned}$$

where c_q and c_p denote two arbitrary integration constants. Among all possible values of c_q and c_p are those that coincide with q_η and p_η for a general $|\eta\rangle$.

Thus we find that the set of solutions of the universal classical equations of motion appropriate to the universal propagator includes *every possible solution* of the classical equations of motion appropriate to the most general coherent-state propagator (with $|\eta\rangle$ having vanishing dispersion as $\hbar \to 0$). Not only does the quantum dynamics (universal propagator) correctly evolve the state vectors in a canonical coherent-state representation for a general $|\eta\rangle$, but the classical dynamics (universal classical equations of motion) correctly evolves the classical phase space points according to the coherent-state induced classical equations imprinted with arbitrary values of the only remnant of the fiducial vector after $\hbar \to 0$, namely its average coordinate and momentum values.

5 Extension to Other Coherent States

We observe that the procedure to introduce a Schrödinger equation and a path integral solution for the universal propagator applies for other sets of coherent states, such as the spin coherent states, the affine [or SU(1,1)] coherent states, etc.[6] In each of these cases it becomes possible to introduce an appropriate universal propagator just by following the procedure we have given for the canonical coherent state case.

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PHASE-SPACE QUANTUM MECHANICS STUDY OF TWO IDENTICAL PARTICLES IN AN EXTERNAL OSCILLATORY POTENTIAL

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Abstract

We use this simple example to show how the formalism of Moyal works when it is applied to systems of identical particles. The symmetric and antisymmetric Moyal propagators are evaluated for this case; from them, the correct energy levels of energy are obtained, as well as the Wigner functions for the symmetric and antisymmetric states of the two identical particle system. Finally, the solution of the Bloch equation is straightforwardly obtained from the expressions of the Moyal propagators.

1 Phase-space Q M formalism

The original ideas of this approach to Q M are due to Weyl [1], Wigner [2] and Moyal [3]. States and observables are no longer operators on a Hilbert space but functions on an adequate phase space. The Weyl mapping relates both formalisms: given a function f defined over the phase space R^{2n}, the corresponding operator $\hat{F}$ is given by

$$\hat{F} = W(f) = \frac{1}{(2\pi\hbar)^n} \int_{R^{2n}} f(\mathbf{u})\, \Pi(\mathbf{u})\, d\mathbf{u}; \qquad \mathbf{u} = (\mathbf{q}, \mathbf{p}). \tag{1}$$

Reciprocally, given an operator $\hat{A}$ the associated function in the phase space is

$$f_{\hat{A}}(\mathbf{u}) = tr\left\{\hat{A}\Pi(\mathbf{u})\right\} = W^{-1}\left(\hat{A}\right). \tag{2}$$

As we can see, a central role is played by the "Grossman-Royer" operators [4, 5]:

$$[\Pi(\mathbf{q},\mathbf{p})\Psi](\boldsymbol{\eta}) = 2^n \exp\left[\frac{2i}{\hbar}\,\mathbf{p}(\boldsymbol{\eta}-\mathbf{q})\right] \Psi(2\mathbf{q}-\boldsymbol{\eta}). \tag{3}$$

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The twisted product of two functions is defined as the non-commutative operation that corresponds to the product of operators:

$$
\begin{aligned}
(f \times g)(\mathbf{u}) &= W^{-1}\Big(W(f)\,W(g)\Big) \\
&= \frac{1}{(\pi\hbar)^{2n}} \int_{R^{4n}} f(\mathbf{v})g(\mathbf{w}) \exp\left[\frac{2i}{\hbar}(\mathbf{u}\mathrm{J}\mathbf{v} + \mathbf{v}\mathrm{J}\mathbf{w} + \mathbf{w}\mathrm{J}\mathbf{u})\right] d\mathbf{v}\, d\mathbf{w},
\end{aligned}
\tag{4}
$$

where the matrix J is simply

$$
\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}, \tag{5}
$$

being I the n-dimensional identity matrix.

In the Schrödinger representation of quantum mechanics, the information about dynamics is contained in the evolution operator $\hat{U}(t)$. Its counterpart in this formalism is the "Moyal propagator", defined as

$$
\Xi(\mathbf{u}, t) = W^{-1}\Big(\hat{U}(t)\Big). \tag{6}
$$

It verifies Schrödinger equation:

$$
i\hbar \frac{\partial \Xi}{\partial t} = H \times \Xi. \tag{7}
$$

The Fourier transform of this function with respect to t gives the spectral projections parametrized by E:

$$
\Gamma(\mathbf{u}, E) = \frac{1}{2\pi\hbar} \int_R \Xi(\mathbf{u}, t)\, e^{iEt/\hbar}\, dt. \tag{8}
$$

If the Hamiltonian is time independent, the support on E of Γ coincides with the spectrum of $\hat{H}$ [6]. If E_0 belongs to the discrete spectrum of $\hat{H}$, $\Gamma(\mathbf{u}, E_0)$ is, but for a constant factor, the Wigner function of the orthogonal projector into the proper subspace E_0 [6]:

$$
\mathcal{W}_\psi(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^n} W^{-1}\Big(|\psi\rangle\langle\psi|\Big) = \frac{1}{(2\pi\hbar)^n} \int_{R^n} e^{i\mathbf{p}\mathbf{y}/\hbar} \psi^*(\mathbf{q} + \mathbf{y}/2)\psi(\mathbf{q} - \mathbf{y}/2)\, d\mathbf{y}. \tag{9}
$$

2 Phase-space Q M formalism for identical particles

In the standard formalism of quantum mechanics, to deal with a system of N identical particles, we introduce a superselection rule: the space of physical states is a closed subspace of the initial Hilbert space. The Hilbert space is splitted [7]

$$
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_w, \tag{10}
$$

where $\mathcal{H}_+$ is the Hilbert space of the wave functions symmetric under the exchange of any two particles and $\mathcal{H}_-$ the Hilbert space of the antisymmetric functions. The functions in $\mathcal{H}_w$ have no symmetry of this kind. The orthogonal projectors are given by

$$
P_+ = \frac{1}{N!} \sum_{\sigma \in P_N} P_\sigma\,, \qquad P_- = \frac{1}{N!} \sum_{\sigma \in P_N} (-1)^{\pi(\sigma)} P_\sigma\,, \tag{11}
$$

where

$$(P_\sigma\psi)(x_1,\ldots,x_n) = \psi(x_{\sigma(1)},\ldots,x_{\sigma(N)}). \tag{12}$$

for any σ in the group P_N of permutations of N elements; $\pi(\sigma)$ is the parity of σ.

If B is either an observable or an state in $\mathcal{H}$, the corresponding operators for a system of N fermions or bosons are:

$$P_-BP_- \qquad \text{and} \qquad P_+BP_+. \tag{13}$$

If B is invariant under the exchange of particles, we have

$$P_\pm BP_\pm = BP_\pm = P_\pm B. \tag{14}$$

We use the Weyl transformation to translate these ideas into the language of phase space. Therefore, the function for an state or observable when we consider N bosons or fermions is

$$W^{-1}(P_\pm BP_\pm) = W^{-1}(P_\pm)\times W^{-1}(B)\times W^{-1}(P_\pm). \tag{15}$$

Due to the fact that the Weyl map is linear, all we need is the function for any permutation σ. As σ can be written as the product of cyclic permutations with no common elements [8], it is enough to compute the function corresponding to such a cycle. If we consider a general cycle $\sigma = (1,2,3,\ldots,M)$ we get:

$$\tilde{\sigma} = 2^{(M-1)n}\exp\left\{-\frac{2i}{\hbar}\sum_{k=1;l>k}^{M}(-1)^{k+l}\,\mathbf{u}_k\mathrm{J}\mathbf{u}_l\right\}, \quad M \text{ odd}; \tag{16}$$

$$\tilde{\sigma} = (2^{M-1}\,\pi\hbar)^n\delta(\mathbf{u}_1-\mathbf{u}_2+\ldots-\mathbf{u}_M)\exp\left\{-\frac{2i}{\hbar}\sum_{k=1;l>k}^{M}(-1)^{k+l}\,\mathbf{u}_k\mathrm{J}\mathbf{u}_l\right\}, \quad M \text{ even}. \tag{17}$$

As an example, for a two cycle that exchanges the particles i and j we have:

$$\tilde{\sigma}_{ij}(\mathbf{u}_1,\ldots,\mathbf{u}_N) = \tilde{\sigma}_{ij}(\mathbf{u}_i,\mathbf{u}_j) = (2\pi\hbar)^n\,\delta(\mathbf{u}_i-\mathbf{u}_j), \tag{18}$$

and it can be checked that

$$(\tilde{\sigma}_{ij}\times\rho\times\tilde{\sigma}_{ij})(\ldots\mathbf{u}_i,\ldots\mathbf{u}_j,\ldots) = \rho(\ldots\mathbf{u}_j,\ldots\mathbf{u}_i,\ldots). \tag{19}$$

The functions corresponding to the orthogonal projectors for a system of two onedimensional particles are

$$p_+(\mathbf{u}_1,\mathbf{u}_2) = \frac{1}{2}(1+2\pi\hbar\,\delta(\mathbf{u}_1-\mathbf{u}_2)), \tag{20}$$

$$p_-(\mathbf{u}_1,\mathbf{u}_2) = \frac{1}{2}(1-2\pi\hbar\,\delta(\mathbf{u}_1-\mathbf{u}_2)). \tag{21}$$

3 Two onedimensional identical particles under an external oscillatory potential

Along the present section, we intend to presenting an example of particular interest in order to illustrate the preceeding discussion. We shall study the behavior of a two onedimensional particle system subjected to oscillatory forces of the same frequency. If we do not take into account the identity of the particles, the Hamiltonian will be simply:

$$H(\mathbf{u}_1, \mathbf{u}_2) = H(\mathbf{u}_1) + H(\mathbf{u}_2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2). \tag{22}$$

The corresponding Moyal propagator has been already evaluated [6], and is:

$$\Xi(\mathbf{u}_1, \mathbf{u}_2) = \Xi(\mathbf{u}_1)\Xi(\mathbf{u}_2) = \frac{1}{\cos^2 \frac{\omega t}{2}} \exp\left\{-\frac{2i}{\hbar\omega}(H(\mathbf{u}_1) + H(\mathbf{u}_2)) \tan \frac{\omega t}{2}\right\}. \tag{23}$$

Let us now introduce the statistics. As $H(\mathbf{u}_1, \mathbf{u}_2)$ is invariant under permutations of the two particles, the Hamiltonian for our system of two identical particles is:

$$H_\pm(\mathbf{u}_1, \mathbf{u}_2) = (H \times p_\pm)(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}\left\{H(\mathbf{u}_1) + H(\mathbf{u}_2) \mp 2\pi\hbar^3\left(\frac{\delta(\mathbf{u}_1 - \mathbf{u}_2)}{2m(q_1 - q_2)^2} + \frac{m\omega^2}{2}\frac{\delta(\mathbf{u}_1 - \mathbf{u}_2)}{(p_1 - p_2)^2}\right)\right\}; \tag{24}$$

We see that, after symmetrization or antisymmetrization, the Hamiltonian on phase space of our system is not longer (22) but (24). Equation (24) includes (22) plus an extra term. From this term results an extra potential, due to the introduction of the statistics, which has a quite different action depending whether the particles are fermions or bosons. In the first case, this potential is preceeded by a plus sign and, therefore, it is equivalent to a delta barrier preventing that $q_1 = q_2$ and $p_1 = p_2$. This already suggests that both particles cannot remain in the same state and, hence, that they fulfill the Pauli principle. This idea will be confirmed by our calculations for the lowest energy levels. On the contrary, if the particles are bosons the extra term has a minus sign and, consequently, it represents the apparition of a delta well. This delta well would rather favor the presence of particles in the same quantum state. In a clear opposition to the case of fermions, no exclusion principle can exist here.

The symmetrized Moyal propagator is obtained in a similar way:

$$\begin{aligned}\Xi_\pm(\mathbf{u}_1, \mathbf{u}_2) \quad = \quad & \frac{1}{2}\left\{\frac{1}{\cos^2 \frac{\omega t}{2}} \exp\left(\frac{-2i(H(\mathbf{u}_1) + H(\mathbf{u}_2))}{\hbar\omega} \tan \frac{\omega t}{2}\right)\right. \\ & \left. \pm(-i)\frac{\exp\left\{\frac{-i}{\hbar\omega}H(\mathbf{u}_1 + \mathbf{u}_2) \tan \frac{\omega t}{2}\right\}}{\cos \frac{\omega t}{2}} \frac{\exp\left\{\frac{i}{\hbar\omega}H(\mathbf{u}_1 - \mathbf{u}_2) \cot \frac{\omega t}{2}\right\}}{\sin \frac{\omega t}{2}}\right\}. \end{aligned} \tag{25}$$

Comparing with (23) we see that there is also an extra term due to the statistics.

The spectral projections are obtained from (8) and (25); in this case we obtain

$$\begin{aligned}\Gamma_\pm(\mathbf{u}_1, \mathbf{u}_2, E) = & \, 2e^{-2\left(H(\mathbf{u}_1)+H(\mathbf{u}_2)\right)/\hbar\omega} \sum_{k=0}^{\infty}(-1)^k \delta\left(E - \hbar\omega(k+1)\right) \\ & \times \sum_{n=0}^{k}\left[L_n\left(\frac{4H(\mathbf{u}_1)}{\hbar\omega}\right) L_{k-n}\left(\frac{4H(\mathbf{u}_2)}{\hbar\omega}\right) \pm (-1)^{k-n} L_n\left(\frac{2H(\mathbf{u}_1+\mathbf{u}_2)}{\hbar\omega}\right) L_{k-n}\left(\frac{2H(\mathbf{u}_1-\mathbf{u}_2)}{\hbar\omega}\right)\right]. \end{aligned} \tag{26}$$

From here, the well know energy levels are obtained for the fermionic and bosonic cases. Let us notice the coefficient of $\delta(E-\hbar\omega)$, that vanishes for fermions but not for bosons.

We can evaluate the Wigner functions corresponding to states of two particles, in states i and j. Let us write those functions as $\mathcal{W}^a_{ij}$, the corresponding to the antisymmetric state, and $\mathcal{W}^s_{ij}$ the associated to the symmetric state. We then have:

$$\begin{aligned}\Gamma_+(\mathbf{u}_1,\mathbf{u}_2,E) &= (2\pi\hbar)^2[\,\mathcal{W}^s_{00}(\mathbf{q},\mathbf{p})\,\delta\,(E-\hbar\omega)+\mathcal{W}^s_{01}(\mathbf{q},\mathbf{p})\,\delta\,(E-2\hbar\omega)\\ &\qquad +(\mathcal{W}^s_{11}(\mathbf{q},\mathbf{p})+\mathcal{W}^s_{02}(\mathbf{q},\mathbf{p}))\,\delta\,(E-3\hbar\omega)+\cdots], \qquad (27)\\ \Gamma_-(\mathbf{u}_1,\mathbf{u}_2,E) &= (2\pi\hbar)^2[\,\mathcal{W}^a_{01}(\mathbf{q},\mathbf{p})\,\delta\,(E-2\hbar\omega)+\mathcal{W}^a_{02}(\mathbf{q},\mathbf{p})\,\delta\,(E-3\hbar\omega)+\cdots]. \qquad (28)\end{aligned}$$

The coefficients of the δ are the Wigner functions of the orthogonal projector on the corresponding eigenspaces.

To finish, let us solve the Bloch equation, that is, let us find the Wigner function corresponding to the density matrix of the canonical ensemble for the system we are considering. Bloch equation reads simply:

$$\frac{\partial\Omega}{\partial\beta}=-H\times\Omega=-\Omega\times H,\quad \beta=1/kT, \qquad (29)$$

that is, it is Schrödinger's equation with the change $t\longrightarrow -i\hbar\beta$.

But, as we already know the form of the Moyal propagator, we can write inmediately the solution for $\Omega(\mathbf{u}_1,\mathbf{u}_2,\beta)$ by making the change $t\longrightarrow -i\hbar\beta$ in $\Xi_\pm(\mathbf{u}_1,\mathbf{u}_2,t)$. We get:

$$\begin{aligned}\Omega_\pm(\mathbf{u}_1,\mathbf{u}_2,\beta) &= \frac{1}{2}\left\{\frac{1}{\cosh^2\frac{\hbar\omega\beta}{2}}\exp\left(\frac{-2(H(\mathbf{u}_1)+H(\mathbf{u}_2))}{\hbar\omega}\tanh\frac{\hbar\omega\beta}{2}\right)\right. \qquad (30)\\ &\qquad \left.\pm\frac{\exp\left\{\frac{-1}{\hbar\omega}H(\mathbf{u}_1+\mathbf{u}_2)\tanh\frac{\hbar\omega\beta}{2}\right\}}{\cosh\frac{\hbar\omega\beta}{2}}\,\frac{\exp\left\{\frac{-1}{\hbar\omega}H(\mathbf{u}_1-\mathbf{u}_2)\coth\frac{\hbar\omega\beta}{2}\right\}}{\sinh\frac{\hbar\omega\beta}{2}}\right\}. \qquad (31)\end{aligned}$$

After integration of $\Omega(\mathbf{u}_1,\mathbf{u}_2,\beta)$ over the phase space, we get

$$Z_\pm(\beta)=\frac{\exp(\pm\hbar\omega\beta/2)}{8\cosh(\hbar\omega\beta/2)\sinh^2(\hbar\omega\beta/2)}. \qquad (32)$$

From this partition function, we can obtain the thermodynamical quantities, for example the internal energy, the free energy and the entropy

$$E_\pm(\beta)=\frac{\hbar\omega}{2}\left\{\tanh\frac{\hbar\omega\beta}{2}+2\coth\frac{\hbar\omega\beta}{2}\right\}\mp\frac{\hbar\omega}{2}, \qquad (33)$$

$$F_\pm(\beta)=\frac{1}{\beta}\left\{\log\left[\cosh\frac{\hbar\omega\beta}{2}\right]+2\log\left[\sinh\frac{\hbar\omega\beta}{2}\right]+\log 8\right\}\mp\frac{\hbar\omega}{2}, \qquad (34)$$

$$S_\pm(\beta)=k\left\{\frac{\hbar\omega\beta}{2}\left[\tanh\frac{\hbar\omega\beta}{2}+2\coth\frac{\hbar\omega\beta}{2}\right]-\ln\left[\cosh\frac{\hbar\omega\beta}{2}\right]-2\ln\left[\sinh\frac{\hbar\omega\beta}{2}\right]-\ln 8\right\}. \qquad (35)$$

Notice that the entropy is the same in both cases (bosonic and fermionic).

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